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Oscillatory behavior of solutions of differential equations with fractional order

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Abstract: In this paper, we investigated a kind of fractional differential equations with damping term. Using generalized Riccati function, we present some oscillation criteria. Finally, some examples are given to illustrative the main results.

Keywords: Oscillation, fractional differential equation, fractional derivative, damping term.

1 Introduction

Fractional differential equations have been proved to be valuable tools in the modelling of many physical and engineering phenomena such as bioengineering, electromagnetism, electronics, polymer physics, chaos and fractals, electrical networks, traffic systems, signal processing, heat transfer, system identification, industrial robotics, viscous damping, fluid flows, genetic algorithms, economics, etc, [1,2,3,4,5,6]. For the many theories and applications of fractional differential equations, we refer the monographs [7,8,9,10].

Recently, research for oscillation of various equations like ordinary and partial differential equations, difference equations, dynamic equations on time scales, and fractional differential equations has been a hot topic in the literature, and much effort has been done to establish oscillation criteria for these equations [11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26] . In these studies, some attention has been paid to oscillations of fractional differential equations [27,28,29,30,31,32].

In [27], Chen considered the oscillation for a class of fractional differential equation,

$$\left[r(t)\left(D_{-}^{\alpha}y\right)^{\eta}(t)\right]' - q(t)\,f\left(\int_t^{\infty}(v-t)^{-\alpha}y(v)\,dv\right) = 0$$

where $t>0$, D_{-}^{α} is the Liouville right-sided fractional derivative of order α with $0<\alpha<1$, η is a quotient of odd positive integers, $r \in C^1([t_0,\infty),\mathbb{R}_+)$, $q \in C([t_0,\infty),\mathbb{R}_+)$ with $t_0>0$. and the function of f belong to $C(\mathbb{R},\mathbb{R})$, $f(x)/x^{\eta} > K$ for all $K\in\mathbb{R}_+$, $x\neq 0$. By using a generalized Riccati transformation technique and an inequality, the author established some oscillation criteria for the equation.

In [28], Zheng researched oscillation of the equations

$$\left[a(t)\left(D_{-}^{\alpha}x(t)\right)^{\eta}\right]' + p(t)\left(D_{-}^{\alpha}x(t)\right)^{\eta} - q(t)\,f\left(\int_t^{\infty}(\xi-t)^{-\alpha}y(\xi)\,d\xi\right) = 0$$

where $t\in[t_0,\infty)$, $\alpha\in(0,1)$, D_{-}^{α} is the Liouville right-sided fractional derivative of order α. Based on a generalized Riccati function and inequality technique, the author established some oscillation criteria for the equation.

In [29], Han *et al.* have established some oscillation criteria for a class of fractional differential equation:

$$\left[r(t)\,g\left(\left(D_{-}^{\alpha}y\right)(t)\right)\right]' - p(t)\,f\left(\int_t^{\infty}(s-t)^{-\alpha}y(s)\,ds\right) = 0$$

where $t>0$, D_{-}^{α} is the Liouville right-sided fractional derivative of order α with $0<\alpha<1$, r and p are positive continuous functions on $[t_0,\infty)$ for $t_0>0$, f, $g:\mathbb{R}\to\mathbb{R}$ are continuous function with $xf(x)>0$, $xg(x)>0$ for $x\neq 0$, there exists some positive constant k_1, k_2 such that $f(x)/x > k_1$, $x/g(x) > k_2$ for all $x\neq 0$, and $g^{-1}\in C(\mathbb{R},\mathbb{R})$ with $ug^{-1}(u)>0$ for $u\neq 0$, there exist

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positive constant γ_1 such that $g^{-1}(uv) > \gamma_1 g^{-1}(u) g^{-1}(v)$ for $uv \neq 0$. By generalized Riccati transformation technique, oscillation criteria for the equation are obtained.

In [30], Qi and Cheng studied the oscillation of the differential equation with fractional-order derivatives:

$$\left[a(t)\left[r(t) D_{-}^{\alpha} x(t)\right]'\right]' + p(t)\left[r(t) D_{-}^{\alpha} x(t)\right]' - q(t)\int_t^{\infty}(\xi - t)^{-\alpha} x(\xi)\, d\xi = 0$$

where $t \in [t_0, \infty)$, $\alpha \in (0,1)$, D_{-}^{α} is the Liouville right-sided fractional derivative of order α, and $r \in C^2([t_0,\infty),\mathbb{R}_+)$, $a \in C^1([t_0,\infty),\mathbb{R}_+)$, and p, $q \in C([t_0,\infty),\mathbb{R}_+)$ with $t_0 > 0$. The authors established some interval oscillation criteria for the equation by a certain Riccati transformation and inequality technique.

In [31], Xiang *et al.* studied the oscillation behavior of the equation with the form

$$\left[a(t)\left(p(t) + q(t)\left(D_{-}^{\alpha} x\right)(t)\right)^{\eta}\right]' - b(t) f\left(\int_t^{\infty}(s-t)^{-\alpha} x(s)\, ds\right) = 0$$

where $t \geq t_0 > 0$, $\alpha \in (0,1)$, D_{-}^{α} is the Liouville right-sided fractional derivative of order α, η is a quotient of odd positive integers, a, b and q are positive continuous functions on $[t_0,\infty)$ for $t_0 > 0$, p is a nonnegative continuous functions on $[t_0,\infty)$ for $t_0 > 0$, and $f \in C(\mathbb{R},\mathbb{R})$ with $f(x)/x^{\eta} > K$ for all $K \in \mathbb{R}_+$, $x \neq 0$. By using a generalized Riccati transformation technique and an inequality, the authors established some oscillation theorems for the equation.

In [32], Xu researched oscillation of the following fractional differential equations

$$\left[a(t)\left[\left(r(t) D_{-}^{\alpha} x(t)\right)'\right]^{\eta}\right]' - F\left(t, \int_t^{\infty}(v-t)^{-\alpha} x(v)\, dv\right) = 0$$

where $t \in [t_0,\infty)$, $\alpha \in (0,1)$, D_{-}^{α} is the Liouville right-sided fractional derivative of order α, η is a quotient of odd positive integers, $r \in C^2([t_0,\infty),\mathbb{R}_+)$, $a \in C^1([t_0,\infty),\mathbb{R}_+)$ and $F\left(t, \int_t^{\infty}(v-t)^{-\alpha} x(v)\, dv\right) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, there exists a function $q \in C([t_0,\infty),\mathbb{R}_+)$ such that $F\left(t, \int_t^{\infty}(v-t)^{-\alpha} x(v)\, dv\right) / \left(\int_t^{\infty}(v-t)^{-\alpha} x(v)\, dv\right)^{\eta} \geq q(t)$ for $\int_t^{\infty}(v-t)^{-\alpha} x(v)\, dv \neq 0$ and $x \neq 0$, $t \geq t_0$. The author was dealing with the oscillation problem of the equation.

Now, in this study, we are concerned with the oscillation of nonlinear fractional differential equations of the form;

$$\left[a(t)\left[\left(r(t)\left(D_{-}^{\alpha} x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\right]' + p(t)\left[\left(r(t)\left(D_{-}^{\alpha} x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2} - q(t) f\left(\int_t^{\infty}(s-t)^{-\alpha} x(s)\, ds\right) = 0 \quad (1)$$

where $t \in [t_0,\infty)$, $\alpha \in (0,1)$, γ_1 and γ_2 are the quotient of two odd positive number,the function $a \in C^1([t_0,\infty),\mathbb{R}_+)$, $r \in C^2([t_0,\infty),\mathbb{R}_+)$, $q \in C([t_0,\infty),\mathbb{R}_+)$,the function of f belong to $C(\mathbb{R},\mathbb{R})$, $f(x)/x \geq k$ for all $k \in \mathbb{R}_+$, $x \neq 0$, $\alpha \in (0,1)$, and $D_{-}^{\alpha} x(t)$ denotes the Liouville right-side fractional derivative of order α of $x(t)$ defined by

$$D_{-}^{\alpha} x(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{\infty}(s-t)^{-\alpha} x(s)\, ds$$

where $t \in \mathbb{R}_+$ and Γ is the gamma function defined by

$$\Gamma(t) = \int_t^{\infty} s^{t-1} e^{-s} ds, t \in \mathbb{R}_+$$

As usual, a solution $x(t)$ of (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

2 Preliminaries

In this section, we present some background materials from fractional calculus theory, which will be used throughout this paper.

Definition 2.1. [8]:The Liouville right-sided fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$\left(I_{-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^{\infty}\frac{f(t)\, dt}{(t-x)^{1-\alpha}}, \text{ for } t > 0$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 2.2. [8]: The Liouville right-sided fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$\begin{aligned}\left(D_{-}^{\alpha} f\right)(x) &:= (-1)^{\lceil\alpha\rceil}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I_{-}^{\lceil\alpha\rceil-\alpha} f\right)(x) \\ &= (-1)^{\lceil\alpha\rceil}\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \\ &\times \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_x^{\infty}\frac{f(t)\, dt}{(t-x)^{\alpha-\lceil\alpha\rceil+1}}, \text{ for } t > 0\end{aligned}$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$, where $\lceil\alpha\rceil := \min\{z \in \mathbb{Z} : z \geq \alpha\}$ is the ceiling function.

Before our main results, now we state a useful lemma.

Lemma 2.3. [33]: Assume that A and B are nonnegative real numbers. Then,

$$\lambda AB^{\lambda-1}-A^{\lambda}\leq(\lambda-1)B^{\lambda} \tag{2}$$

for all $\lambda>1$.

3 Main Results

In this section, we establish some oscillation criteria for (1). Firstly, for the sake of convenience, we denote $\delta_1(t,t_i)=\int_{t_i}^{t}\left(1/a^{1/\gamma_2}(s)\right)ds$ for $i=0,1,2,3,4,5$; $A(t)=\exp\left(\int_{t_0}^{t}(p(s)/a(s))\,ds\right)$ and let $G(t)=\int_t^{\infty}(s-t)^{-\alpha}x(s)\,ds$ for $\alpha\in(0,1)$, $t>0$. Then, using Definition 2.2., we obtain $G'(t):=-\Gamma(1-\alpha)\left(D_-^{\alpha}x\right)(t)$.

Lemma 3.1. Assume $x(t)$ is an eventually positive solution of (1), and

$$\int_{t_0}^{\infty}\frac{1}{[A(s)a(s)]^{1/\gamma_2}}ds=\infty \tag{3}$$

$$\int_{t_0}^{\infty}\frac{1}{r^{1/\gamma_1}(s)}ds=\infty \tag{4}$$

$$\int_{t_0}^{\infty}\left[\frac{1}{r(\zeta)}\int_{\zeta}^{\infty}[B(\tau)]^{1/\gamma_2}d\tau\right]^{1/\gamma_1}d\zeta=\infty \tag{5}$$

where $B(x)=[A(x)a(x)]^{-1}\int_x^{\infty}A(s)q(s)\,ds$. Then, there exist a sufficiently large T such that $\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'<0$ on $[T,\infty)$ and either $D_-^{\alpha}x(t)<0$ on $[T,\infty)$ or $\lim_{t\to\infty}G(t)=0$.

Proof. From the hypothesis, there exist a t_1 such that $x(t)>0$ on $[t_1,\infty)$, so that $G(t)>0$ on $[t_1,\infty)$, and we have

$$\begin{aligned}&\left[A(t)a(t)\left[\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\right]'\\&=A(t)q(t)f\left(\int_t^{\infty}(v-t)^{-\alpha}x(v)\,dv\right)\\&\geq kA(t)q(t)G(t)>0,\ t>t_1\end{aligned} \tag{6}$$

Then $A(t)a(t)\left[\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}$ is strictly increasing on $[t_1,\infty)$, thus we know that $\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'$ is eventually of one sign. For $t_2>t_1$ is sufficiently large, we claim $\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'<0$ on $[t_2,\infty)$. Otherwise, assume that there exists a sufficiently large $t_3>t_2$ such that $\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'>0$ on $[t_3,\infty)$ .Thus, we get that

$$\begin{aligned}&r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}-r(t_3)\left(D_-^{\alpha}x(t_3)\right)^{\gamma_1}\\&=\int_{t_3}^{t}\frac{\left(A(s)a(s)\left[\left(r(s)\left(D_-^{\alpha}x(s)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\right)^{1/\gamma_2}}{[A(s)a(s)]^{1/\gamma_2}}ds\\&\geq A^{1/\gamma_2}(t_3)a^{1/\gamma_2}(t_3)\left(r(t_3)\left(D_-^{\alpha}x(t_3)\right)^{\gamma_1}\right)'\\&\times\int_{t_3}^{t}\frac{1}{[A(s)a(s)]^{1/\gamma_2}}ds\end{aligned} \tag{7}$$

Then from (3) we have $\lim_{t\to\infty}r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}=+\infty$, which implies that for a certain constant $t_4>t_3$, $D_-^{\alpha}x(t)>0$, $t\in[t_4,\infty)$, then

$$\begin{aligned}G(t)-G(t_4)&=\int_{t_4}^{t}G'(s)\,ds\\&=-\Gamma(1-\alpha)\int_{t_4}^{t}\frac{\left(r(s)\left(D_-^{\alpha}x(s)\right)^{\gamma_1}\right)^{1/\gamma_1}}{r^{1/\gamma_1}(s)}ds\\&\leq-\Gamma(1-\alpha)r^{1/\gamma_1}(t_4)D_-^{\alpha}x(t_4)\\&\times\int_{t_4}^{t}\frac{1}{r^{1/\gamma_1}(s)}ds\end{aligned}$$

by (4) we obtain $\lim_{t\to\infty}G(t)=-\infty$, which contradicts to $G(t)>0$ on $[t_1,\infty)$. So we have $\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'<0$ on $[t_2,\infty)$, and $D_-^{\alpha}x(t)$ is eventually of one sign. Now we assume $D_-^{\alpha}x(t)>0$ on $[t_5,\infty)$ where $t_5>t_4$ is sufficiently large. So, $G'(t)<0$, $t\in[t_5,\infty)$ and we have $\lim_{t\to\infty}G(t)=\beta\geq0$. We claim that $\beta=0$. Otherwise, assume $\beta>0$; then $G(t)\geq\beta$ on $[t_5,\infty)$, and $f(G(t))>k\beta\geq M$ for $M\in\mathbb{R}_+$, by (6), we have

$$\begin{aligned}&\left[A(t)a(t)\left[\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\right]'\\&=q(t)f\left(\int_t^{\infty}(v-t)^{-\alpha}x(v)\,dv\right)\\&\geq kA(t)q(t)G(t)>MA(t)q(t),\ t>t_5\end{aligned} \tag{8}$$

Substituting t with s in (8) and integrating it with respect to s from t to ∞ leads to

$$\begin{aligned}&-A(t)a(t)\left[\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\\&\geq M\int_t^{\infty}A(s)q(s)\,ds\\&-\lim_{t\to\infty}A(t)a(t)\left[\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\end{aligned}$$

That is

$$-A(t)a(t)\left[\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\geq M\int_t^{\infty}A(s)q(s)\,ds$$

which means

$$\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)' \leq -M^{1/\gamma_2}\left[B(t)\right]^{1/\gamma_2} \tag{9}$$

Substituting t with τ in (9) and integrating it with respect to τ from t to ∞ yields

$$-r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1} \leq -M^{1/\gamma_2}\int_t^{\infty}\left[B(\tau)\right]^{1/\gamma_2}d\tau$$

That is

$$G'(t) \leq -M^{1/\gamma_1\gamma_2}\Gamma(1-\alpha)\left[\frac{1}{r(t)}\int_t^{\infty}\left[B(\tau)\right]^{1/\gamma_2}d\tau\right]^{1/\gamma_1} \tag{10}$$

Substituting t with ζ in (10) and integrating it with respect to ζ from t_5 to t, we have

$$G(t)-G(t_5) \leq -M^{1/\gamma_1\gamma_2}\Gamma(1-\alpha) \times \int_{t_5}^{t}\left[\frac{1}{r(\zeta)}\int_{\zeta}^{\infty}\left[B(\tau)\right]^{1/\gamma_2}d\tau\right]^{1/\gamma_1}d\zeta$$

By (5), we have $\lim_{t\to\infty}G(t)=-\infty$, which contradicts to the fact that $G(t)>0$. Then we get that $\beta=0$, which is $\lim_{t\to\infty}G(t)=0$. The proof is complete.

Lemma 3.2. Assume that $x(t)$ is an eventually positive solution of (1) such that

$$\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'<0,\quad D_{-}^{\alpha}x(t)<0 \tag{11}$$

on $[t_1,\infty)$, where $t_1>t_0$ is sufficiently large. Then, for $t\geq t_1$, we have

$$G'(t) \geq -\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_1)A^{1/\gamma_1\gamma_2}(t)a^{1/\gamma_1\gamma_2}(t) \times \frac{\left[\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{1/\gamma_1}}{r^{1/\gamma_1}(t)} \tag{12}$$

Proof. Assume that x is an eventually positive solution of (1). So, by (6), we obtain that $a(t)\left[\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\right]^{\gamma_2}$ is strictly increasing on $[t_1,\infty)$. Then,

$$\begin{aligned}
&r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\\
&\leq r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}-r(t_1)\left(D_{-}^{\alpha}x(t_1)\right)^{\gamma_1}\\
&=\int_{t_1}^{t}\frac{\left(A(s)a(s)\left[\left(r(s)\left(D_{-}^{\alpha}x(s)\right)^{\gamma_1}\right)'\right]^{\gamma_2}\right)^{1/\gamma_2}}{\left[A(s)a(s)\right]^{1/\gamma_2}}ds\\
&\leq A^{1/\gamma_2}(t)a^{1/\gamma_2}(t)\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\\
&\times\int_{t_1}^{t}\frac{1}{\left[A(s)a(s)\right]^{1/\gamma_2}}ds\\
&=A^{1/\gamma_2}(t)a^{1/\gamma_2}(t)\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\delta_1(t,t_1).
\end{aligned} \tag{13}$$

That is,

$$D_{-}^{\alpha}x(t) \leq \left[\frac{A^{1/\gamma_2}(t)a^{1/\gamma_2}(t)\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\delta_1(t,t_1)}{r(t)}\right]^{1/\gamma_1} \tag{14}$$

So, the proof is complete.

Theorem 3.3. Assume (3)-(5) and $\gamma_1\gamma_2=1$ hold. If there exists two functions $\phi\in C^1([t_0,\infty),\mathbb{R}_+)$ and $\rho\in C^1([t_0,\infty),[0,\infty))$ such that

$$\begin{aligned}
&\int_{t_2}^{t}\left\{kA(s)q(s)\phi(s)\right.\\
&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(t)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\
&\left.+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\right\}ds\\
&=\infty
\end{aligned} \tag{15}$$

for all sufficiently large T, then every solution of (1) is oscillatory or satisfies $\lim_{t\to\infty}G(t)=0$.

Proof. Suppose the contrary that $x(t)$ is non-oscillatory solution of (1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.1., we have $\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'<0$, $t\in[t_2,\infty)$, where $t_2>t_1$ is sufficiently large, and either $D_{-}^{\alpha}x(t)<0$ on $[t_2,\infty)$ or $\lim_{t\to\infty}G(t)=0$. If we take $D_{-}^{\alpha}x(t)<0$ on $[t_2,\infty)$. Define the following generalized Riccati function:

$$\omega(t)=\phi(t) \times\left\{-\frac{A(t)a(t)\left(\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\right)^{\gamma_2}}{G(t)}+\rho(t)\right\} \tag{16}$$

For $t\in[t_2,\infty)$, we have

$$\begin{aligned}
\omega'(t)=&-\phi'(t)\frac{A(t)a(t)\left(\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\right)^{\gamma_2}}{G(t)}\\
&-\phi(t)\left\{\frac{A(t)a(t)\left(\left(r(t)\left(D_{-}^{\alpha}x(t)\right)^{\gamma_1}\right)'\right)^{\gamma_2}}{G(t)}\right\}'\\
&+\phi'(t)\rho(t)+\phi(t)\rho'(t)
\end{aligned}$$

So,

$$\omega'(t) = \frac{\phi'(t)}{\phi(t)}\omega(t) - \phi(t)\frac{G(t)\left(A(t)a(t)\left(\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right)^{\gamma_2}\right)'}{G^2(t)} + \phi(t)\frac{G'(t)A(t)a(t)\left(\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right)^{\gamma_2}}{G^2(t)} + \phi(t)\rho'(t)$$
$$= \frac{\phi'(t)}{\phi(t)}\omega(t) - \phi(t)\frac{A(t)q(t)f(G(t))}{G(t)} + \phi(t)\frac{G'(t)A(t)a(t)\left(\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)'\right)^{\gamma_2}}{G^2(t)} + \phi(t)\rho'(t)$$

Using (12), we obtain

$$\omega'(t) \le \frac{\phi'(t)}{\phi(t)}\omega(t) - kA(t)q(t)\phi(t) - \phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\left(\frac{\omega(t)}{\phi(t)} - \rho(t)\right)^2 + \phi(t)\rho'(t) \tag{17}$$

That is

$$\omega'(t) \le \frac{\phi'(t)}{\phi(t)}\omega(t) - kA(t)q(t)\phi(t) - \phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\frac{\omega^2(t)}{\phi^2(t)} + \phi(t)\frac{2\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)\rho(t)}{r^{1/\gamma_1}(t)}\frac{\omega(t)}{\phi(t)} - \phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\rho^2(t) + \phi(t)\rho'(t)$$

So, we have

$$\omega'(t) \le -kA(t)q(t)\phi(t) - \frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)\phi(t)}\omega^2(t) + \frac{2\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)\rho(t) + r^{1/\gamma_1}(t)\phi'(t)}{r^{1/\gamma_1}(t)\phi(t)}\omega(t) - \phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\rho^2(t) + \phi(t)\rho'(t) \tag{18}$$

In (18), setting $\lambda = 2$, $A = \left(\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)\phi(t)}\right)^{1/2}\omega(t)$ and $B = \frac{2\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)\rho(t)+r^{1/\gamma_1}(t)\phi'(t)}{2\left(r^{1/\gamma_1}(t)\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)\right)^{1/2}}$, using Lemma 2.3., we have

$$\omega'(t) \le -kA(t)q(t)\phi(t) + \frac{\left[2\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)\rho(t) + r^{1/\gamma_1}(t)\phi'(t)\right]^2}{4r^{1/\gamma_1}(t)\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)} - \phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\rho^2(t) + \phi(t)\rho'(t) \tag{19}$$

Substituting t with s in (19), and integration both sides of (19) with respect to s from t_2 to t yields

$$\int_{t_2}^{t}\left\{kA(s)q(s)\phi(s) - \frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(t) + r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)} + \phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s) + \phi(s)\rho'(s)\right\}ds \le \omega(t_2) < \infty$$

which contradicts to (15), so proof is complete.

Theorem 3.4. Assume (3)-(5) and $\gamma_1\gamma_2 = 1$ hold. Furthermore, suppose that ϕ, ρ are defined as in Theorem 3.3. and there exists a function $H \in C(D,\mathbb{R})$, where $D := \{(t,s)\,|\,t \ge s \ge t_0\}$, such that $H(t,t) = 0$, for $t \ge t_0$, $H(t,s) > 0$, for $t > s \ge t_0$, and H has a non-positive continuous partial derivative $H_s'(t,s)$ and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t}H(t,s)\left\{kA(s)q(s)\phi(s) - \frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s) + r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)} + \phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s) + \phi(s)\rho'(s)\right\}ds\right\} = \infty \tag{20}$$

for all sufficiently large T, then every solution of (1) is oscillatory or satisfies $\lim_{t\to\infty}G(t) = 0$.

Proof. Suppose the contrary that $x(t)$ is non-oscillatory solution of (1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1) such that $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.1., we have $\left(r(t)\left(D_-^{\alpha}x(t)\right)^{\gamma_1}\right)' < 0$, $t \in [t_2,\infty)$, where $t_2 > t_1$

is sufficiently large, and either $D_t^{\alpha}x(t)<0$ on $[t_2,\infty)$ or $\lim_{t\to\infty}G(t)=0$. Let $\omega(t)$, be defined as in Theorem 3.3.. Thus we have (19). So,

$$\begin{aligned}&kA(t)q(t)\phi(t)\\&-\frac{\left[2\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)\rho(t)+r^{1/\gamma_1}(t)\phi'(t)\right]^2}{4r^{1/\gamma_1}(t)\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}\\&+\phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\rho^2(t)+\phi(t)\rho'(t)\\&\le-\omega'(t)\end{aligned}\tag{21}$$

Substituting t with s in (21), multiplying both sides by $H(t,s)$ and then integrating it with respect to s from t_2 to t, we get that

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)\{kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg\}ds\\&\le-\int_{t_2}^{t}H(t,s)\omega'(s)ds\\&=-G(t,t)\omega(t)+G(t,t_2)\omega(t_2)+\int_{t_2}^{t}G_s'(t,s)\omega(s)\Delta s\\&\le G(t,t_2)\omega(t_2)\\&\le G(t,t_0)\omega(t_2)\end{aligned}\tag{22}$$

and then,

$$\begin{aligned}I=&\int_{t_0}^{t}H(t,s)\\&\times\{kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma^{1/\gamma_1}(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma^{1/\gamma_1}(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma^{1/\gamma_1}(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg\}ds\end{aligned}$$

$$\begin{aligned}I=&\int_{t_0}^{t_2}H(t,s)\\&\times\{kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg\}ds\\&+\int_{t_2}^{t}H(t,s)\\&\times\{kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg\}ds\\&\le H(t,t_0)\omega(t_2)\\&+H(t,t_0)\int_{t_0}^{t_2}|kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg|ds\end{aligned}$$

So,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\Bigg\{\int_{t_0}^{t}H(t,s)\{kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg\}ds\Bigg\}\\&\le\omega(t_2)+\int_{t_0}^{t_2}|kA(s)q(s)\phi(s)\\&-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\\&+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\Bigg|ds\\&<\infty\end{aligned}$$

which contradicts (20). So the proof is complete.

Using Theorem 3.3. and Theorem 3.4., we can derive a lot of oscillation criteria with respect to choose H, ϕ and ρ. For instance, we can choose $H(t,s)=(t-s)^{\lambda}$, or $H(t,s)=\ln\left(\frac{t}{s}\right)$, we obtain the following corollaries.

Corollary 3.5. Under the conditions of Theorem 3.4. and

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^{\lambda}}\left\{\int_{t_0}^{t}(t-s)^{\lambda}\left\{kA(s)q(s)\phi(s)-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\right\}ds\right\}=\infty \tag{23}$$

Then every solution of (1) is oscillatory or satisfies $\lim_{t\to\infty}G(t)=0$.

Corollary 3.6 Under the conditions of Theorem 3.4. and

$$\limsup_{t\to\infty}\frac{1}{\ln(t)-\ln(t_0)}\left\{\int_{t_0}^{t}(\ln(t)-\ln(s))\left\{kA(s)q(s)\phi(s)-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\right\}ds\right\}=\infty \tag{24}$$

Then every solution of (1) is oscillatory or satisfies $\lim_{t\to\infty}G(t)=0$.

4 Examples

In this section, we present some examples that apply the main results established.

Example 4.1. Consider the following fractional differential equation

$$\left[t^2\left(\left[\left(D_-^{\alpha}x(t)\right)^{1/2}\right]'\right)^2\right]'-t^{-2}\left(\int_t^{\infty}(s-t)^{-\alpha}x(s)\,ds\right)\times\sin^2\left(\int_t^{\infty}(s-t)^{-\alpha}x(s)\,ds\right)=0,\ t\geq 1 \tag{25}$$

This corresponds to (1) with $t_0=1$; $\gamma_1=\frac{1}{2}$; $\gamma_2=2$; $\alpha\in(0,1)$; $a(t)=t^2$; $r(t)=1$; $p(t)=0$; $q(t)=t^{-2}$ and $f(x)/x=\left(1+\sin^2 x\right)\geq 1=k$. On the other hand,

$$\delta_1(t,t_2)=\int_1^t\frac{1}{s}ds=\ln t$$

which implies $\lim_{t\to\infty}\delta_1(t,t_2)=\infty$, and so, (3) holds. Then, there exists a sufficiently large $T>t_2$ such that $\delta_1(t,t_2)>1$ on $[T,\infty)$. In (4),

$$\int_1^{\infty}\frac{1}{r^{1/\gamma_1}(s)}ds=\int_1^{\infty}ds=\infty \tag{26}$$

In (5),

$$\int_{t_0}^{\infty}\left[\frac{1}{r(\zeta)}\int_{\zeta}^{\infty}[B(\tau)]^{1/\gamma_2}d\tau\right]^{1/\gamma_1}d\zeta=\int_1^{\infty}\left[\int_{\zeta}^{\infty}\left[\frac{1}{\tau^2}\int_{\tau}^{\infty}\frac{1}{s^2}ds\right]^{1/2}d\tau\right]^2 d\zeta=\infty \tag{27}$$

Letting $\phi(t)=t$, $\rho(t)=0$ in (15),

$$\begin{aligned}S&=\int_{t_2}^{t}\left\{kA(s)q(s)\phi(s)-\frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(t)+r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}+\phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s)+\phi(s)\rho'(s)\right\}ds\\&\geq\int_1^t\left\{\frac{1}{s}-\frac{1}{4s\Gamma(1-\alpha)\delta_1^2(s,t_2)}\right\}ds\end{aligned}$$

$$\begin{aligned}S&\geq\int_1^t\left\{1-\frac{1}{4\Gamma(1-\alpha)\delta_1^2(s,t_2)}\right\}\frac{1}{s}ds\\&=\int_1^T\left\{1-\frac{1}{4\Gamma(1-\alpha)\delta_1^2(s,t_2)}\right\}\frac{1}{s}ds+\int_T^t\left\{1-\frac{1}{4\Gamma(1-\alpha)\delta_1^2(s,t_2)}\right\}\frac{1}{s}ds\\&\geq\int_{t_2}^T\left\{1-\frac{1}{4\Gamma(1-\alpha)\delta_1^2(s,t_2)}\right\}\frac{1}{s}ds+\int_T^t\left\{1-\frac{1}{4\Gamma(1-\alpha)}\right\}\frac{1}{s}ds\\&=\infty\end{aligned}$$

So, (25) is oscillatory by Theorem 3.3..

Example 4.2. Consider the following fractional differential equation

$$\left[t^{1/5}\left(\left[\left(D_-^{\alpha}x(t)\right)^5\right]'\right)^{1/5}\right]'+t^{-1}\left(\left[\left(D_-^{\alpha}x(t)\right)^5\right]'\right)^{1/5}-t^{-2}\left(\int_t^{\infty}(s-t)^{-\alpha}x(s)\,ds\right)\times\exp\left(\left(\int_t^{\infty}(s-t)^{-\alpha}x(s)\,ds\right)^2\right)=0,\quad t\geq 2 \tag{28}$$

This corresponds to (1) with $t_0=2$; $\alpha\in(0,1)$; $\gamma_1=5$; $\gamma_2=\frac{1}{5}$; $a(t)=t^{1/5}$; $r(t)=1$; $p(t)=t^{-1}$; $q(t)=t^{-2}$ and $f(x)/x=\exp\left(x^2\right)\geq 1=k$. Moreover,

$1 \leq A(t) = \exp\left(\int_2^t s^{-6/5} ds\right) \leq \exp\left(5/\sqrt[5]{2}\right)$. On the other hand,

$$\delta_1(t,t_2) = \int_{t_0}^{t} \frac{1}{[A(s)a(s)]^{1/\gamma_2}} ds \geq \exp\left(-25/\sqrt[5]{2}\right) \int_2^t \frac{1}{s} ds$$

which implies $\lim_{t\to\infty} \delta_1(t,t_2) = \infty$ and so (3) holds. Then, there exists a sufficiently large $T > t_2$ such that $\delta_1(t,t_2) > 1$ on $[T,\infty)$. In (4),

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma_1}(s)} ds = \int_2^{\infty} ds = \infty \tag{29}$$

In (5),

$$\int_{t_0}^{\infty} \left[\frac{1}{r(\zeta)} \int_{\zeta}^{\infty} [B(\tau)]^{1/\gamma_2} d\tau\right]^{1/\gamma_1} d\zeta \tag{30}$$

$$\geq \exp\left(-5/\sqrt[5]{2}\right) \int_2^{\infty} \left[\int_{\zeta}^{\infty} \left[\tau^{-1/5} \int_{\tau}^{\infty} s^{-2} ds\right]^5 d\tau\right]^{1/5} d\zeta = \infty$$

Letting $\phi(t) = t$, $\rho(s) = 0$ and $\lambda = 1$ in (23), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{(t-t_0)^{\lambda}} \left\{ \int_{t_0}^{t} (t-s)^{\lambda} \left\{ kA(s)q(s)\phi(s) - \frac{\left[2\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)\rho(s) + r^{1/\gamma_1}(s)\phi'(s)\right]^2}{4r^{1/\gamma_1}(s)\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)} + \phi(s)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}{r^{1/\gamma_1}(s)}\rho^2(s) + \phi(s)\rho'(s) \right\} ds \right\} \\
&\geq \limsup_{t\to\infty} \frac{1}{t-2} \times \left\{ \int_2^t (t-s) \left\{ s^{-1} - \frac{1}{4s\Gamma(1-\alpha)\delta_1^{1/5}(s,t_2)} \right\} ds \right\} \\
&= \limsup_{t\to\infty} \frac{1}{t-2} \times \left\{ \int_2^t (t-s) \left\{ 1 - \frac{1}{4\Gamma(1-\alpha)\delta_1^{1/5}(s,t_2)} \right\} \frac{1}{s} ds \right\} \\
&= \limsup_{t\to\infty} \frac{1}{t-2} \left\{ \begin{array}{c} \int_2^T (t-s)\left\{1 - \frac{1}{4\Gamma(1-\alpha)\delta_1^{1/5}(s,t_2)}\right\} \frac{1}{s} ds \\ + \int_T^t (t-s)\left\{1 - \frac{1}{4\Gamma(1-\alpha)\delta_1^{1/5}(s,t_2)}\right\} \frac{1}{s} ds \end{array} \right\} \\
&\geq \limsup_{t\to\infty} \frac{1}{t-2} \left\{ \begin{array}{c} \int_2^T (t-s)\left\{1 - \frac{1}{4\Gamma(1-\alpha)\delta_1^{1/5}(s,t_2)}\right\} \frac{1}{s} ds \\ + \int_T^t (t-s)\left\{1 - \frac{1}{4\Gamma(1-\alpha)}\right\} \frac{1}{s} ds \end{array} \right\} \\
&= \infty
\end{aligned}$$

So (23) holds, and then we deduce that (28) is oscillatory by Corollary 3.5..

5 Conclusion

In this study, we are concerned with the oscillation for a class of nonlinear fractional differential equations. As one can see, by the aid of Liouville right-sided fractional derivative definition, we have a correlation between first order derivative of the $G(t)$ and fractional order derivative of $G(t)$. By using the correlation, inequality, integration average technique, and Riccati transformation, we are established some oscillation criteria. Finally we give examples.

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