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Oscillatory Behavior of Three Dimensional α -Fractional Delay Differential Systems

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Abstract: In the present work we study the oscillatory behavior of three dimensional α -fractional nonlinear delay differential system. We establish some sufficient conditions that will ensure all solutions are either oscillatory or converges to zero, by using the inequality technique and generalized Riccati transformation. The newly derived criterion are also used to establish a new class of systems with delay which are not covered in the existing study of literature. Further, we constructed some suitable illustrations.

Keywords: oscillation; nonlinear differential system; delay differential system; α -fractional derivative

1. Introduction

In the literature there are many advanced strategies in the expansion of ordinary and partial differential equations of fractional order and they have been used as excellent sources and tools in order to model many phenomena in the different fields of engineering, science and technology. Further, these tools are also used in fields such as chemical processes, polymer rheology, mathematical biology, industrial robotics, viscoelasticity, and many more, see the monographs [1–7].

At the end of the nineteenth century, Henry Poincare initiated the method and used the qualitative analysis of nonlinear systems of integer order differential equations. Since then, there has been significant development in the theory of oscillation of integer order differential systems [8–18].

In a study [19], Vreeke et al. applied the differential systems in the application of physics in order to solve the problem of a nuclear reactor which involved two temperature feedback. In the current literature there are many established results in the oscillation theory of classical differential systems (see [20–24]). However, in the nonlinear fractional differential system development is relatively slow due to the occurrence of nonlocal behavior of fractional derivatives that possess weakly singular kernels.

In 2014, Khalil et al. introduced the idea of conformable fractional derivative as a kind of local derivative with no memory (see [25–27]). By following the idea of Khalil, an interesting application of the conformable fractional derivative in physics was discussed and the action principle for particles under the frictional forces were formulated, see [28].

The idea of conformable fractional derivatives was generalized by Katugampola, and today it is known as the Katugampola fractional derivative. Nowadays, many researchers have interest in this type of derivative for their useful properties (see [29–31]). In this respect, we list the contributions of

Spanikova [32], Sadhasivam [33] and Chatzarakis [34] where the oscillation of α -fractional nonlinear three dimensional delay differential systems were also studied.

Now we study oscillatory behavior of the following system having the form

$$\begin{aligned} D^\alpha (u(t)) &= p(t)g(v(\sigma(t))), \\ D^\alpha (v(t)) &= -q(t)h(w(t)), \\ D^\alpha (w(t)) &= r(t)f(u(\delta(t))), \quad t \geq t_0, \end{aligned} \quad (1)$$

where $0 < \alpha \leq 1$, D^α denotes the α -fractional derivative respect to t .

Based on the following assumptions:

- (A₁) $p(t) \in C^{2\alpha}([t_0, \infty), \mathbb{R}^+)$, $q(t) \in C^\alpha([t_0, \infty), \mathbb{R}^+)$, $r(t) \in C([t_0, \infty), \mathbb{R}^+)$, $p(t)$, $q(t)$ and $r(t)$ are not identically zero on any interval of $[T_0, \infty)$, $T_0 \geq t_0$, $r(t)$ and $q(t)$ are decreasing and positive;
 (A₂) $g \in C^\alpha(\mathbb{R}, \mathbb{R})$, $vg(v) > 0$, $D^\alpha g(v) \geq l' > 0$, $h \in C^\alpha(\mathbb{R}, \mathbb{R})$, $wh(w) > 0$, $D^\alpha h(w) \geq m' > 0$, $f \in C(\mathbb{R}, \mathbb{R})$, $yf(y) > 0$ and $\frac{f(y)}{y} \geq k > 0$ for $y \neq 0$;
 (A₃) $\sigma(t) \leq t$ with $D^\alpha \sigma(t) \geq l > 0$, $\delta(t) \leq t$ and satisfies $\lim_{t \rightarrow \infty} \sigma(t) = \infty$, $\lim_{t \rightarrow \infty} \delta(t) = \infty$;
 (A₄) The case will be considered as

$$\int_{t_0}^{\infty} s^{\alpha-1} \frac{1}{b(s)} ds = \infty, \int_{t_0}^{\infty} s^{\alpha-1} \frac{1}{a(s)} ds = \infty,$$

where $b(t) = \frac{1}{q(t)}$, $a(t) = \frac{1}{p(t)}$ and $c(t) = l^2 l' m' r(t)$, $a(t)$, $b(t)$ and $c(t)$ are positive real-valued continuous functions with $b(t)t^{1-\alpha} < 1$.

The solution implies that, it is a vector-valued function such that $U(t) = (u(t), v(t), w(t))$ with $T_1 = \min \{\delta(t_1), \sigma(t_1)\}$ for some $t_1 \geq t_0$ which has the property such that $b(t)D^\alpha (a(t)D^\alpha u(t)) \in C^\alpha([T_1, \infty), \mathbb{R})$ and satisfies the system (1) on $[T_1, \infty)$. Denote by \mathbb{P} , the set of all solutions $U(t)$ of (1) which exist on some half line $[T_1, \infty)$, $T_1 > t_0$. The researchers only focus to the nontrivial solutions of system (1) and satisfy $\sup \{|u(\xi)| + |v(\xi)| + |w(\xi)|, t \leq \xi < \infty\} > 0$ for any $t \geq T_1$. We make a standing hypothesis that (1) has such a solution.

A proper solution $U(t) \in \mathbb{P}$ for the system (1) is called oscillatory if all the components are oscillatory, otherwise it is nonoscillatory. Further, the system (1) is said to be oscillatory if all proper solutions oscillate.

The main goal of this paper is to establish some new oscillation criteria for the system (1) by making use of the generalized Riccati transformation and inequality technique. The study is structured as follows. In Section 2, we recall some preliminary concepts relative to the α -fractional derivative. In Section 3, some new conditions for the oscillatory behavior of the solutions of system (1) were presented. Illustrative examples are included in the final part of the paper in order to demonstrate the efficiency of new theorems.

2. Preliminaries

We begin this section with the following definition of the operator D^α .

Definition 1. [30] Let $y : [0, \infty) \rightarrow \mathbb{R}$, then α -fractional derivative of y is defined by

$$D^\alpha (y)(t) := \lim_{\epsilon \rightarrow 0} \frac{y(te^{\epsilon t^{-\alpha}}) - y(t)}{\epsilon} \quad \text{for } t > 0 \text{ and } \alpha \in (0, 1]. \quad (2)$$

If y is differentiable α -times in some $(0, a)$ with $a > 0$, $\lim_{t \rightarrow 0^+} D^\alpha (y)(t)$ exists, then we have

$$D^\alpha (y)(0) := \lim_{t \rightarrow 0^+} D^\alpha (y)(t).$$

α -fractional derivative satisfies the following properties. [30]

Let $\alpha \in (0, 1]$ and g, h be α -differentiable for $t > 0$. Then

- (p₁) $D^\alpha(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$.
- (p₂) $D^\alpha(C) = 0$ for all constant functions, $g(t) = C$.
- (p₃) $D^\alpha(gh) = gD^\alpha(h) + hD^\alpha(g)$.
- (p₄) $D^\alpha\left(\frac{g}{h}\right) = \frac{hD^\alpha(g) - gD^\alpha(h)}{h^2}$.
- (p₅) $D^\alpha(g \circ h)(t) = g'(h(t))D^\alpha h(t)$, for g is differentiable at $h(t)$.
- (p₆) If g is differentiable, then $D^\alpha(g)(t) = t^{1-\alpha} \frac{dg}{dt}(t)$.

Definition 2. [30] Let $a \geq 0, t \geq a$ and a function y defined on $(a, t]$ with $\alpha \in \mathbb{R}$. Then, α -fractional integral as follows

$$I_a^\alpha(y)(t) := \int_a^t \frac{y(x)}{x^{1-\alpha}} dx \tag{3}$$

provided improper integral exists.

3. Main Results

In this section, the oscillatory behavior of solutions for the system (1) under certain conditions are established. Next we give the following lemmas that will be used in our further discussion.

Lemma 1. If $U(t) \in \mathbb{P}$ is a nonoscillatory solution for (1), then the component function $x(t)$ is always nonoscillatory.

Lemma 2. Suppose that (A_1) and (A_4) holds. Then there exists a $t_1 \geq t_0$ such that either

(I) $u(t) > 0, D^\alpha u(t) > 0, D^\alpha(a(t)D^\alpha u(t)) > 0$ for $t \geq t_1$.

or

(II) $u(t) > 0, D^\alpha u(t) < 0, D^\alpha(a(t)D^\alpha u(t)) > 0$ for $t \geq t_1$ holds.

Proof. Let $u(t)$ be an eventually positive solution for (1) on $[t_0, \infty)$. Now, system (1) will be reduced to the following inequality

$$D^\alpha \left(\frac{1}{q(t)} D^\alpha \left(\frac{1}{p(t)} D^\alpha u(t) \right) \right) + l^2 l' m' r(t) f(u(\delta(\sigma(t)))) \leq 0, t \geq t_1, \tag{4}$$

which implies,

$$D^\alpha (b(t)D^\alpha (a(t)D^\alpha u(t))) + c(t)f(u(\delta(\sigma(t)))) \leq 0, t \geq t_1. \tag{5}$$

From (5), we get $D^\alpha (b(t)D^\alpha (a(t)D^\alpha u(t))) \leq 0$ for $t \geq t_0$. Then $b(t)D^\alpha (a(t)D^\alpha u(t))$ is decreasing on (t_0, ∞) . Thus the proof completes on using the Lemma 3.2 in [34]. \square

The following notations are employed in the sequel.

$$(A_\alpha)_* := \liminf_{t \rightarrow \infty} t \int_t^\infty s^{\alpha-1} A^\alpha(s) ds \text{ and } (B_\alpha)_* := \liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t s^{\alpha+1} A_\alpha(s) ds, \tag{6}$$

where $A_\alpha(t) = \frac{k}{2} \frac{c(t)}{a(t)} \frac{\delta(\sigma(t))-T}{t} (\delta(\sigma(t)))^\alpha$.

$$d := \liminf_{t \rightarrow \infty} tw(t) \text{ and } D := \limsup_{t \rightarrow \infty} tw(t). \tag{7}$$

Theorem 1. Suppose that $(A_1) - (A_4)$ hold. Assume also that

$$\int_{t_2}^{\infty} c(s)(s - T)\delta(\sigma(s))ds = \infty, \tag{8}$$

there exists a positive function $\rho \in C^\alpha([0, \infty); \mathbb{R}_+)$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left(s^{\alpha-1} \rho(s) A_\alpha(s) - \frac{1}{4} \frac{(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds = \infty. \tag{9}$$

Then every solution of system (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution $(u(t), v(t), w(t))$ on $[t_0, \infty)$. From Lemma 1, $u(t)$ is always nonoscillatory. Without loss of generality, we shall assume that $u(t) > 0, u(\delta(t)) > 0$ and $u(\delta(\sigma(t))) > 0$ for $t \geq T \geq t_0$, since similar arguments can be made for $u(t) < 0$ eventually. Suppose that Case (I) of Lemma 2 holds for $t \geq t_1$. Define

$$w(t) = \rho(t) \frac{b(t) D^\alpha(a(t) D^\alpha u(t))}{a(t) D^\alpha u(t)}, \quad t \geq t_1. \tag{10}$$

Thus $w(t) > 0$, differentiating α times with respect to t , using (5) and (A_2) , we have

$$D^\alpha w(t) \leq \frac{D^\alpha \rho(t)}{\rho(t)} w(t) - \frac{k\rho(t)c(t)}{a(t)} \frac{u(\delta(\sigma(t)))}{D^\alpha u(t)} - \frac{1}{\rho(t)b(t)} w^2(t). \tag{11}$$

Now, let $z_1(t, T) = (t - T)$, $z_2(t, T) = \frac{(t-T)^2}{2}$ and define $U(t) := (t - T)t^{1-\alpha}u(t) - z_2(t, T)D^\alpha u(t)$. Then $U(T) = 0$ and differentiating the above, we get

$$D^\alpha U(t) = t^{1-\alpha} \left(t^{1-\alpha} u(t) + (t - T)(1 - \alpha)t^{-\alpha} u(t) + (t - T)t^{1-\alpha} u'(t) - z_2'(t, T) D^\alpha u(t) - z_2(t, T) (D^\alpha u(t))' \right),$$

which implies

$$U'(t) \geq t^{1-\alpha} u(t) - z_2(t, T) (D^\alpha u(t))'. \tag{12}$$

By Taylor's Theorem, we have

$$\int_T^t s^{1-\alpha} u'(s) ds = z_1(t, T) D^\alpha u(T) + \int_T^t z_1(t, s) (D^\alpha u(s))' ds,$$

since $D^\alpha(a(t)D^\alpha u(t))$ is decreasing, we get

$$t^{1-\alpha} u(t) \geq t^{1-\alpha} u(T) + z_1(t, T) D^\alpha u(T) + (D^\alpha u(t))' \int_T^t z_1(t, s) ds.$$

Thus $U'(t) > 0$ on $[T, \infty)$. From this we get $U(t) > 0$ on $[T, \infty)$, which implies that

$$\frac{u(t)}{D^\alpha u(t)} > \frac{z_2(t, T)}{(t - T)t^{1-\alpha}} = \frac{t - T}{2} t^{1-\alpha}, \quad t \in [T, \infty). \tag{13}$$

Next, define $V(t) := D^\alpha u(t) - t(D^\alpha u(t))'$. In view of the fact that $D^\alpha V(t) = -t^{2-\alpha}(D^\alpha u(t))''$, which implies $V'(t) = -t^{2-\alpha}(D^\alpha u(t))'' > 0$ for $t \in [T, \infty)$, therefore $V(t)$ is strictly increasing on $[T, \infty)$.

We claim that there is a $t_1 \in [T, \infty)$ such that $V(t) > 0$ on $[t_1, \infty)$. Suppose not, $V(t) < 0$ on $[t_1, \infty)$. Hence,

$$D^\alpha \left(\frac{D^\alpha u(t)}{t} \right) = -\frac{t^{1-\alpha}}{t^2} (t(D^\alpha u(t))' - D^\alpha u(t)),$$

which gives

$$\left(\frac{D^\alpha u(t)}{t} \right)' = -\frac{1}{t^2} V(t) > 0, \quad t \in [t_1, \infty).$$

Choose $t_2 \in (t_1, \infty)$, for $t \geq t_2, \delta(\sigma(t)) \geq \delta(\sigma(t_2))$. Since, $\frac{D^\alpha u(t)}{t}$ is strictly increasing,

$$\frac{D^\alpha u(\delta(\sigma(t)))}{\delta(\sigma(t))} \geq \frac{D^\alpha u(\delta(\sigma(t_2)))}{\delta(\sigma(t_2))} := m > 0,$$

the Equation (13) implies that

$$u(\delta(\sigma(t))) \geq \frac{t-T}{2} t^{1-\alpha} m \delta(\sigma(t)). \tag{14}$$

Now, integrating (5) from t_2 to t , using (A_2) and inequality in (14), we have

$$\int_{t_2}^t \left((b(s)D^\alpha(a(s)D^\alpha u(s)))' + \frac{km}{2} c(s)(s-T)\delta(\sigma(s)) \right) ds \leq 0.$$

Then

$$b(t_2)D^\alpha(a(t_2)D^\alpha u(t_2)) \geq \frac{km}{2} \int_{t_2}^t c(s)(s-T)\delta(\sigma(s)) ds,$$

which contradicts to (8). Hence $V(t) > 0$ on $[t_1, \infty)$. Accordingly,

$$t^{1-\alpha} \left(\frac{D^\alpha u(t)}{t} \right)' = -\frac{t^{1-\alpha}}{t^2} (t(D^\alpha u(t))' - D^\alpha u(t)) = -\frac{t^{1-\alpha}}{t^2} V(t) < 0, \quad t \in (t_1, \infty),$$

which gives $t(D^\alpha u(t))' < D^\alpha u(t)$. Then $\delta(\sigma(t)) \leq \delta(t) \leq t$,

$$\frac{D^\alpha u(\delta(\sigma(t)))}{\delta(\sigma(t))} \geq \frac{D^\alpha u(t)}{t}, \tag{15}$$

since $\frac{D^\alpha u(t)}{t}$ is strictly increasing. Using (13) and (15) in (11), we get

$$D^\alpha w(t) \leq \frac{D^\alpha \rho(t)}{\rho(t)} w(t) - \frac{k\rho(t)c(t)}{ta(t)} \frac{(\delta(\sigma(t)))^\alpha (\delta(\sigma(t)) - T)}{2} - \frac{1}{\rho(t)b(t)} w^2(t). \tag{16}$$

Therefore

$$D^\alpha w(t) \leq -\frac{k\rho(t)c(t)}{ta(t)} \frac{(\delta(\sigma(t)))^\alpha (\delta(\sigma(t)) - T)}{2} + \frac{1}{4} b(t) \frac{(D^\alpha \rho(t))^2}{\rho(t)},$$

using (6) and (p_6) , we get

$$w'(t) \leq -t^{\alpha-1}\rho(t)A_\alpha(t) + \frac{1}{4} \frac{(\rho'(t))^2}{\rho(t)} t^{1-\alpha} b(t). \quad (17)$$

Integrating,

$$\int_{t_1}^t \left(s^{\alpha-1}\rho(s)A_\alpha(s) - \frac{1}{4} \frac{(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds \leq w(t_1),$$

which contradicts the hypothesis (9). \square

We now derive various oscillatory criteria on using the earlier results and we can generalize the Philos type kernel. Let us define a class of functions Ω . Consider

$$\mathbb{D}_0 = \{(t, s) : t > s \geq t_0\}, \text{ and } \mathbb{D} = \{(t, s) : t \geq s \geq t_0\}.$$

The function $H \in C(\mathbb{D}, \mathbb{R})$ belongs to the class Ω , if

(T₁) $H(t, t) = 0$ for $t \geq t_0$, and $H(t, s) > 0$ for $(t, s) \in \mathbb{D}_0$.

(T₂) The nonpositive partial derivative $\frac{\partial H}{\partial s}$ exist on \mathbb{D}_0 such that $h(t, s) = H(t, s) \frac{\rho'(s)}{\rho(s)} + \frac{\partial H}{\partial s}(t, s)$.

Theorem 2. Assume that $(A_1) - (A_4)$ hold. Further there exists $\rho \in C^\alpha([0, \infty); \mathbb{R}_+)$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \left(H(t, s) s^{\alpha-1} \rho(s) A_\alpha(s) - \frac{1}{4} \frac{\rho(s) b(s)}{H(t, s)} s^{1-\alpha} H^2(t, s) \right) ds = \infty. \quad (18)$$

Then each solution of system (1) is oscillatory.

Proof. As we proceed in the proof of Theorem 1 and from (16), we have the inequality

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - t^{\alpha-1} \rho(t) A_\alpha(t) - \frac{t^{\alpha-1}}{\rho(t) b(t)} w^2(t). \quad (19)$$

Integrating,

$$\begin{aligned} & \int_{t_1}^t H(t, s) s^{\alpha-1} \rho(s) A_\alpha(s) ds \\ & \leq \int_{t_1}^t H(t, s) \frac{\rho'(s)}{\rho(s)} w(s) ds - \int_{t_1}^t H(t, s) w'(s) ds - \int_{t_1}^t H(t, s) \frac{s^{\alpha-1}}{\rho(s) b(s)} w^2(s) ds, \\ & \leq H(t, t_1) w(t_1) + \int_{t_1}^t \left(H(t, s) \frac{\rho'(s)}{\rho(s)} + \frac{\partial H}{\partial s}(t, s) \right) w(s) ds - \int_{t_1}^t H(t, s) \frac{s^{\alpha-1}}{b(s) \rho(s)} w^2(s) ds, \\ & \leq H(t, t_1) w(t_1) + \int_{t_1}^t \left(w(s) h(t, s) - H(t, s) \frac{s^{\alpha-1}}{\rho(s) b(s)} w^2(s) \right) ds, \end{aligned}$$

$$\leq H(t, t_1)w(t_1) + \int_{t_1}^t \frac{1}{4} \frac{\rho(s)b(s)}{H(t, s)} s^{1-\alpha} h^2(t, s) ds.$$

From this we conclude that

$$\int_{t_1}^t \left(H(t, s)s^{\alpha-1}\rho(s)A_\alpha(s) - \frac{1}{4} \frac{\rho(s)b(s)}{H(t, s)} s^{1-\alpha} h^2(t, s) \right) ds \leq H(t, t_1)w(t_1).$$

Since $0 < H(t, s) \leq H(t, t_1)$ for $t > s > t_1$, we have $0 < \frac{H(t, s)}{H(t, t_1)} \leq 1$, hence

$$\frac{1}{H(t, t_1)} \int_{t_1}^t \left(H(t, s)s^{\alpha-1}\rho(s)A_\alpha(s) - \frac{1}{4} \frac{\rho(s)b(s)}{H(t, s)} s^{1-\alpha} h^2(t, s) \right) ds \leq w(t_1).$$

Letting $t \rightarrow \infty$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \left(H(t, s)s^{\alpha-1}\rho(s)A_\alpha(s) - \frac{1}{4} \frac{\rho(s)b(s)}{H(t, s)} s^{1-\alpha} h^2(t, s) \right) ds \leq w(t_1).$$

Therefore assumption (18) is contradicted. Thus every solution of (1) oscillates. \square

We immediately obtain the following oscillation result for (1).

Theorem 3. Assume that (A_1) – (A_4) hold. Also assume that there exists a function $\rho \in C^\alpha([0, \infty); \mathbb{R}_+)$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \left(H(t, s)s^{\alpha-1}\rho(s)A_\alpha(s) - \frac{1}{4} \frac{H(t, s)(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds = \infty. \tag{20}$$

Then every solution of system (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 1, multiplying inequality (17) by $H(t, s)$ and integrating, we get

$$\int_{t_1}^t \left(H(t, s)s^{\alpha-1}\rho(s)A_\alpha(s) - \frac{1}{4} \frac{H(t, s)(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds \leq - \int_{t_1}^t H(t, s)w'(s) ds \leq H(t, t_1)w(t_1).$$

Taking limsup as $t \rightarrow \infty$, and hence

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \left(s^{\alpha-1}\rho(s)A_\alpha(s)H(t, s) - \frac{1}{4} \frac{H(t, s)(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds \leq w(t_1),$$

which contradicts the hypothesis in (20). \square

The following theorem is to be proved using the techniques employed in the previous theorems.

Theorem 4. Suppose that the assumptions (A_1) – (A_4) and (8) hold. Further assume also that Case (I) of Lemma 2 holds, then

$$(A_\alpha)_* \leq d - t^{\alpha-1} \frac{1}{b(s)} d^2, \quad (21)$$

and

$$(B_\alpha)_* \leq D - D^2. \quad (22)$$

Proof. Let $u(t)$ be a nonoscillatory solution of (5) such that $u(t) > 0$, $u(\delta(t)) > 0$ and $u(\delta(\sigma(t))) > 0$ for $t \geq T > t_0$, consider the case (I) of Lemma 2 holds, $u(t)$ satisfies the inequality $D^\alpha (b(t)D^\alpha (a(t)D^\alpha u(t))) \leq 0$, $t \in [T, \infty)$. Define Riccati transformation

$$w(t) = \frac{b(t)D^\alpha (a(t)D^\alpha u(t))}{a(t)D^\alpha u(t)}.$$

Thus $w(t) > 0$, differentiating α times with respect to t , using (5) and (A_2) , we have

$$D^\alpha w(t) \leq -\frac{kc(t)}{a(t)} \frac{u(\delta(\sigma(t)))}{D^\alpha u(t)} - \frac{1}{b(t)} w^2(t).$$

By using (15), (13) and (6), we obtain the above inequality

$$w'(t) + t^{\alpha-1} A_\alpha(t) + t^{\alpha-1} \frac{1}{b(t)} w^2(t) \leq 0. \quad (23)$$

Given that $A_\alpha(t) > 0$ and $w(t) > 0$, which gives $w'(t) \leq 0$ and

$$-b(t)(w'(t)t^{1-\alpha}/w^2(t)) > 1.$$

which yields that

$$\left(\frac{1}{w(t)} \right)' > t^{\alpha-1} \frac{1}{b(t)}.$$

Integrating the above inequality and denote $t_1^{\alpha-1} \frac{1}{b(t_1)} = M$, we have

$$M(t - t_1)w(t) < 1. \quad (24)$$

which implies that

$$\lim_{t \rightarrow \infty} w(t) = 0, \quad \lim_{t \rightarrow \infty} tw(t) = 0. \quad (25)$$

From (9) and (24), $0 < d < 1$ and $0 < D < 1$. Even though if $d = 0$ and $D = 0$, there is nothing to prove. Now, to claim (21). Integrating (23) from t to ∞ and use (25), we get

$$w(t) \geq \int_t^\infty s^{\alpha-1} A_\alpha(s) ds + \int_t^\infty s^{\alpha-1} \frac{1}{b(s)} w^2(s) ds.$$

Multiplying by t and taking \liminf as $t \rightarrow \infty$, by (25), $d \geq (A_\alpha)_*$. For given $\epsilon > 0$, there exists a $t_2 \geq t_1$ as

$$d - \epsilon < tw(t) < d + \epsilon \text{ and } t \int_t^\infty s^{\alpha-1} A_\alpha(s) ds \geq (A_\alpha)_* - \epsilon, t \geq t_2. \tag{26}$$

Again from (26),

$$\begin{aligned} tw(t) &\geq t \int_t^\infty s^{\alpha-1} A_\alpha(s) ds + t \int_t^\infty s^{\alpha-1} \frac{1}{b(s)} w^2(s) ds \\ &\geq t \int_t^\infty s^{\alpha-1} A_\alpha(s) ds + t^\alpha \frac{1}{b(t)} \int_t^\infty \frac{(sw(s))^2}{s^2} ds \\ &\geq t \int_t^\infty s^{\alpha-1} A_\alpha(s) ds + t^\alpha \frac{1}{b(t)} (d - \epsilon)^2 \int_t^\infty \frac{1}{s^2} ds \\ &= t \int_t^\infty s^{\alpha-1} A_\alpha(s) ds + t^{\alpha-1} \frac{1}{b(t)} (d - \epsilon)^2. \end{aligned} \tag{27}$$

Therefore from (26) and (27), $d \geq (A_\alpha)_* - \epsilon + (d - \epsilon)^2$. Then

$$d \geq (A_\alpha)_* + t^{\alpha-1} \frac{1}{b(t)} d^2,$$

since ϵ is arbitrarily small. Next to prove that (22). Multiply (23) by s^2 , integrating from t_1 to t , and integration by parts follows that

$$\begin{aligned} \int_{t_1}^t s^{\alpha+1} A_\alpha(s) ds &\leq - \int_{t_1}^t s^2 w'(s) ds - \int_{t_1}^t s^{\alpha+1} \frac{1}{b(s)} w^2(s) ds \\ &\leq -t^2 w(t) + t_1^2 w(t_1) + 2 \int_{t_1}^t s w(s) ds - \int_{t_1}^t s^{\alpha+1} \frac{1}{b(s)} w^2(s) ds, \end{aligned}$$

implies

$$t^2 w(t) \leq t_1^2 w(t_1) - \int_{t_1}^t s^{\alpha+1} A_\alpha(s) ds + \int_{t_1}^t \left(2s w(s) - s^{\alpha+1} \frac{1}{b(s)} w^2(s) \right) ds. \tag{28}$$

Thus, we obtain

$$\begin{aligned} tw(t) &\leq \frac{t_1^2 w(t_1)}{t} - \frac{1}{t} \int_{t_1}^t s^{\alpha+1} A_\alpha(s) ds + \frac{1}{t} \int_{t_1}^t s^{1-\alpha} b(s) ds, \\ &\leq \frac{t_1^2 w(t_1)}{t} - \frac{1}{t} \int_{t_1}^t s^{\alpha+1} A_\alpha(s) ds + \frac{1}{t} t^{1-\alpha} b(t) \int_{t_1}^t ds. \end{aligned} \tag{29}$$

By (A_4) , (29) imply that

$$tw(t) \leq \frac{t_1^2 w(t_1)}{t} - \frac{1}{t} \int_{t_1}^t s^{\alpha+1} A_\alpha(s) ds + \frac{1}{t} (t - t_1).$$

Thus

$$\limsup_{t \rightarrow \infty} tw(t) \leq 1 - \liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t s^{\alpha+1} A_\alpha(s) ds.$$

Hence from (6), (7), $D \leq 1 - (B_\alpha)_*$. For any $\epsilon > 0$, there exists a $t_2 \geq t_1$ such that

$$D - \epsilon < tw(t) < D + \epsilon \text{ and } \frac{1}{t} \int_{t_0}^t s^{\alpha+1} A_\alpha(s) ds > (B_\alpha)_* - \epsilon, t \geq t_2. \tag{30}$$

Now, from (28) and (30) we get

$$D \leq -(B_\alpha)_* + \epsilon(D + \epsilon)(2 - D + \epsilon), t \geq t_2,$$

since ϵ is arbitrarily small, we have

$$(B_\alpha)_* \leq D - D^2,$$

which proves (22). \square

Lemma 3. Suppose that (A_1) – (A_4) and (8) hold. Also assume that Case (II) of Lemma 2 holds. If

$$\int_{t_2}^\infty \eta^{\alpha-1} \frac{1}{a(\eta)} \left(\int_\eta^\infty \int_\mu^\infty s^{\alpha-1} c(s) ds d\mu \right) d\eta = \infty. \tag{31}$$

Then $\lim_{t \rightarrow \infty} u(t) = 0$.

Proof. We consider the Case (II) of Lemma 2, $D^\alpha u(t) < 0, D^\alpha(a(t)D^\alpha u(t)) > 0$ for $t \geq t_1$. Since $u(t)$ is positive and decreasing, we have $\lim_{t \rightarrow \infty} u(t) = d' \geq 0$. Suppose not, $d' > 0$. Given that $u(\delta(\sigma(t))) \leq \delta(t) \leq t$, then $u(\delta(\sigma(t))) \geq u(t) > d'$ for $t \geq t_2 \geq t_1$ sufficiently large, $u(t)$ is decreasing. Integrating (5) from t to ∞ and using $u(\delta(\sigma(t))) \geq d'$, we get

$$\int_t^\infty \left(b(s)D^\alpha (a(s)D^\alpha u(s)) \right)' ds \leq - \int_t^\infty ks^{\alpha-1} c(s)u(\delta(\sigma(s))) ds \leq -kd' \int_t^\infty s^{\alpha-1} c(s) ds,$$

then,

$$b(t)D^\alpha (a(t)D^\alpha u(t)) \geq kd' \int_t^\infty s^{\alpha-1} c(s) ds.$$

By (A_4) , we get

$$(a(t)D^\alpha u(t))' \geq kd' \frac{1}{b(t)t^{1-\alpha}} \int_t^\infty s^{\alpha-1} c(s) ds \geq kd' \int_t^\infty s^{\alpha-1} c(s) ds.$$

Again integrating, we obtain

$$-a(t)D^\alpha u(t) \geq kd' \int_t^\infty \int_\mu^\infty s^{\alpha-1} c(s) ds d\mu,$$

this implies that,

$$-u'(t) \geq kd' t^{\alpha-1} \frac{1}{a(t)} \int_t^\infty \int_\mu^\infty s^{\alpha-1} c(s) ds d\mu.$$

By integrating, once again it is get as

$$u(t_2) \geq kd' \int_{t_2}^\infty \left(\eta^{\alpha-1} \frac{1}{a(\eta)} \int_\eta^\infty \int_\mu^\infty s^{\alpha-1} c(s) ds d\mu \right) d\eta,$$

which contradicts to (31). Thus $d' = 0$ and hence $\lim_{t \rightarrow \infty} u(t) = 0$. \square

From Theorem 4, Nehari type oscillation criteria for (1).

Theorem 5. Assume that $(A_1) - (A_4)$, (8) and (31) hold. If

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \left(ks^{\alpha+1} \frac{c(s)}{a(s)} \frac{u(\delta(\sigma(s))) - T}{s} (u(\delta(\sigma(s))))^\alpha \right) ds > \frac{1}{2}, \quad (32)$$

then $u(t)$ is oscillatory or satisfies $u(t) = 0$ as $t \rightarrow \infty$.

4. Examples

In this section, we provide some examples in order to see the effect of the main results.

Example 1. Consider $\frac{1}{2}$ -fractional delay differential system

$$\begin{aligned} D^{\frac{1}{2}}(u(t)) &= \frac{1}{\sqrt{t}} g(v(\frac{t}{2})) \\ D^{\frac{1}{2}}(v(t)) &= -\frac{1}{\sqrt{t}} h(w(t)), \\ D^{\frac{1}{2}}(w(t)) &= \frac{1}{\sqrt{t}} f(u(\frac{t}{2})), \quad t \geq t_0, \end{aligned} \quad (33)$$

where $C_1 = \cos(\ln 2)$, $C_2 = \sin(\ln 2)$, $A_1 = \cos(\ln 4)$, $A_2 = \sin(\ln 4)$.

Here $\alpha = \frac{1}{2}$, $p(t) = \frac{1}{a(t)} = \frac{1}{\sqrt{t}}$, $q(t) = \frac{1}{b(t)} = \frac{1}{\sqrt{t}}$, $r(t) = \frac{1}{\sqrt{t}}$, $f(u) = A_1 \sqrt{1-u^2} - A_2 u$, $g(v) = v$ and $h(w) = w$.

It is easy to see that

$$\begin{aligned} D^\alpha g(v) &= \frac{1}{\sqrt{t}} (C_1 + C_2) \geq l' > 0, \\ D^\alpha h(w) &= \frac{1}{\sqrt{t}} (C_1 - C_2) \geq m' > 0, \\ f(u)/u &= A_1 \sqrt{\frac{1}{u^2} - 1} - A_2 \geq 0.2579 = k > 0, \end{aligned}$$

since $u^2 < 1$, $\sigma(t) = \delta(t) = \frac{t}{2}$ and $D^\alpha \sigma(t) = \frac{\sqrt{t}}{2} \geq l > 0$, $c(t) = \frac{C_1^2 - C_2^2}{4\sqrt{t}}$, $A_\alpha(t) = \frac{0.2579}{16} \frac{t-T}{\sqrt{t}}$. Now it is considered as,

$$\int_{t_2}^\infty c(s)(s-T)\delta(\sigma(s))ds = \frac{C_1^2 - C_2^2}{4} \int_{t_2}^\infty \frac{(s-T)s}{\sqrt{s}} \frac{1}{4} ds \rightarrow \infty \text{ as } t \rightarrow \infty.$$

By taking $\rho(t) = 16/k$ then $\rho'(t) = 0$. Consider

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(s^{\alpha-1} \rho(s) A_\alpha(s) - \frac{1}{4} \frac{(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds \\ = \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(s^{-\frac{1}{2}} \frac{16}{k} \frac{k(\frac{s}{4} - T)}{16} \frac{1}{\sqrt{s}} \right) ds \\ = \limsup_{t \rightarrow \infty} \frac{1}{4} \int_{t_1}^t \left(\frac{s - 4T}{s} \right) ds \rightarrow \infty \text{ as } t \rightarrow \infty. \end{aligned}$$

Since, each of the conditions are verified in Theorem 1, all solutions of (33) are oscillatory. Thus $(u(t), v(t), w(t)) = (\sin(\ln t), C_1 \cos(\ln t) - C_2 \sin(\ln t), C_1 \sin(\ln t) + C_2 \cos(\ln t))$ is one such solution.

Note: The decreasing condition imposed on $q(t)$ and $r(t)$ is only a sufficient condition, however it is not a necessary one. The following example ensures the oscillatory behavior of the system (34) even though $q(t)$ and $r(t)$ is nondecreasing.

Example 2. Consider $\frac{1}{3}$ -fractional following differential system

$$\begin{aligned} D^{\frac{1}{3}}(u(t)) &= \frac{t^{\frac{2}{3}}}{1 + \frac{3}{4} \cos^{\frac{5}{3}}(t)} g(v(t - 2\pi)), \\ D^{\frac{1}{3}}(v(t)) &= -t^{\frac{2}{3}} w(t), \\ D^{\frac{1}{3}}(w(t)) &= \frac{t^{\frac{2}{3}}}{1 + \cos^2(t)} f(u(t - \frac{3\pi}{2})), \quad t \geq t_0. \end{aligned} \tag{34}$$

Here $\alpha = \frac{1}{3}$, $p(t) = \frac{1}{a(t)} = \frac{t^{\frac{2}{3}}}{1 + \frac{3}{4} \cos^{\frac{5}{3}}(t)}$, $q(t) = \frac{1}{b(t)} = t^{\frac{2}{3}}$, $r(t) = \frac{t^{\frac{2}{3}}}{1 + \cos^2(t)}$, $f(u) = u(1 + u^2)$, $g(v) = v(1 + \frac{3}{4} v^{\frac{5}{3}})$ and $h(w) = w$. It is easy to see that $D^\alpha g(v) = v^{\frac{1}{3}} + 2v^2 \geq 1 = l' > 0$ such that $y^2 > 1$, $y' > \frac{1}{3}$, $D^\alpha h(w) \geq 1 = m' > 0$, $f(u)/u = 1 + u^2 \geq 1 = k > 0$, $\sigma(t) = t - 2\pi$, $\delta(t) = t - \frac{3\pi}{2}$ and $D^\alpha \sigma(t) = t^{\frac{2}{3}} \geq l$ such that $t_1 = l^{\frac{2}{3}}$, $t \geq t_1$, $c(t) = l^2 \frac{t^{\frac{2}{3}}}{1 + \cos^2(t)}$, $A_\alpha(t) = \frac{l^2}{2} \frac{1 + \frac{3}{4} \cos^{\frac{5}{3}}(t)}{1 + \cos^2(t)} \frac{t - \frac{3\pi}{2} - T}{t} (t - \frac{3\pi}{2})^{\frac{1}{3}}$. Now consider,

$$\begin{aligned} \int_{t_2}^{\infty} c(s)(s - T)\delta(\sigma(s))ds &= \int_{t_2}^{\infty} l^2 \frac{s^{\frac{2}{3}}}{1 + \cos^2(s)} (s - T)(s - \frac{3\pi}{2}) ds \\ &\geq \frac{l^2}{2} \int_{t_2}^{\infty} s^{\frac{2}{3}} (s - T)(s - \frac{3\pi}{2}) ds \rightarrow \infty \text{ as } t \rightarrow \infty. \end{aligned}$$

If we take $\rho(t) = 1$ then $\rho'(t) = 0$. Consider

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(s^{\alpha-1} \rho(s) A_\alpha(s) - \frac{1}{4} \frac{(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds \\ &= \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(s^{-\frac{2}{3}} l^2 \frac{s^{\frac{2}{3}}}{1 + \cos^2(s)} \frac{1}{2} \frac{1 + \frac{3}{4} \cos^{\frac{5}{3}}(s)}{1 + \cos^2(s)} \frac{s - \frac{3\pi}{2} - T}{s} (s - \frac{3\pi}{2})^{\frac{1}{3}} \right) ds \\ &\geq \limsup_{t \rightarrow \infty} \frac{7l^2}{16} \int_{t_1}^t \left(\left(1 - \frac{3\pi - T}{s}\right) (s - \frac{3\pi}{2})^{\frac{1}{3}} \right) ds \rightarrow \infty \text{ as } t \rightarrow \infty. \end{aligned}$$

Theorem 1 are satisfying the new conditions arriving at the solution for (34) is oscillatory and it is given as $(u(t), v(t), w(t)) = (\sin t, \cos t, \sin t)$.

Example 3. Consider the $\frac{1}{2}$ -fractional differential system

$$\begin{aligned} D^{\frac{1}{2}}(u(t)) &= e^{2t} t^{\frac{1}{2}} g(v(t-1)), \\ D^{\frac{1}{2}}(v(t)) &= -e^{-2t} t^{\frac{1}{2}} w(t), \\ D^{\frac{1}{2}}(w(t)) &= (et)^{\frac{1}{2}} f(u(t - \frac{1}{2})), \quad t \geq t_0. \end{aligned} \tag{35}$$

Here $\alpha = \frac{1}{2}$, $\frac{1}{a(t)} = p(t) = e^{2t} t^{\frac{1}{2}}$, $\frac{1}{b(t)} = q(t) = e^{-2t} t^{\frac{1}{2}}$, $r(t) = (et)^{\frac{1}{2}}$, $g(v) = v$, $h(w) = w$ and $f(u) = u$. Now it is easy to check that $D^\alpha g(v) = v^{\frac{1}{2}} = e^{-\frac{t}{2}} = l' > 0$, $D^\alpha h(w) = w^{\frac{1}{2}} = e^{\frac{t}{2}} = m' > 0$, $f(u)/u = 1 = k > 0$, $\sigma(t) = t - 1$, $\delta(t) = t - \frac{1}{2}$ and $D^\alpha \sigma(t) = t^{\frac{1}{2}} \geq l$ such that $t_1 = l^{\frac{1}{2}}$ for $t \geq t_1$, $c(t) = l^2 (et)^{\frac{1}{2}}$, $A_\alpha(t) = \frac{l^2}{2} e^{\frac{1}{2}} e^{2t} (t - \frac{1}{2} - T) (t - \frac{1}{2})^{\frac{1}{2}}$. Now,

$$\int_{t_2}^\infty c(s)(s - T)\delta(\sigma(s))ds = \int_{t_2}^\infty l^2 (es)^{\frac{1}{2}} (s - T) (s - \frac{1}{2}) ds = l^2 e^{\frac{1}{2}} \int_{t_2}^\infty s^{\frac{1}{2}} (s - T) (s - \frac{1}{2}) ds \rightarrow \infty.$$

Taking $\rho(t) = \frac{1}{t^{\frac{7}{2}} e^{2t}}$ then $\rho'(t) = -\frac{t^{\frac{7}{2}}}{2t^7 e^{2t}} (4t + 7)$. Consider

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(s^{\alpha-1} \rho(s) A_\alpha(s) - \frac{1}{4} \frac{(\rho'(s))^2}{\rho(s)} s^{1-\alpha} b(s) \right) ds \\ &= \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(s^{-\frac{1}{2}} \frac{l^2 e^{\frac{1}{2}}}{s^{\frac{7}{2}} e^{2s}} \frac{1}{2} e^{2s} (s - \frac{1}{2} - T) (s - \frac{1}{2})^{\frac{1}{2}} - \frac{(4s + 7)^2}{4s^9 e^{4s}} e^{4s} s^{\frac{7}{2}} \right) ds \\ &\leq \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(\frac{l^2 e^{\frac{1}{2}}}{2s^{\frac{7}{2}}} s - \frac{s^2}{4s^{\frac{11}{2}}} \right) ds \leq \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(\frac{l^2 e^{\frac{1}{2}}}{2s^{\frac{5}{2}}} - \frac{1}{4s^{\frac{7}{2}}} \right) ds < \infty. \end{aligned}$$

Here further the condition (9) of the above Theorem 1 seems to be not satisfied, in view of the fact that (A_4) fails to hold, and hence the system (35) is not oscillatory. In fact, $(u(t), v(t), w(t)) = (e^t, e^{-t}, e^t)$ it is a solution for (35), and nonoscillatory.

Remark 1. The results obtained in this article further can be extended to a neutral system with forced term

$$\begin{aligned} D^\alpha (u(t) + p(t)u(\delta(t))) &= a(t)h_1(v(\tau(t))), \\ D^\alpha (v(t)) &= -b(t)h_2(w(t)), \\ D^\alpha (w(t)) &= c(t)h_3(u(\sigma(t))) + e(t), \quad t \geq t_0, \end{aligned}$$

for the cases

$$\int_{t_0}^{\infty} a(s)d_\alpha s < \infty, \int_{t_0}^{\infty} b(s)d_\alpha s = \infty,$$

and

$$\int_{t_0}^{\infty} a(s)d_\alpha s < \infty, \int_{t_0}^{\infty} b(s)d_\alpha s < \infty.$$

5. Conclusions

Through this article, we have derived some new oscillation results for a certain class of nonlinear three-dimensional α -fractional differential systems by using the Riccati transformation and inequality technique. This work extends and also improves some classical results in the literature [16,18,32] to the α -fractional systems and studied the oscillation criteria. Further, the present results are essentially new and, in order to illustrate the validity of the obtained results, we have provided three examples.

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References

1. Abbas, S.; Benchohra, M.; N'Guerekata, G.M. *Topics in Fractional Differential Equations*; Springer: New York, NY, USA, 2012.
2. Daftardar-Gejji, V. *Fractional Calculus Theory and Applications*; Narosa Publishing House Pvt. Ltd.: Kolkata, India, 2014.
3. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific Pub. Co.: Singapore, 2000.
4. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science: Amsterdam, The Netherlands, 2006.
5. Milici, C.; Draganescu, G. *Introduction to Fractional Calculus*; Lambert Academic Publishing: Saarbrücken, Germany, 2016.
6. Samko, S.G.; Kilbas, A.A.; Mirchev, O.I. *Fractional Integrals and Derivatives, Theory and Applications*; Gordon and Breach Science Publishers: Singapore, 1993.
7. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
8. Akin, E.; Dosla, Z.; Lawrence, B. Oscillatory properties for three-dimensional dynamic systems. *Nonlinear Anal.* **2008**, *69*, 483–494. [[CrossRef](#)]
9. Akin, E.; Dosla, Z.; Lawrence, B. Almost oscillatory three-dimensional dynamical systems. *Adv. Differ. Equ.* **2012**, *2012*, 46. [[CrossRef](#)]
10. Dosoudilova, M.; Lomtadze, A.; Sremr, J. *Oscillatory Properties of Solutions to Two-Dimensional Emden-Fowler Type Equations*; International Workshop Qualitde; 2014. Available online: http://www.rmi.ge/eng/QUALITDE-2014/Dosoudilova_et_al_workshop_2014.pdf (accessed on 1 December 2018)
11. Huo, H.F.; Li, W.T. Oscillation criteria for certain two-dimensional differential systems. *Int. J. Appl. Math.* **2001**, *6*, 253–261.

12. Kordonis, I.-G.E.; Philos, C.G. On the oscillation of nonlinear two-dimensional differential systems. *Proc. Am. Math. Soc.* **1998**, *126*, 1661–1667. [[CrossRef](#)]
13. Li, W.T.; Cheng, S.S. Limiting behaviours of non-oscillatory solutions of a pair of coupled nonlinear differential equations. *Proc. Edinb. Math. Soc.* **2000**, *43*, 457–473. [[CrossRef](#)]
14. Lomtadze, A.; Sremr, J. On oscillation and nonoscillation of two-dimensional linear differential systems. *Georgian Math. J.* **2013**, *20*, 573–600. [[CrossRef](#)]
15. Sadhasivam, V.; Deepa, M.; Nagajothi, N. On the oscillation of nonlinear fractional differential systems. *Int. J. Adv. Res. Sci. Eng. Technol.* **2017**, *4*, 4861–4867.
16. Thandapani, E.; Selvaraj, B. Oscillatory behavior of solutions of three-dimensional delay difference systems. *Rodov. Mater.* **2004**, *13*, 39–52.
17. Troy, W.C. Oscillations in a third order differential equation modeling a nuclear reactor. *SIAM J. Appl. Math.* **1977**, *32*, 146–153. [[CrossRef](#)]
18. Schmeidel, E. Oscillation of nonlinear three-dimensional difference systems with delays. *Math. Biochem.* **2010**, *135*, 163–170.
19. Vreeke, S.D.; Sandquist, G.M. Phase plane analysis of reactor kinetics. *Nuclear Sci. Eng.* **1970**, *42*, 259–305. [[CrossRef](#)]
20. Bainov, D.; Mishev, D. *Oscillatory Theory of Operator-Differential Equations*; World Scientific: Singapore, 1987.
21. Erbe, L.H.; Kong, Q.K.; Zhang, B.G. *Oscillation Theory for Functional Differential Equations*; Marcel Dekker: New York, NY, USA, 1995.
22. Padhi, S.; Pati, S. *Theory of Third Order Differential Equations*; Electronic Journal of Differential Equations; Springer: New Delhi, India, 2014.
23. Saker, S.H. *Oscillation Theory of Delay Differential and Difference Equations*; VDM Verlag Dr. Muller Aktiengesellschaft and Co.: Mansoura, Egypt, 2010.
24. Kiguradze, I.T.; Chaturia, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Acad. Publ.: Dordrecht, The Netherlands, 1993.
25. Abdeljawad, T. On conformable fractional calculus. *J. Comput. Appl. Math.* **2015**, *279*, 57–66. [[CrossRef](#)]
26. Atangana, A.; Baleanu, D.; Alsaedi, A. New properties of conformable derivative. *Open Math.* **2015**, *13*, 889–898. [[CrossRef](#)]
27. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivatives. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [[CrossRef](#)]
28. Lazo, M.J.; Torres, D.F. Variational calculus with conformable fractional derivatives. *IEEE/CAA J. Autom. Sin.* **2017**, *4*, 340–352. [[CrossRef](#)]
29. Anderson, D.R.; Ulness, D.J. Properties of the Katugampola fractional derivative with potential application in quantum mechanics. *J. Math. Phys.* **2015**, *56*, 063502. [[CrossRef](#)]
30. Katugampola, U.N. A new fractional derivative with classical properties. *arXiv* **2014**, arXiv:1410.6535.
31. Katugampola, U.N. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **2014**, *6*, 1–15.
32. Spanikova, E. Oscillatory properties of solutions of three-dimensional differential systems of neutral type. *Czechoslov. Math. J.* **2000**, *50*, 879–887. [[CrossRef](#)]
33. Sadhasivam, V.; Kavitha, J.; Deepa, M. Existence of solutions of three-dimensional fractional differential systems. *Appl. Math.* **2017**, *8*, 193–208. [[CrossRef](#)]
34. Chatzarakis, G.E.; Deepa, M.; Nagajothi, N.; Sadhasivam, V. On the oscillation of three-dimensional α -fractional differential systems. *Dyn. Syst. Appl.* **2018**, *27*, 873–893.

