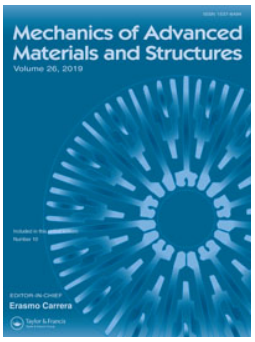

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On the free vibration of orthotropic and inhomogeneous with spatial coordinates plates resting on the inhomogeneous viscoelastic foundation

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ABSTRACT
The paper developed the closed-form solution for the free vibration problem of inhomogeneous orthotropic rectangular plates (IHORPs) resting on the inhomogeneous viscoelastic foundation (IHVEF). The Young's moduli and density of the orthotropic plate vary continuously with respect to three spatial coordinates, while the characteristics of the viscoelastic foundation vary depending on the in-plane coordinates. The relevant motion equation is obtained using the classical plate theory (CPT) and solved using method of separation of variables. The influences of inhomogeneity of orthotropic materials, inhomogeneity of viscoelastic and elastic foundations on the non-dimensional frequencies (NDFs) of plates are studied in detail.

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Inhomogeneous orthotropic plates; vibration; inhomogeneous viscoelastic foundation; frequencies

1. Introduction

An essential part of design and design calculation is the task of ensuring vibration and stability. It is very important to take a detailed account of the physic-mechanical factors, i.e. taking into account the real properties of the material and the working conditions of the products. One of these defining properties is the heterogeneity of the construction, both natural and technological, manifested in the process of manufacturing, working off and operating individual parts or the structure as a whole. Under a heterogeneous body in the general case is understood a body, mechanical, rigid, thermo-physical and other characteristics of which change in a certain way in its volume. If the formulation of the problems of the mechanics of inhomogeneous bodies does not cause any particular difficulties, since the general equations of the theory and stresses and deformations remain valid in this case, the methods for solving problems of a particularly dynamic nature, for objects with variable rigidity, density and curvature, are far from perfect in engineering terms And require intensive research. The basic knowledge on the changes of the material properties is given in the work of Kolchin and Favarion [1]. Significant contributions made in the future to consider different types of consideration of the inhomogeneity of the materials are given in references [2]–[12]. In many engineering applications, plates made of traditional and new composites are in contact with various elastic media and may have significant and inevitable effects on their vibration and stability behaviors. The various elastic and visco-elastic foundation models and their influences on the behavior on the structural elements are included in the works of Pasternak [13], Kerr [14], Vlasov and Leont'ev [15], Bazhenov [16], Selvaduarı [17], Levinson and Bharata [18], Gorbunov-Possadov et al. [19], Pronk [20] and Tsudik [21].

Various studies have been carried out to analyze the vibration of homogenous orthotropic plates resting on the homogeneous elastic and viscoelastic elastic foundations, using various approaches. Omurtag et al. [22] used mixed finite element formulation based on the Gateaux differential Kirchhoff plates resting on elastic foundation Sun [23] considered Kirchoff plate model on viscoelastic foundation, Sharma [24] proposed Pasternak foundation model to study vibration of orthotropic plate, Ferreira et al. [25] used wavelet collocation to analyze plates on the Winkler elastic foundation, Arani and Jalaei [26] developed visco-Pasternak model to study of orthotropic graphene sheet, Tornabene [27] and Sofiyev et al. [28] presented the non-linear elastic foundation model to the study of vibration shells.

Significantly increase the use of heterogeneous composite plates, as the bearing structural elements which in contact with various operating environments, have significantly increased the urgency of solving their vibration problems. Because of this urgency, some attempts have been made to study the influences of homogeneous elastic or viscoelastic foundations on the vibration behaviors of the inhomogeneous orthotropic plates [29]–[40].

Some of above mentioned studies suggest that the heterogeneity changes only in the direction of thickness, in some studies only in the longitudinal direction and in some studies together in the transverse and longitudinal directions. However, in the above studies, the elastic or viscoelastic foundation models are considered homogeneous. In the literature, the number of publications related to the interaction of composite plates with the heterogeneous foundations is limited [41]–[46].

To the best of the author's knowledge, literature lacks in analytical modeling for the vibration characteristics of inhomogeneous orthotropic plates resting on the inhomogeneous

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viscoelastic foundations. The proposed study describes closed form solution of the free vibration problem of inhomogeneous orthotropic plates resting on the inhomogeneous viscoelastic foundation. It is assumed that the Young's moduli and density of plates vary continuously with respect to the spatial coordinates and the viscoelastic foundation parameters vary continuously with respect to the in-plane coordinates, while the Poisson's ratio of the materials are assumed to be unchanged. A new closed-form solution for the free vibration problem of IHORPs resting on the IHVEF is obtained.

2. Formulation of the problem

Consider a thin IHORP of the length a, the width b and the thickness h. The plate is resting on the IHVEF. The mid-plane being $x_3 = 0$ and the origin of the coordinate system is at one corners of the orthotropic plate as shown in Figure 1. The x_1 and x_2 axes are taken along the principle directions of orthotropy and x_3 axis is normal to the them. The load-deflection relationship of the IHVEF is assumed to be [13]–[21], [41]–[45]

$$R = K_1(x_1, x_2)u_3 + K_2(x_1, x_2)\frac{\partial^2 u_3}{\partial t^2} \tag{1}$$

where R is the force per unit area, $K_1(x_1, x_2)$ and $K_2(x_1, x_2)$ are characteristics of the IHVEF, which are determined by experiments, u_3 is the deflection and t is time variable.

It is assumed that the Young's moduli, $E_1(X_1, X_2, X_3)$ and $E_2(X_1, X_2, X_3)$ in the x_1 and x_2 directions, shear modulus $G_{12}(X_1, X_2, X_3)$ and density $\rho(X_1, X_2, X_3)$ of the IHORP depending on the spatial coordinates x_1, x_2 and x_3, and the Poisson's ratios ν_{12} and ν_{21} are assumed to be constant [2]–[12].

$$\begin{aligned} &E_1 = E_{01} f_1(X_1, X_2) f_2(X_3), \\ &E_2 = E_{02} f_1(X_1, X_2) f_2(X_3), \quad G_{12} = G_{012} f_1(X_1, X_2) f_2(X_3), \\ &\rho = \rho_0 \psi_1(X_1, X_2)\psi_2(X_3), \quad \nu_{12} = const; \quad \nu_{21} = const. \end{aligned} \tag{2}$$

where E_{01} and E_{02} are the Young's moduli in the x_1 and x_2 directions, G_{012} is the shear modulus and ρ_0 is the density of the homogeneous orthotropic rectangular plate (HORP), respectively. Here $X_1 = x_1/a$, $X_2 = x_2/b$, $X_3 = x_3/h$ are the dimensionless variables; $f_1(X_1, X_2)$ and $\psi_1(X_1, X_2)$ with their partial derivatives up to the second order are continuous functions and characterize the change of the Young's moduli ad density, respectively, in the x_1 and x_2 directions; $f_2(X_3)$ and $\psi_2(X_3)$ are continuous functions and characterize the change of the Young's moduli and density, respectively, in the x_3 direction.

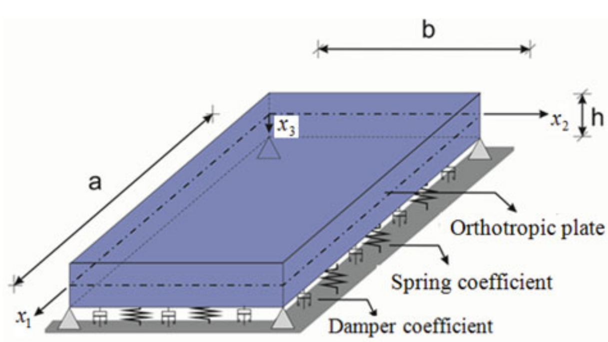

Figure 1. The inhomogeneous orthotropic rectangular plate on the viscoelastic foundation and the coordinate system.

3. Governing equation

Based on the CPT and using (2), the constitutive equations of the IORPs may be expressed as [1]:

$$\begin{aligned} \sigma_{11} &= \frac{E_{01} f_1(X_1, X_2)}{1 - \nu_{12}\nu_{21}} f_2(X_3)\left(\varepsilon_{11} + \nu_{12}\varepsilon_{22}\right), \\ \sigma_{22} &= \frac{E_{02} f_1(X_1, X_2)}{1 - \nu_{12}\nu_{21}} f_2(X_3)\left(\varepsilon_{22} + \nu_{21}\varepsilon_{11}\right) \\ \sigma_{12} &= G_{012} f_1(X_1, X_2) f_2(X_3)\varepsilon_{12} \end{aligned} \tag{3}$$

For continuously IHORPs, the Kirchhoff-Love hypotheses are valid and have [48]

$$\varepsilon_{11} = \varepsilon_{11}^0 - x_3\chi_{11}, \quad \varepsilon_{22} = \varepsilon_{22}^0 - x_3\chi_{22}, \quad \varepsilon_{12} = \varepsilon_{12}^0 - x_3\chi_{12} \tag{4}$$

where $\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0$ are strains in the mid-plane, $\chi_{11}, \chi_{22}, \chi_{12}$ are the curvatures of the middle plane and u_1, u_2, u_3 are the displacement components in which are related as follows:

$$\begin{aligned} &\varepsilon_{11}^0 = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22}^0 = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{12}^0 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \\ &\chi_{11} = \frac{\partial^2 u_3}{\partial {x_1}^2}, \quad \chi_{22} = \frac{\partial^2 u_3}{\partial {x_2}^2}, \quad \chi_{12} = 2\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \end{aligned} \tag{5}$$

The force and moment resultants are defined in terms of the normal and shear stresses as [47], [48],

$$\begin{aligned} &(T_{11}, \ T_{22}, \ T_{12}, \ M_{11}, \ M_{22}, \ M_{12}) \\ &= \int_{-h/2}^{h/2} (\sigma_{11}, \sigma_{22}, \sigma_{12}, \ x_3\sigma_{11}, \ x_3\sigma_{22}, \ x_3\sigma_{12}) dx_3 \end{aligned} \tag{6}$$

The resultant forces are everywhere equal to zero, because there are no external forces, $T_{11}, \ T_{22}, \ T_{12}$, in the plane of the plate. This case allows us to write the following conditions:

$$\begin{aligned} &A_1\left(\varepsilon_{11}^0 + \nu_{12}\varepsilon_{22}^0\right) - A_2\left(\chi_{11} + \nu_{12}\chi_{22}\right) = 0, \\ &A_1\left(\varepsilon_{22}^0 + \nu_{21}\varepsilon_{11}^0\right) - A_2\left(\chi_{22} + \nu_{21}\chi_{11}\right) = 0 \quad A_1\varepsilon_{12}^0 - A_2\chi_{12} = 0. \end{aligned} \tag{7}$$

where the following definitions apply:

$$A_1 = \int_{-h/2}^{h/2} f_2(X_3)dz, \quad A_2 = \int_{-h/2}^{h/2} x_3 \, f_2(X_3)dx_3 \tag{8}$$

The moments are expressed in terms of the lateral deflection, u_3, by substituting (3) into expression (6):

$$\begin{aligned} m_{11} &= \Pi D_{01} f_1(X_1, X_2)\left(\frac{\partial^2 u_3}{\partial {x_1}^2} + \nu_{12}\frac{\partial^2 u_3}{\partial {x_2}^2}\right), \\ m_{22} &= \Pi D_{02} f_1(X_1, X_2)\left(\frac{\partial^2 u_3}{\partial {x_2}^2} + \nu_{21}\frac{\partial^2 u_3}{\partial {x_1}^2}\right) \\ m_{12} &= 2\Pi D_{0T} f_1(X_1, X_2)\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \end{aligned} \tag{9}$$

where Π is the inhomogeneity parameter depending on the dimensionless thickness parameter, $D_{01}, \ D_{02}, \ D_{0T}$ are flexural rigidities of the homogeneous orthotropic rectangular plate

(HORP) and are defined as:

$$\Pi = \frac{12}{h^3}\left(\frac{A_2^2}{A_1} - A_3\right), \quad D_{01} = \frac{E_{01}h^3}{12(1-\nu_{12}\nu_{21})},$$
$$D_{02} = \frac{E_{02}h^3}{12(1-\nu_{12}\nu_{21})}, \quad D_{0T} = \frac{G_{012}h^3}{12} \tag{10}$$

in which

$$A_3 = \int_{-h/2}^{h/2} {x_3}^2 f_2(x_3)dx_3 \tag{11}$$

With considering Eqs. (1) and (2), the governing equation of motion of the IHORPs resting on the IHVEF may be expressed as [28], [37]–[45]:

$$\frac{\partial^2 m_{11}}{\partial {x_1}^2} + 2\frac{\partial^2 m_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 m_{22}}{\partial {x_2}^2} - K_1(X_1,X_2)u_3 - [K_2(X_1,X_2) + \bar{\rho}_0\psi_1(X_1,X_2)]\frac{\partial^2 u_3}{\partial t^2} = 0 \tag{12}$$

where the following notation is used:

$$\bar{\rho}_0 = \rho_0 h \int_{-1/2}^{1/2} \psi_2(X_3)dX_3 \tag{13}$$

Writing expression (9) into Eq. (12), is transformed into the following form

$$L(u_3) - K_1(X_1,X_2)u_3 - [K_2(X_1,X_2) + \bar{\rho}_0\psi_1(X_1,X_2)]\frac{\partial^2 u_3}{\partial t^2} = 0 \tag{14}$$

where $L(u_3)$ is differential operator and defined as:

$$\begin{aligned} L(u_3) = f_1(X_1,X_2)\Bigg[D_1\frac{\partial^4 u_3}{\partial {x_1}^4} + D_2\frac{\partial^4 u_3}{\partial {x_2}^4} &+ (D_1\nu_{21} + \nu_{12}D_2 + 4D_T)\frac{\partial^4 u_3}{\partial {x_1}^2\partial {x_2}^2}\Bigg] \\ &+ 2D_1\frac{\partial f_1(X_1,X_2)}{\partial x_1}\left(\frac{\partial^3 u_3}{\partial {x_1}^3} + \nu_{12}\frac{\partial^3 u_3}{\partial x_1\partial {x_1}^2}\right) \\ &+ D_1\frac{\partial^2 f_1(X_1,X_2)}{\partial {x_1}^2}\left(\frac{\partial^2 u_3}{\partial {x_1}^2} + \nu_{12}\frac{\partial^2 u_3}{\partial {x_2}^2}\right) \\ &+ 2D_2\frac{\partial f_1(X_1,X_2)}{\partial x_1}\left(\frac{\partial^3 u_3}{\partial {x_2}^3} + \nu_{21}\frac{\partial^3 u_3}{\partial {x_1}^2\partial x_2}\right) \\ &+ D_2\frac{\partial^2 f_1(X_1,X_2)}{\partial {x_2}^2}\left(\frac{\partial^2 u_3}{\partial {x_2}^2} + \nu_{21}\frac{\partial^2 u_3}{\partial {x_1}^2}\right) \\ &+ 4D_T\Bigg[\frac{\partial f_1(X_1,X_2)}{\partial x_1}\frac{\partial^3 u_3}{\partial x_1\partial {x_2}^2} + \frac{\partial f_1(X_1,X_2)}{\partial x_2}\frac{\partial^3 u_3}{\partial x_2\partial {x_1}^2} \\ &+ \frac{\partial^2 f_1(X_1,X_2)}{\partial x_1\partial x_2}\frac{\partial^2 u_3}{\partial x_1\partial x_2}\Bigg] \end{aligned} \tag{15}$$

which the following notations are used:

$$(D_1, D_2, D_T) = \Pi\left(D_1^0,\ D_2^0,\ D_T^0\right) \tag{16}$$

As $f_1(X_1,X_2) = \psi_1(X_1,X_2) = 1$, from Eq. (14) is obtained the equation of motion for the plates made of inhomogeneous orthotropic materials in which elastic properties vary only in the thickness direction, z.

If $f_1(X_1,X_2) = \psi_1(X_1,X_2) = 1$ and $f_2(X_3) = \psi_2(X_3) = 1$, from Eq. (14) is obtained equation of motion for the homogeneous orthotropic plates.

4. Solution of governing equation

As can be seen, Eq. (14) is complex, and is difficult to find an exact solution. Therefore, to the solution of the Eq. (14) for the homogeneous boundary conditions, we use method of separation of variables and method of Bubnov-Galerkin.

At the first stage, for the harmonic solution of the Eq. (14), the displacement, u_3, is assumed to be [48]:

$$u_3 = U_3(x_1,x_2)e^{i\omega t} \tag{17}$$

where $i = \sqrt{-1}$, $U_3(x_1,x_2)$ is unknown function, should satisfy the boundary conditions and ω is the angular frequency of the free vibration.

Substituting (17) into Eq. (14), we obtain:

$$\bar{L}(U_3) - K_1(X_1,X_2)U_3 + \omega^2[K_2(X_1,X_2) + \bar{\rho}_0\psi_1(X_1,X_2)]\,U_3 = 0 \tag{18}$$

where

$$\begin{aligned} \bar{L}(U_3) = f_1(X_1,X_2)\Bigg[D_1\frac{\partial^4 U_3}{\partial {x_1}^4} + D_2\frac{\partial^4 U_3}{\partial {x_2}^4} &+ (D_1\nu_{12} + \nu_{21}D_2 + 4D_T)\frac{\partial^4 U_3}{\partial {x_1}^2\partial {x_2}^2}\Bigg] \\ &+ 2D_1\frac{\partial f_1(X_1,X_2)}{\partial x_1}\left(\frac{\partial^3 U_3}{\partial {x_1}^3} + \nu_{12}\frac{\partial^3 U_3}{\partial x_1\partial {x_2}^2}\right) \\ &+ D_1\frac{\partial^2 f_1(X_1,X_2)}{\partial {x_1}^2}\left(\frac{\partial^2 U_3}{\partial {x_1}^2} + \nu_{12}\frac{\partial^2 U_3}{\partial {x_2}^2}\right) \\ &+ 2D_2\frac{\partial f_1(X_1,X_2)}{\partial x_2}\left(\frac{\partial^3 U_3}{\partial {x_2}^3} + \nu_{21}\frac{\partial^3 U_3}{\partial {x_1}^2\partial x_2}\right) \\ &+ D_2\frac{\partial^2 f_1(X_1,X_2)}{\partial {x_2}^2}\left(\frac{\partial^2 U_3}{\partial {x_2}^2} + \nu_{21}\frac{\partial^2 U_3}{\partial {x_1}^2}\right) \\ &+ 4D_T\Bigg[\frac{\partial f_1(X_1,X_2)}{\partial x_1}\frac{\partial^3 U_3}{\partial x_1\partial {x_2}^2} + \frac{\partial f_1(X_1,X_2)}{\partial x_2}\frac{\partial^3 U_3}{\partial {x_1}^2\partial x_2} \\ &+ \frac{\partial^2 f_1(X_1,X_2)}{\partial x_1\partial x_2}\frac{\partial^2 U_3}{\partial x_1\partial x_2}\Bigg] \end{aligned} \tag{19}$$

In the second stage, we will solve the Eq. (18) using the Bubnov-Galerkin method and $U_3(x_1,x_2)$ represents in the following form:

$$U_3(x_1,x_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} A_{ij}\varphi_i(x_1)\eta_j(x_2) \tag{20}$$

where A_{ij} are unknown constants and each $\varphi_i(x_1)$ and $\eta_j(x_2)$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k)$ must satisfy the corresponding boundary conditions.

The error function is written in the following way:

$$\Gamma(x_1,x_2) = \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{A_{ij}\left\langle L(\varphi_i,\eta_j) - K_1(X_1,X_2)\varphi_i(x_1)\eta_j(x_2) + \omega^2[K_2(X_1,X_2) + \bar{\rho}_0\psi_1(X_1,X_2)]\varphi_i(x_1)\eta_j(x_2)\right\rangle\right\} \tag{21}$$

where $L(\varphi_i, \eta_j)$ is the differential operator and expressed as

$$L\left(\varphi_i, \eta_j\right) = f_1(X_1, X_2)\left[D_1\frac{d^4\varphi_i}{dx_1{}^4}\eta_j + D_2\frac{d^4\eta_j}{dx_2{}^4}\varphi_i + (D_1\nu_{21} + D_2\nu_{12} + 4D_T)\frac{d^2\varphi_i}{dx_1{}^2}\frac{d^2\eta_j}{dx_2{}^2}\right] + 2D_1\frac{\partial f_1(X_1, X_2)}{\partial x_1}\left(\frac{d^3\varphi_i}{dx_1{}^3}\eta_j + \nu_{12}\frac{d\varphi_i}{dx_1}\frac{d^2\eta_j}{dx_2{}^2}\right) + D_1\frac{\partial^2 f_1(X_1, X_2)}{\partial x_1{}^2}\left(\frac{d^2\varphi_i}{dx_1{}^2}\eta_j + \nu_{12}\frac{d^2\eta_j}{dx_2{}^2}\varphi_i\right) + 2D_2\frac{\partial f_1(X_1, X_2)}{\partial x_2}\left(\frac{d^3\eta_j}{dx_2{}^3}\varphi_i + \nu_{21}\frac{d^2\varphi_i}{dx_1{}^2}\frac{d\eta_j}{dx_2}\right) + D_2\frac{\partial^2 f_1(X_1, X_2)}{\partial x_2{}^2}\left(\frac{d^2\eta_j}{dx_2{}^2}\varphi_i + \nu_{21}\frac{d^2\varphi_j}{dx_1{}^2}\eta_i\right) + 4D_T\left[\frac{\partial f_1(X_1, X_2)}{\partial x_1}\frac{d\varphi_i}{dx_1}\frac{d^2\eta_j}{dx_2{}^2} + \frac{\partial f_1(X_1, X_2)}{\partial x_2}\frac{d^2\varphi_i}{dx_1{}^2}\frac{d\eta_j}{dx_2} + \frac{\partial^2 f_1(X_1, X_2)}{\partial x_1\partial x_2}\frac{d\varphi_i}{dx_1}\frac{d\eta_j}{dx_2}\right] \tag{22}$$

The orthogonalization conditions have the following form:

$$\Lambda \equiv \int_0^a\int_0^b\sum_{i=1}^n\sum_{j=1}^k\left\{A_{ij}\left\langle f_1(X_1, X_2)\left[D_1\frac{d^4\varphi_i}{dx_1{}^4}\eta_j + D_2\frac{d^4\eta_j}{dx_2{}^4}\varphi_i + (D_1\nu_{21} + D_2\nu_{12} + 4D_T)\frac{d^2\varphi_i}{dx_1{}^2}\frac{d^2\eta_j}{dx_2{}^2}\right] + 2D_1\frac{\partial f_1(X_1, X_2)}{\partial x_1}\left(\frac{d^3\varphi_i}{dx_1{}^3}\eta_j + \nu_{12}\frac{d\varphi_i}{dx_1}\frac{d^2\eta_j}{dx_2{}^2}\right) + D_1\frac{\partial^2 f_1(X_1, X_2)}{\partial x_1{}^2}\left(\frac{d^2\varphi_i}{dx_1{}^2}\eta_j + \nu_{12}\frac{d^2\eta_j}{dx_2{}^2}\varphi_i\right) + 2D_2\frac{\partial f_1(X_1, X_2)}{\partial x_2}\left(\frac{d^3\eta_j}{dx_2{}^3}\varphi_i + \nu_{21}\frac{d^2\varphi_i}{dx_1{}^2}\frac{d\eta_j}{dx_2}\right) + D_2\frac{\partial^2 f_1(X_1, X_2)}{\partial x_2{}^2}\left(\frac{d^2\eta_j}{dx_2{}^2}\varphi_i + \nu_{21}\frac{d^2\phi_j}{dx_1{}^2}\eta_i\right) + 4D_T\left[\frac{\partial f_1(X_1, X_2)}{\partial x_1}\frac{d\varphi_i}{dx_1}\frac{d^2\eta_j}{dx_2{}^2} + \frac{\partial f_1(X_1, X_2)}{\partial x_2}\frac{d^2\varphi_i}{dx_1{}^2}\frac{d\eta_j}{dx_2} + \frac{\partial^2 f_1(X_1, X_2)}{\partial x_1\partial x_2}\frac{d\varphi_i}{dx_1}\frac{d\eta_j}{dx_2}\right] - K_1(X_1, X_2)\varphi_i(x_1)\eta_j(x_2) + \omega^2\left[K_2(X_1, X_2) + \bar{\rho}_0\psi_1(X_1, X_2)\right]\varphi_i(x_1)\eta_j(x_2)\right\rangle\right\} \times \varphi_p(x_1)\eta_q(x_2)dx_1dx_2 = 0, \quad (p, q = 1, 2, \ldots) \tag{23}$$

The values of ω^2 can be determined from the system of linear homogeneous algebraic equations (23). For the existence of a nontrivial solution, the principal determinant must vanish:

$$|\Lambda| = 0 \tag{24}$$

The Eq. (24) is the system of the nonlinear algebraic equations. However, in engineering practice are limited to the first approximation, although the solution of this equation in an arbitrary approximation does not cause special difficulty.

In the first approximation, from Eq. (21), we obtain

$$\int_0^a\int_0^b\Gamma(x_1, x_2)\varphi_1(x_1)\eta_1(x_2)dx_1dx_2 = 0 \tag{25}$$

It assumed that Young's moduli, shear modulus and density of the IHORPs vary with the spatial coordinates by the following functional relations:

$$f_1(X_1, X_2) = 1 + \alpha_1X_1 + \alpha_2X_2, \quad f_2(X_3) = 1 + \alpha_3X_3,$$
$$\psi_1(X_1, X_2) = 1 + \beta_1X_1 + \beta_2X_2, \quad \psi_2(X_3) = 1 + \beta_3X_3 \tag{26}$$

where α_i and $\beta_i (i = 1, 2, 3)$ inhomogeneity parameters for material are changed from 0 to 1.

The IHVEF parameters vary with the in-plane coordinates by the following functional relations:

$$K_1(X_1, X_2) = K_{01}\left(1 + \delta_1X_1 + \delta_2X_2\right),$$
$$K_2(X_1, X_2) = K_{02}\left(1 + \delta_1X_1 + \delta_2X_2\right) \tag{27}$$

where $\delta_j (j = 1, 2)$ inhomogeneity parameters for the viscoelastic foundation are changed from 0 to 1.

Since the boundaries of a rectangular plate are simply-supported, the approximation function is sought as [48]

$$\varphi_1(x) = \sin\frac{m\pi x}{a}, \quad \eta_1(y) = \frac{\sin n\pi y}{b} \tag{28}$$

where m and n are integers.

Substituting expressions (26)–(28) into Eq. (25), after integrating we obtain expression for the frequency (in rad/s) of IHORPs resting on the IHVEF:

$$\omega = \sqrt{\frac{1}{2}\frac{K_{01}(\delta_1 + \delta_2 + 2) - \Pi(\alpha_1 + \alpha_2 + 2)\left[m_1^4D_{01} + n_1^4D_{02} + m_1^2n_1^2(\nu_{21}D_{01} + \nu_{12}D_{02} + 4D_{0T})\right]}{K_{02}(\delta_1 + \delta_2 + 1) + \rho_0h(\beta_1 + \beta_2 + 1)}} \tag{29}$$

The expression for the non-dimensional frequencies (NDFs) for IHORPs resting on the IHVEF is defined as:

$$\omega_1 = \omega\frac{a^2}{h}\sqrt{\frac{\rho_0}{E_{01}}} \tag{30}$$

As $\alpha_i = \beta_i = \delta_j = 0$ $(i = 1, 2, 3;\ j = 1, 2)$, the frequency coincides with the frequency of the HORPs resting on the homogeneous viscoelastic foundation.

As $\alpha_i = \beta_i = 0$ $(i = 1, 2, 3)$ and $K_{01} = K_{02} = 0$, the frequency coincides with the frequency of the HORPs without an elastic foundation and expressed as:

$$\omega_0 = \sqrt{\frac{m_1^4D_{01} + n_1^4D_{02} + m_1^2n_1^2(\nu_{21}D_{01} + \nu_{12}D_{02} + 4D_{0T})}{\rho_0h}} \tag{31}$$

5. Numerical results and discussion

5.1. Comparative studies

The magnitudes of the NDF of HORPs without an elastic foundation with $a/h = 100$ are compared with the results

Table 1. Comparison of the values of NDF for HORPs with the results of Thai and Kim [33].

		$\omega_1 = \omega a^2/h\sqrt{\rho_0/E_{02}}$	
a/b	E_{01}/E_{02}	Thai and Kim [32]	Present study
0.5	10	9.3421	9.34211
	25	14.4578	14.4578
	40	18.1876	18.1876
2.0	10	17.1364	17.1364
	25	20.3682	20.3682
	40	23.1622	23.1622

Table 2. Comparison of the NDF for IHORPs with the results of Xia et al. [7] for $a/h = 20$ and $a = 2b$.

	$\omega_1 = \omega a^2\sqrt{\rho_0 h/D_0}$	
α	Xia et al. [7]	Present study
1	48.540	49.348
3	66.722	69.788

of Thai and Kim [33] using CPT and presented in Table 1. The Levy type solution is used in the study of Thai and Kim [33]. The expression (31) is used for comparison. The following orthotropic material properties is used in the comparison: $E_{01}/E_{02} = 10,\ 25,\ 40;\ G_{012} = 0.5E_{02};\ \ \nu_{12} = 0.25;\ \rho_0 = 1$. The values of the NDF obtained in this study are in a good agreement with those obtained in the study of the Thai and Kim [33].

In Table 2, the magnitudes of NDF for the inhomogeneous (along the x_1 axis) isotropic rectangular plate (IHIRP) are compared with the results of Xia et al. [7]. The expression (29) is used in the comparison by using, $\alpha_1 = \alpha - 1,\ \ \alpha_2 = \alpha_3 = \beta_i = \delta_j = 0,\ \ K_{01} = K_{02} = 0\ (i = 1, 2, 3;\ j = 1, 2)$ and $E_{01} = E_{02} = E_0,\ \nu_{12} = \nu_{21} = \nu_0$. The Young's modulus of the IHIRP changes depending on the coordinate x_1 as $E = E_0 f(X_1)$, in which $f(X_1) = 1 + X_1(\alpha - 1)$ is the inhomogeneity function, $\alpha = \frac{f(1)}{f(0)}$ is parameter that $f(0)$ and $f(1)$ are the magnitudes of the $f(X_1)$ at the left and right ends of the plate, respectively. The homogeneous material properties are given by: $E_0 = 3 \times 10^7 Pa$, $\nu_0 = 0.3,\ \ G_{012} = \frac{E_0}{2(1+\nu_0)},\ \rho_0 = 1\ kg/m^3$ and $D_0 = \frac{E_0 h^3}{12(1-\nu_0^2)}$. It is seen from Table 2 that the values of the NDF of the IHIRP are in good agreement with those of Xia et al. [7]. The reason for the difference between NDF values is the use of SDT in the study of Xia et al. [7].

The values of the NDF, $\omega_1 = \frac{\omega b^2}{\pi^2}\sqrt{\rho_0 h/D_0}$, for the thin homogeneous isotropic square plates resting on the homogeneous Winkler elastic foundation are compared with the studies of Leissa [49] and Ferreira et al. [50] and tabulated in Table 3. The following plate and foundation parameters are used:

Table 3. Comparison of NDFs for the thin square plates on the homogeneous Winkler elastic foundation.

$\bar{K}_1$	References	ω_1
10^2	Leissa [47]	2.2420
	Ferreira et al. [50]	2.2414
	Present study	2.2420
5×10^2	Leissa [47]	3.0221
	Ferreira et al. [50]	3.0216
	Present study	3.0220

Table 4. Comparison of NDFs for IHRPs (in x_1 direction) resting on the homogeneous Winkler elastic foundation for different a/h.

	ω_1			
	Present study		Ref. [40]	
a/h	75	100	75	100
$\alpha = 1.5,\ \beta = 1$	3.771	4.410	3.772	4.409
$\alpha =\ \ \beta = 1.5$	3.391	3.964	3.390	3.964

$E_0 = 2 \times 10^{11}(Pa),\ \nu_0 = 0.3,\ \ \rho_0 = 1,\ \ b/h = 100$, and $K_1 = \frac{K_{01}a^4}{D_0},\ K_{02} = 0$. The expression (29) is used in the comparison for the case, $\alpha_i = \ \beta_i = \delta_j = 0,\ \ K_{02} = 0\ (i = 1, 2, 3;\ j = 1, 2)$ and $E_{01} = E_{02} = E_0,\ \nu_{12} = \nu_{21} = \nu_0$. The magnitudes of the NDF are in excellent agreement with the results of Refs. [49], [50] for thin plates.

The values of NDFs of IHORPs resting on the homogeneous Winkler elastic foundation are compared with the results of Ref. [40] for different a/h ($a/b = 0.5,\ \ b = 1$ m, $K_{01} = 2.5 \times 10^6\ \mathrm{N/m^3}$) and are presented in Table 4. The following homogeneous and inhomogeneous (in x_1 direction) orthotropic material properties are used [40], [48]:

$$E_{01} = 138.6\,\mathrm{GPa},\ \ E_{02} = 8.27\,\mathrm{GPa};\ \ G_{012} = 4.12\,\mathrm{GPa},$$
$$\nu_{12} = 0.26,\ \ \rho_0 = 1824\,\mathrm{kg/m^3}$$
$$E_1(X_1) = E_{01}\alpha^{X_1}, E_2(X_1) = E_{02}\alpha^{X_1},\ \ G_{12}(X_1) = G_{012}\alpha^{X_1},$$
$$\rho(X_1) = \rho_0\beta^{X_1}$$

Here α and β are the variation parameter of Young's (or shear) modulus and density, respectively, in the x_1 direction and they are defined as $\alpha = f(1)/f(0)$ and $\beta = \psi(1)/\psi(0)$ in which $f(0)$ and $f(1)$ are the values of function, $f(X_1)$, $\psi(0)$ and $\psi(1)$ are the values of the function, $\psi(X_1)$, on the $x_1 = 0$ and $x_1 = a$ ends of the plate, respectively. For comparison, formula (15) is used in the Ref. [40]. It is seen that a very good agreement between the present results and in the literature are achieved.

5.2. *Study of influences of inhomogeneity and viscoelastic foundations on the NDFs*

This section presents new numerical calculations and analyzes related to the influences of inhomogeneity in different directions on the NDFs of the rectangular plates resting on the inhomogeneous viscoelastic and elastic foundations.

The elastic properties for the homogeneous boron-epoxy are given by [48]

$$E_{01} = 206.9\,\mathrm{GPa},\ \ E_{02} = 20.69\,\mathrm{GPa};$$
$$G_{012} = 6.9\,\mathrm{GPa},\ \ \nu_{12} = 0.25,\ \ \ \ \rho_0 = 1950\,\mathrm{kg/m^3}$$

The changes in the NDFs of HORPs and IHORPs with and without homogeneous and inhomogeneous viscoelastic and elastic foundations versus the ratio, a/h, are presented in Tables 5 and 6, and Figures 2–4. In the following analyzes, the values of the NDFs of unconstrained homogeneous orthotropic plates are compared with IHORPs with and without homogeneous and inhomogeneous elastic and viscoelastic foundations for different cases of inhomogeneity. It seems that the values of the NDFs of HORPs and IHORPs-linear profiles (inhomogeneity functions are linear functions of the coordinates x_1, x_2

Table 5. Distribution of NDFs of the HORPs and IHORPs resting on the elastic and viscoelastic foundations versus the a/h (a/b = 0.5, b = 1 m).

	ω_1			
	$\alpha_i = \beta_i = 0$ (i = 1, 2, 3) $\delta_j = 0$ (i = 1, 2)	$\alpha_1 = 1,\ \alpha_2 = \alpha_3 = 0$ $\beta_i = \delta_j = 0$	$\alpha_1 = \alpha_2 = 1,\ \alpha_3 = 0$ $\beta_i = \delta_j = 0$	$\alpha_i = 1$ $\beta_i = \delta_j = 0$
a/h	$(K_{01}, K_{02}) = (0, 0)$			
25	2.931	3.590	4.145	3.969
50	2.931	3.590	4.145	3.969
75	2.931	3.590	4.145	3.969
100	2.931	3.590	4.145	3.969
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
25	2.963	3.616	4.168	3.993
50	3.178	3.795	4.324	4.155
75	3.700	4.241	4.720	4.566
100	4.547	4.997	5.410	5.276
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
25	2.644	25	2.644	25
50	2.584	50	2.584	50
75	2.782	75	2.782	75
100	3.195	100	3.195	100
	$\alpha_i = \beta_i = \delta_1 = 0$	$\alpha_1 = 1,\ \alpha_2 = \alpha_3 = 0$ $\beta_i = \delta_1 = 0$	$\alpha_1 = \alpha_2 = 1,$ $\alpha_3 = \beta_i = \delta_1 = 0$	$\alpha_i = 1,$ $\beta_i = \delta_1 = 0$
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
25	2.979	25	2.979	25
50	3.295	50	3.295	50
75	4.030	75	4.030	75
100	5.169	100	5.169	100
	$\alpha_i = \beta_i = 0$ $\delta_1 = \delta_2 = 1$	$\alpha_1 = \delta_1 = \delta_2 = 1,$ $\alpha_2 = \alpha_3 = \beta_i = 0$	$\alpha_1 = \alpha_2 = 1,\ \alpha_3 = 0$ $\beta_i = 0,\ \delta_1 = \delta_2 = 1$	$\alpha_i = 1,\ \beta_i = 0,$ $\delta_1 = \delta_2 = 1$
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
25	2.435	2.961	3.407	3.265
50	2.394	2.802	3.158	3.044
75	2.721	3.016	3.284	3.197
100	3.277	3.484	3.681	3.617

and x_3) without elastic foundations remain unchanged, as a/h increases. If the material properties linearly change along the x_1, x_2 and x_3 directions, the effects of the inhomogeneity on the NDFs of the unconstrained ($\delta_j = 0$, $j = 1, 2$; $K_{01} = K_{02} = 0$) orthotropic rectangular plates are considerable and remain unchanged (see, Tables 5 and 6, and Figure 2).

It is appearing that the values of the NDFs of HORP and IHORPs resting on the homogeneous and inhomogeneous Winkler elastic foundation (HWEF and IHWEF) increases, with increasing of the ratio, a/h. If the inhomogeneity linearly changes along the x_1, x_2 or x_3 directions, the effects of the inhomogeneity on the frequency parameters of the orthotropic

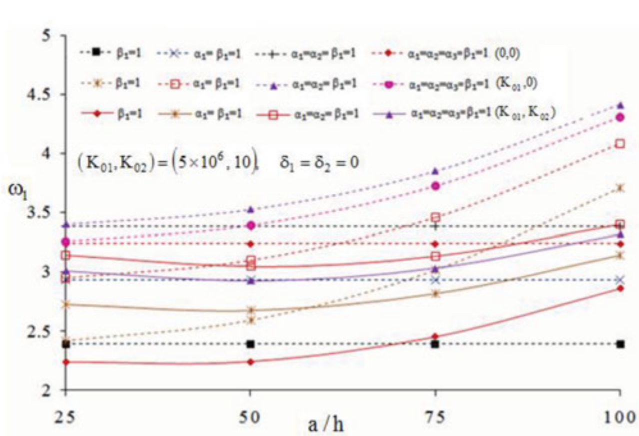

Figure 2. Variation of NDFs for the HORPs and IHORPs with and without the HWEF and HVEF versus the ratio a/h (a/b = 0.5, b = 1 m).

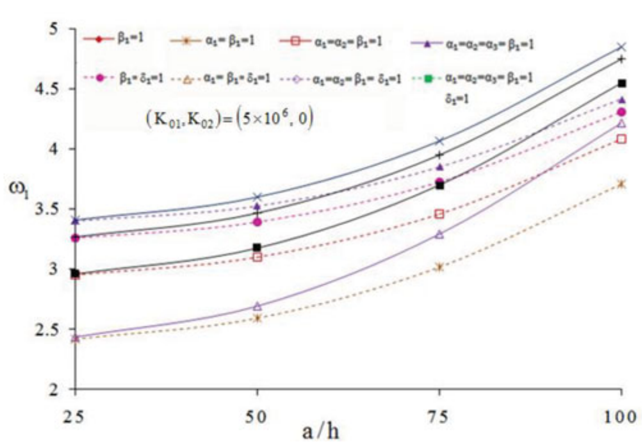

Figure 3. Variation of NDFs of the HORPs and IHORPs resting on the HWEF and IHWEF and elastic foundations versus the ratio, a/h (a/b = 0.5, b = 1 m).

Table 6. Distribution of NDFs for HORPs and IHORPs resting on the HWEF, IHWEF, HVEF and IHVEF and elastic foundations versus the a/h (a/b = 0.5, b = 1 m).

	ω_1			
	$\alpha_i = \delta_j = \beta_3 = 0$, $\beta_1 = \beta_2 = 1$	$\alpha_1 = \beta_1 = \beta_2 = 1$, $\alpha_2 = \alpha_3 = \beta_3 = \delta_j = 0$	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\alpha_3 = \beta_3 = \delta_j = 0$	$\alpha_i = \beta_1 = \beta_2 = 1$, $\beta_3 = \delta_j = 0$
a/h	$(K_{01}, K_{02}) = (0, 0)$			
25	2.072	2.539	2.931	2.806
50	2.072	2.539	2.931	2.806
75	2.072	2.539	2.931	2.806
100	2.072	2.539	2.931	2.806
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
25	2.095	2.557	2.947	2.823
50	2.248	2.683	3.057	2.938
75	2.616	2.999	3.338	3.229
100	3.215	3.534	3.825	3.731
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
25	1.973	25	1.973	25
50	2.005	50	2.005	50
75	2.223	75	2.223	75
100	2.614	100	2.614	100
	$\alpha_i = \beta_3 = \delta_2 = 0$, $\beta_1 = \beta_2 = \delta_1 = 1$	$\alpha_2 = \alpha_3 = \beta_3 = \delta_2 = 0$, $\alpha_1 = \beta_1 = \beta_2 = \delta_1 = 1$	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\alpha_3 = \beta_3 = \delta_2 = 0, \delta_1 = 1$	$\beta_1 = \beta_2 = \alpha_i = 1$, $\beta_3 = \delta_2 = 0, \delta_1 = 1$
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
25	1.973	2.407	2.775	2.658
50	2.005	2.394	2.728	2.621
75	2.223	2.549	2.837	2.744
100	2.614	2.873	3.110	3.033
	$\alpha_i = \beta_3 = 0$, $\beta_1 = \beta_2 = \delta_j = 1$	$\alpha_1 = \beta_1 = \beta_2 = \delta_j = 1$, $\alpha_2 = \alpha_3 = \beta_3 = 0$	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\alpha_3 = \beta_3 = 0, \delta_j = 1$	$\alpha_i = \delta_j = 1$, $\beta_1 = \beta_2 = 1, \beta_3 = 0$
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
25	1.889	2.298	2.644	2.534
50	1.959	2.293	2.584	2.491
75	2.304	2.554	2.782	2.708
100	2.844	3.024	3.194	3.139

rectangular plates resting on the homogeneous Winkler elastic foundation (HWEF) ($\delta_1 = \delta_2 = 0$, $K_{01} = 5 \times 10^6$, $K_{02} = 0$) reduce for $\alpha_1 = 1, \beta_i = 0$ $(i = 1, 2, 3)$; $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ and $\alpha_i = 1$, $\beta_i = 0$ (see, Table 5); the effect of inhomogeneity is (−18.3%) for $\alpha_1 = 0, \beta_1 = 1$; this effect increases for $\alpha_1 = \beta_1 = 1$; changes irregularly for $\alpha_1 = \alpha_2 = \beta_1 = 1$ and $\alpha_i = \beta_1 = 1$ (see, Figure 2); the influence of inhomogeneity is (−29.3%) for $\alpha_1 = 0$, $\beta_1 = \beta_2 = 1$; increase for $\alpha_1 = \beta_1 = \beta_2 = 1$; $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ and $\alpha_i = \beta_1 = \beta_2 = 1$ (see, Table 6), since a/h increases from 25 to 100.

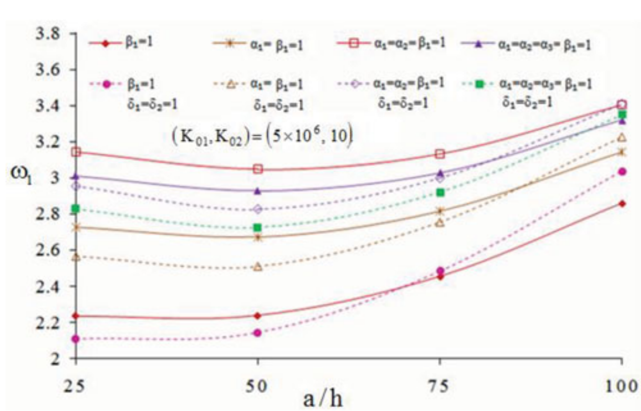

Figure 4. Variation of NDFs of the HORPs and IHORPs resting on the HVEF and IHVEF and elastic foundations versus the a/h (a/b = 0.5, b = 1 m).

When plates resting on the inhomogeneous Winkler elastic foundation (IHWEF) ($\delta_1 = 1$, $\delta_2 = 0$, $K_{01} = 5 \times 10^6$, $K_{02} = 0$) the influences of inhomogeneity decrease for $\alpha_1 = 1$, $\beta_i = 0$; $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ and $\alpha_i = 1$, $\beta_i = 0$ (see, Table 5); the effect of inhomogeneity is (−18.4%) for $\alpha_i = 0$, $\beta_1 = 1$, increases for $\alpha_1 = \beta_1 = 1$; changes between 14.85% and (−2.86%) for $\alpha_1 = \alpha_2 = \beta_1 = 1$; changes irregularly for $\alpha_i = \beta_1 = 1$ (see, Figure 3); the effect of inhomogeneity is (−29.27%) for $\alpha_1 = 0$; $\beta_1 = \beta_2 = 1$; increment for $\alpha_1 = \beta_1 = \beta_2 = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ and $\alpha_i = \beta_1 = \beta_2 = 1$ (see, Table 6), since a/h increases from 25 to 100.

As the HVEF effect is taken into account, the values of the NDF are significantly reduced compared to the case without an elastic foundation, whereas, the effects of inhomogeneity on the values of NDF vary with the change of the ratio, a/h. For instance the effects of the inhomogeneity on the NDFs for IHORPs resting on the homogeneous viscoelastic foundation (HVEF) ($\delta_1 = \delta_2 = 0$, $K_{01} = 5 \times 10^6$, $K_{02} = 10$) reduce for $\alpha_1 = 1, \beta_i = 0$; $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ and $\alpha_i = 1, \beta_i = 0$ (see, Table 5); the influences of the inhomogeneity diminish from (−15.43%) to (−10.45%) for $\alpha_i = 0$, $\beta_1 = 1$, changes between 3.21% and (−1.6%) for $\alpha_1 = \beta_1 = 1$, diminish from 18.95% to

6.54% for $\alpha_1 = \alpha_2 = \beta_1 = 1$ and from 13.96% to 3.91% for $\alpha_i = \beta_1 = 1$ (see, Figure 2); as Table 6 shows, the effects of the inhomogeneity decreases for $\alpha_i = 0$ $\beta_1 = \beta_2 = 1$, increases for $\alpha_1 = \beta_1 = \beta_2 = 1$; changes irregularly for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ and $\alpha_i = \beta_1 = \beta_2 = 1$, since a/h increases from 25 to 100.

If the cases of IHVEF and HVEF are compared, the values of the NDFs for the IHVEF are reduced. Consideration of the effect of IHVEF, changes the influence of the inhomogeneity on the values of NDFs depending on the change of the a/h ratio. For instance the plates resting on the inhomogeneous viscoelastic foundation, i.e., on the IHVEF ($\delta_1 = \delta_2 = 1$, $K_{01} = 5 \times 10^6$, $K_{02} = 10$) the effects inhomogeneity decrease for $\alpha_1 = 1$, $\beta_i = 0$; $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ and $\alpha_i = 1$, $\beta_i = 0$ (Table 5); this effect decreases for $\alpha_1 = 0$; $\beta_1 = 1$, $\alpha_1 = \alpha_2 = \beta_1 = 1$; $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = 1$, and changes irregularly for $\alpha_1 = \beta_1 = 1$, (see, Figure 4); the effect of inhomogeneity decreases from (−22.42%) to (−13.21%) for $\alpha_i = 0$, $\beta_1 = \beta_2 = 1$, increases from (−5.63%) to (−7.72%) for $\alpha_1 = \beta_1 = \beta_2 = 1$; changes between 8.58% and (−2.53%) for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$; changes between 4.07% and (−4.21%) for $\alpha_i = \beta_1 = \beta_2 = 1$ (see, Table 6), since a/h increases from 25 to 100.

As can be seen from Tables 5 and 6, and Figures 2–4, the effects of inhomogeneity on the values of the NDFs of plates resting on the IHVEF are greater than the plates resting on the HVEF and the difference varies from 2% to 4%. In addition, the effect of the inhomogeneity on the NDF does not depend on the change in the ratio a/h, and remains approximately the same (-18.4%) at $\alpha_i = 0$ and $\beta_i = 1$ ($i = 1, 2, 3$), when the orthotropic plate rests on the HWEF and IHWEF. When α_i and β_i change together, the effects of the inhomogeneity on the NDFs of the orthotropic plate rests on the HWEF and IHWEF vary with the change in the ratio a/h, but the inhomogeneity effect decreases compared to the case of $\alpha_i = 0$ and $\beta_i = 1$ ($i = 1, 2, 3$).

When the values of the NDFs of HORP on the IHWEF are compared with the non-dimensional frequency values of the plates resting on the HWEF, the influence of the inhomogeneity of WEF on the values of the NDF increases from 0.54% to 13.68%, due to an increase in the ratio a/h.

As the values of the frequencies of HORP on the IHVEF are compared with the NDF values of the plates resting on the HVEF, the influence of the inhomogeneity of viscoelastic foundation on the values of the NDF changes between (−7.59%) and (2.57%) due to an increase in the ratio a/h from 25 to 100.

While the influences of HWEF and IHWEF on the NDFs increase, the effects of HVEF and IHVEF on the NDFs change irregularly in the HORP and IHORPs, as the ratio a/h increases. It is noted that the influence of heterogeneity of elastic and viscoelastic foundations enhances the influence of foundations on the NDFs.

If the inhomogeneity function is a linear function, the numerical results and responses for $\beta_1 = 0$ and $\beta_1 = 1$ become the same with all $\alpha_i (i = 1, 2, 3)$, since the integral with the coefficient β_3 is equal to zero.

It is seen from Tables 7 and 8, and Figures 5–7 that the values of the NDFs of HORPs and IHORPs-linear profiles with and without elastic foundations increase, as the ratio, a/b increases. As the inhomogeneity linearly change along the x_1, x_2 and x_3 directions, the effect of the inhomogeneity on the NDFs of the unconstrained ($\delta_j = 0$, $j = 1, 2$; $K_{01} = K_{02} = 0$) plates is significantly and remains constant. For example, the greatest effect are; 41.4% which occurs at $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ (see, Table 7); (−18.36%) which occurs in $\alpha_i = 0$, $\beta_1 = 1$ (see, Figure 5); (−29.3%) which occurs in $\alpha_i = 0$, $\beta_1 = \beta_2 = 1$ (see, Table 8), since a/b increases from 0.5 to 2.0.

If the inhomogeneity changes linearly along the x_1, x_2 and x_3 directions, the greatest effect of the inhomogeneity on the NDFs of the IHORPs resting on the HWEF ($\delta_1 = \delta_2 = 0$, $K_{01} = 5 \times 10^6$, $K_{02} = 0$) occurs in $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ and decreases

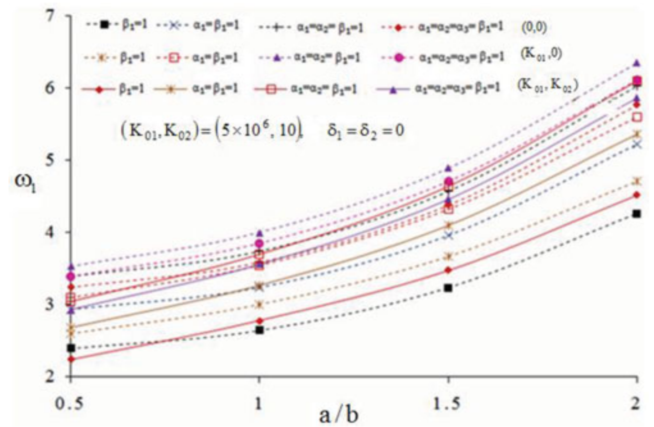

Figure 5. Distribution of NDFs for the HORPs and IHORPs with and without the HWEF and HVEF against the a/b (a/h = 50, b = 1 m).

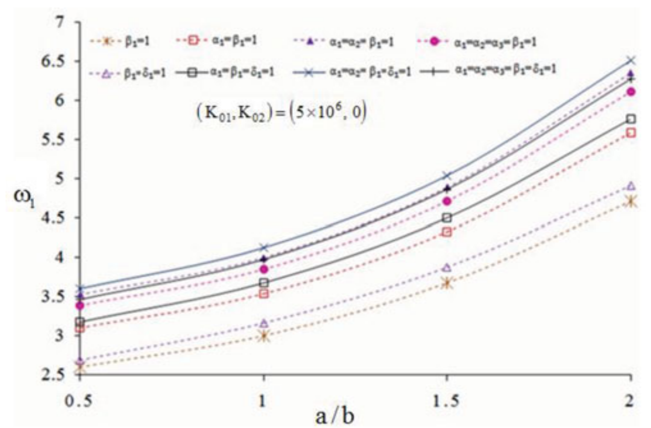

Figure 6. Distribution of NDFs of the HORPs and IHORPs resting on the HWEF and IHWEF and elastic foundations against the a/b (a/h = 50, b = 1 m).

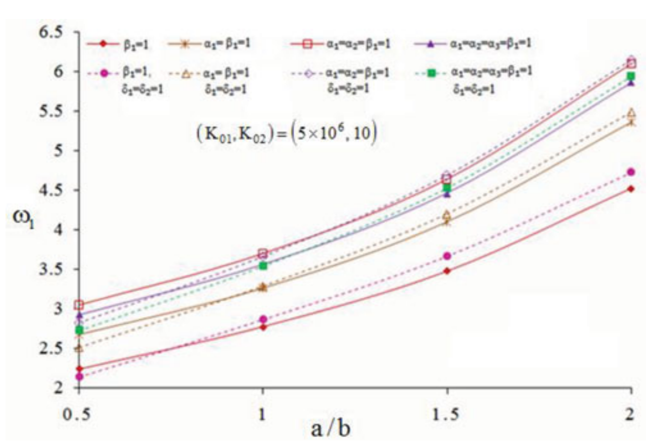

Figure 7. Distribution of NDFs of the HORPs and IHORPs resting on the HVEF and IHVEF and elastic foundations against the a/b (a/h = 50, b = 1 m).

Table 7. Distribution of NDFs of HORPs and IHORPs resting on the elastic and viscoelastic foundations against the a/b (a/h = 50, b = 1 m).

	ω_1			
	$\alpha_i = \beta_i = 0$ (i = 1, 2, 3) $\delta_j = 0$ (i = 1, 2)	$\alpha_1 = 1,\ \alpha_2 = \alpha_3 = 0$ $\beta_i = \delta_j = 0$	$\alpha_1 = \alpha_2 = 1,\ \alpha_3 = 0$ $\beta_i = \delta_j = 0$	$\alpha_i = 1$ $\beta_i = \delta_j = 0$
a/b	$(K_{01}, K_{02}) = (0, 0)$			
0.5	2.931	3.590	4.145	3.969
1.0	3.237	3.964	4.577	4.383
1.5	3.957	4.847	5.596	5.358
2.0	5.216	6.388	7.376	7.062
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
0.5	3.178	3.795	4.324	4.155
1.0	3.674	4.328	4.896	4.715
1.5	4.493	5.293	5.987	5.765
2.0	5.766	6.844	7.775	7.478
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
0.5	2.584	3.085	3.515	3.378
1.0	3.278	3.862	4.368	4.206
1.5	4.152	4.892	5.533	5.328
2.0	5.428	6.444	7.320	7.040
	$\alpha_i = \beta_i = \delta_1 = 0$	$\alpha_1 = 1,\ \alpha_2 = \alpha_3 = 0$ $\beta_i = \delta_1 = 0$	$\alpha_1 = \alpha_2 = 1,$ $\alpha_3 = \beta_i = \delta_1 = 0$	$\alpha_i = 1,$ $\beta_i = \delta_1 = 0$
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
0.5	3.295	3.893	4.410	4.245
1.0	3.874	4.500	5.048	4.872
1.5	4.739	5.503	6.174	5.959
2.0	6.022	7.062	7.967	7.677
	$\alpha_i = \beta_i = 0$ $\delta_1 = \delta_2 = 1$	$\alpha_1 = \delta_1 = \delta_2 = 1,$ $\alpha_2 = \alpha_3 = \beta_i = 0$	$\alpha_1 = \alpha_2 = 1,\ \alpha_3 = 0$ $\beta_i = 0,\ \delta_1 = \delta_2 = 1$	$\alpha_i = 1,\ \beta_i = 0,$ $\delta_1 = \delta_2 = 1$
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
0.5	2.394	2.802	3.158	3.044
1.0	3.304	3.792	4.224	4.085
1.5	4.292	4.925	5.486	5.305
2.0	5.592	6.488	7.275	7.022

from 36.06% to 34.84%, whereas, as the plates resting on the IHWEF ($\delta_1 = 1,\ \delta_2 = 0,\ K_{01} = 5 \times 10^6,\ K_{02} = 0$), the greatest influence of inhomogeneity occurs in $\alpha_1 = \alpha_2 = 1$, $\beta_i = 0$ and decreases from 33.84% to 32.3% (see, Table 7), since a/b increases from 0.5 to 2.0.

As the HVEF effect is taken into account, the values of the NDF are reduced compared to the case without an elastic foundation, whereas, the effects of inhomogeneity on the values of NDF vary with the change of the ratio, a/b. The greatest effect of the inhomogeneity on the NDFs for IHORPs resting on the HVEF (i.e., $\delta_1 = \delta_2 = 0,\ K_{01} = 5 \times 10^6,\ K_{02} = 10$) occurs at $\alpha_1 = \alpha_2 = 1,\ \beta_i = 0$ and decreases from 36.03% to 34.86% (see, Table 7); the greatest effect occurs at $\alpha_1 = \alpha_2 = \beta_1 = 1$ and decreases from 17.96% to12.25% which occurs at (see, Figure 5); the greatest effect changes between (−22.41%) and (−27.19%) which occurs at $\alpha_i = 0,\ \beta_1 = \beta_2 = 1$ (see, Table 8), whereas, the plates resting on the IHVEF ($\delta_1 = \delta_2 = 1,\ K_{01} = 5 \times 10^6,\ K_{02} = 10$) the greatest effects inhomogeneity occurs in $\alpha_1 = \alpha_2 = 1,\ \beta_i = 0$ and increase from 31.91% to 30.1% (see, Table 7); the greatest effect changes between 28.13% and 9.09% which occurs at $\alpha_1 = \alpha_2 = \beta_1 = 1$ (see, Fig. 7); the greatest effect changes between (−18.17%) and (−25.38%) which occurs at $\alpha_i = 0,\ \beta_1 = \beta_2 = 1$ (see, Table 7), since a/b increases from 0.5 to 2.0.

As can be seen from Tables 7 and 8, and Figures 5–7 that, the effects of inhomogeneity on the values of the frequencies of plates resting on the inhomogeneous viscoelastic foundation are greater than the plates resting on the homogeneous viscoelastic foundation and the difference varies from 2% to 4%. In addition, the effect of the inhomogeneity on the frequency does not depend on the change in the ratio a/b, and remains approximately the same (−29.3%) at $\alpha_i = 0$ and $\beta_i = 1$ (i = 1, 2, 3), when the orthotropic plate rests on the homogeneous and inhomogeneous Winkler elastic foundations. When α_i and β_i change together, the effect of the inhomogeneity on the frequencies of the orthotropic plate rests on the homogeneous and inhomogeneous Winkler elastic foundations varies with the change in the ratio a/b, but the inhomogeneity effect decreases compared to the case of $\alpha_i = 0$ and $\beta_i = 1$ (i = 1, 2, 3).

The effect of material inhomogeneity on the frequency values increases due to an increase in the ratio a / b, when taking into account the effects of HVEF and IHVEF. This effect varies from 2% to 5% compared to the frequencies of the plates located on HWEF and IHWEF, whereas varies from 4% to 10% for the plates resting on IHVEF, since the ratio a/b increases from 0.5 to 2.

If the values of the frequencies of HORP on the IHWEF are compared with the frequency values of the plates resting on the

Table 8. Distribution of NDFs for HORPs and IHORPs resting on the HWEF, IHWEF, HVEF and IHVEF and elastic foundations against the a/b (a/h = 50, b = 1 m).

	ω_1			
	$\alpha_i = \delta_j = \beta_3 = 0$, $\beta_1 = \beta_2 = 1$	$\alpha_1 = \beta_1 = \beta_2 = 1$, $\alpha_2 = \alpha_3 = \beta_3 = \delta_j = 0$	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\alpha_3 = \beta_3 = \delta_j = 0$	$\alpha_i = \beta_1 = \beta_2 = 1$, $\beta_3 = \delta_j = 0$
a/b	$(K_{01}, K_{02}) = (0, 0)$			
0.5	2.073	2.539	2.931	2.806
1.0	2.289	2.803	3.237	3.099
1.5	2.798	3.427	3.957	3.789
2.0	3.688	4.517	5.216	4.994
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
0.5	2.248	2.683	3.057	2.938
1.0	2.599	3.061	3.462	3.334
1.5	3.177	3.743	4.234	4.077
2.0	4.077	4.840	5.498	5.287
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
0.5	2.005	2.394	2.728	2.621
1.0	2.446	2.882	3.260	3.139
1.5	3.050	3.593	4.064	3.913
2.0	3.952	4.692	5.329	5.126
	$\alpha_i = \beta_3 = \delta_2 = 0$, $\beta_1 = \beta_2 = \delta_1 = 1$	$\alpha_2 = \alpha_3 = \beta_3 = \delta_2 = 0$, $\alpha_1 = \beta_1 = \beta_2 = \delta_1 = 1$	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\alpha_3 = \beta_3 = \delta_2 = 0, \delta_1 = 1$	$\beta_1 = \beta_2 = \alpha_i = 1$, $\beta_3 = \delta_2 = 0, \delta_1 = 1$
a/b	$(K_{01}, K_{02}) = (5 \times 10^6, 0)$			
0.5	2.330	2.753	3.119	3.002
1.0	2.739	3.182	3.570	3.445
1.5	3.351	3.891	4.365	4.213
2.0	4.258	4.993	5.633	5.428
	$\alpha_i = \beta_3 = 0$, $\beta_1 = \beta_2 = \delta_j = 1$	$\alpha_1 = \beta_1 = \beta_2 = \delta_j = 1$, $\alpha_2 = \alpha_3 = \beta_3 = 0$	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\alpha_3 = \beta_3 = 0, \delta_j = 1$	$\alpha_i = \delta_j = 1$, $\beta_1 = \beta_2 = 1, \beta_3 = 0$
a/h	$(K_{01}, K_{02}) = (5 \times 10^6, 10)$			
0.5	1.959	2.293	2.584	2.491
1.0	2.564	2.942	3.278	3.170
1.5	3.249	3.728	4.152	4.016
2.0	4.173	4.841	5.428	5.240

HWEF, the influence of the inhomogeneity of WEF on the values of the frequency changes between 3.68% and 5.48% due to an increase in the ratio a/b.

When the values of the frequencies of HORP on the IHVEF are compared with the frequency values of the plates resting on the HVEF, the influence of the inhomogeneity of viscoelastic foundation on the values of the frequency changes between (−7.35%) and 3.02% due to an increase in the ratio a/b from 0.25 to 2.

The effects of HVEF and IHVEF on the NDFs change irregularly in the HORP and IHORPs, whereas, the influences of HWEF and IHWEF on the NDFs increase, as the ratio a/b increases.

6. Conclusions

The paper developed the closed-form solution for the free vibration problem of IHORPs resting on the IHVEF. The Young's moduli and density of the orthotropic plate vary continuously with respect to the three spatial coordinates, while the characteristics of the viscoelastic foundation vary depending on the in-plane coordinates. The relevant motion equation is obtained using the CPT and solved using method of separation of variables and then Bubnov–Galerkin method. The frequencies obtained in this study were confirmed in comparison with studies in the literature. The influences of inhomogeneity of the orthotropic materials, the inhomogeneity of viscoelastic and elastic foundations on the NDFs of the plates are studied in detail.

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