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On numerical solution of the time-fractional diffusion-wave equation with the fictitious time integration method

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Abstract. In this work we offer a robust numerical algorithm based on the Lie group to solve the time-fractional diffusion-wave (TFDW) equation. Firstly, we use a fictitious time variable ξ to convert the related variable $u(x,t)$ into a new space with one extra dimension. Then by using a composition of the group preserving scheme (GPS) and a semi-discretization of new variable, we approximate the solutions of the problem. Finally, various numerical experiments are performed to illustrate the power and accuracy of the given method.

1 Introduction

The fractional calculus is one of the most applicable and significant extensions of the classical derivatives. Utilizing the models based on fractional derivatives in multiple subdivisions of science and engineering [1,2] is one of the main research fields of many related scientific problems. The main advantage of fractional models in comparison with classical models is their precision in the model explanation and this was our main reason in investigating the TFDW equation:

$$\begin{aligned}
{}^{C}\mathcal{D}^{\alpha}_{0^+,t}u(x,t)+\frac{\partial u}{\partial t}(x,t)&=\frac{\partial^2 u}{\partial x^2}(x,t)+\mathcal{P}(x,t),\quad (\mathbf{x},t)\in\Omega\subset\mathbb{R}^2,\\
u(x,0)&=g_1(x),\quad x\in\Omega_{\mathbf{x}},\\
u(x,T)&=g_2(x),\quad x\in\Omega_{\mathbf{x}},\\
u(0,t)&=h_1(t),\quad t\in\Omega_t,\\
u(b,t)&=h_2(t),\quad t\in\Omega_t,
\end{aligned}\tag{1}$$

where Ω_t and $\Omega_{\mathbf{x}}$ are boundaries of $\Omega:=\{(x,t): a\le x\le b,\ 0\le t\le T\}$ in t and x, respectively. Moreover, ${}^{C}\mathcal{D}^{\alpha}_{0^+,t}u(x,t)$ is the Caputo fractional derivative with order of α defined as

$${}^{C}\mathcal{D}^{\alpha}_{0^+,t}u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t(t-\sigma)^{m-\alpha-1}\frac{\partial^m u(x,\sigma)}{\partial\sigma^m}\mathrm{d}\sigma, & m-1<\alpha<m,\\[2mm] \dfrac{\partial u(x,t)}{\partial t^m}, & \alpha=m,\end{cases}\tag{2}$$

where $1<\alpha\le 2$. The fractional diffusion-wave equation is investigated by many authors in the literature [3–10]. Because of the importance of the fractional models, some kinds of fractional derivatives are introduced until now. Some of the most important definitions are the Caputo derivative, the Riemann-Liouville derivative [11,12], the Caputo-Fabrizio derivative [13–16] and Atangana-Baleanu derivative [17–19].

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Moreover, in the last decades, many researchers have paid attentions to solve the fractional partial differential equations (FPDEs). Among few analytical methods applied for FPDEs it can be mentioned the fractional sub-equation method [20–25], Lie-symmetry method [26–31], (G'/G)-expansion method [32,33] and some other methods [34–36]. On the other hand, there are some approximation methods for FPDEs, *e.g.*, geometric approaches [37–39], finite-difference scheme [40–43], Tau method [44–46], the unified method [48]. Also, some soliton solutions, rational solutions and lump solutions are applied for this type of equations [49–51].

2 The fictitious time integration method (FTIM)

The FTIM is a novel integration method for partial differential equations which was introduced and extended by Liu and other authors [52–54]. The construction of finding the approximate solution of eq. (1) by using the TFIM is discussed as follows.

With regard to the Caputo fractional derivative (2) and $0 \leq \alpha < 1$ for eq. (1) we have

$$\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{u_{\sigma\sigma}(x,\sigma)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + u_t - u_x x - \mathcal{P}(x,t) = 0. \tag{3}$$

Now, we multiply eq. (3) by the parameter ρ as a fictitious damping coefficient which can help us to increase the stability of the method:

$$\frac{\rho}{\Gamma(2-\alpha)}\int_0^t \frac{u_{\sigma\sigma}(x,\sigma)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \rho u_t - \rho u_{xx} - \rho\mathcal{P}(x,t) = 0. \tag{4}$$

Imposing the following transformation in eq. (4):

$$\omega(x,t,\xi) = (1+\xi)^l u(x,t), \quad 0 < l \leq 1, \tag{5}$$

results in

$$\frac{\rho}{(1+\xi)^l}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\xi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \omega_t(x,t,\xi) - \omega_{xx}(x,t,\xi)\right] - \rho\mathcal{P}(x,t) = 0. \tag{6}$$

Considering

$$\frac{\partial\omega}{\partial\xi} = l(1+\xi)^{l-1}u(x,t), \tag{7}$$

eq. (6), can be written as

$$\frac{\partial\omega}{\partial\xi} = \frac{\rho}{(1+\xi)^l}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\xi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \omega_t - \omega_{xx}\right] - \rho\mathcal{P}(x,t) + l(1+\xi)^{l-1}u. \tag{8}$$

Equation (8) can be converted to a new type of functional PDE for ω, by setting $u = \omega/(1+\xi)^l$

$$\frac{\partial\omega}{\partial\xi} = \frac{\rho}{(1+\xi)^l}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\xi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \omega_t - \omega_{xx}\right] - \rho\mathcal{P}(x,t) + \frac{k\omega}{1+\xi}\,. \tag{9}$$

Using

$$\frac{\partial}{\partial\xi}\left(\frac{\omega}{(1+\xi)^l}\right) = \frac{\omega_\xi}{(1+\xi)^l} - \frac{l\omega}{(1+\xi)^{1+l}}\,, \tag{10}$$

and multiplying the integrating factor $1/(1+\xi)^l$ on both sides of eq. (9), one obtains

$$\frac{\partial}{\partial\xi}\left(\frac{\omega}{(1+\xi)^l}\right) = \frac{\rho}{(1+\xi)^{2l}}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\xi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \omega_t - \omega_{xx}\right] - \frac{\rho\mathcal{P}(x,t)}{(1+\xi)^l}\,. \tag{11}$$

Using again the transformation $u = \frac{\omega}{(1+\xi)^l}$, we get

$$u_\xi = \frac{\rho}{(1+\xi)^l}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{u_{ss}(x,\sigma,\xi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + u_t(x,t,\xi) - u_{xx}(x,t,\xi) - \mathcal{P}(x,t)\right]. \tag{12}$$

We have to emphsize that ξ plays the fictitious coordinate role which enable us to embed eq. (3) into a new PDE form in a space called 3-space, denoted R^3. Moreover, by an initial guess $u(x,t,0)$, for all $\xi \geq 0$, $u = u(x,t,\xi)$ is an undetermined function related to the conditions in eq. (1).

Suppose $u_i^j(\xi) := u(x_i, t_j, \xi)$ and $\mathcal{P}_i^j := \mathcal{P}(x_i, t_j)$ as the discrete values of u and $\mathcal{P}$ at (x_i, t_j). By using these notations, eq. (12) is converted to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u_i^j(\xi) = \frac{\rho}{(1+\xi)^l}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^{t_j}\frac{u_{\sigma\sigma}(x_i,\sigma,\xi)}{(t_j-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \frac{u_i^{j+1}(\xi)-u_i^j(\xi)}{\Delta t} - \frac{u_{i+1}^j(\xi)-2u_i^j(\xi)+u_{i-1}^j(\xi)}{\Delta x^2} - \mathcal{P}_i^j\right]. \tag{13}$$

Full discretization of the above equation is needed to calculate the approximation of the following integral:

$$\int_0^{t_j}\frac{u_{\sigma\sigma}(x_i,\sigma,\xi)}{(t_j-\sigma)^{\alpha-1}}\mathrm{d}\sigma \approx \frac{u(x_i,t_3,\xi)-2u(x_i,t_2,\xi)+u(x_i,t_1,\xi)}{\Delta t^2(t_j-t_1)^{\alpha-1}} + \sum_{l=2}^{j-1}\frac{u(x_i,t_{l+1},\xi)-2u(x_i,t_l,\xi)+u(x_i,t_{l-1},\xi)}{\Delta t^2(t_j-t_l)^{\alpha-1}}, \tag{14}$$

where $\Delta x = \frac{b-a}{n_1}$, $\Delta t = \frac{T}{n_2}$, $x_i = a + i\Delta x$ and $t_j = j\Delta t$.

Considerig $\mathbf{u} = (u_1^1, u_1^2, \ldots, u_{n_1}^{n_2})^T$, eq. (13) can be written as

$$\mathbf{u}' = \varpi(\mathbf{u},\xi), \quad \mathbf{u}\in\mathbb{R}^N, \quad \xi\in\mathbb{R}, \tag{15}$$

where Θ denotes the ij-component of right-hand side of eq. (13), and the total number of utilized grid points in Ω is denoted by $N = n_1 \times n_2$. Now, we can use an ODE integrator to solve eq. (15). We use the GPS [39], to approximate the solution of eq. (15).

3 The GPS for the extracted system of ODEs

Consider the dynamical system (15), corresponding to the TFDW equation. Let

$$\mathbf{n} := \frac{\mathbf{u}}{\|\mathbf{u}\|}, \tag{16}$$

where $\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} > 0$ is the Euclidean norm of $\mathbf{u}$. From eqs. (15) and (16) we have

$$\dot{\mathbf{n}} := \frac{\varpi(\mathbf{u},\xi)}{\|\mathbf{u}\|} - \left(\frac{\varpi(\mathbf{u},\xi)}{\|\mathbf{u}\|}\cdot\mathbf{n}\right)\mathbf{n}. \tag{17}$$

Moreover, utilizing eqs. (15) and (16) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\|\mathbf{u}\| = \frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{\mathbf{u}\cdot\mathbf{u}} = \dot{\mathbf{u}}\cdot\mathbf{n} = \varpi(\mathbf{u},\xi)\cdot\mathbf{n}. \tag{18}$$

Using eqs. (17) and (18), we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{bmatrix}\mathbf{u}\\ \|\mathbf{u}\|\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{N\times N} & \frac{\varpi(\mathbf{u},\xi)}{\|\mathbf{u}\|}\\ \frac{\varpi^T(\mathbf{u},\xi)}{\|\mathbf{u}\|} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \|\mathbf{u}\|\end{bmatrix}. \tag{19}$$

Clearly, the primary equation (15) can be derived from the first row equation in (19), but from the excess of the secondary equation we can obtain a Minkowskian construction of the added state variables of $\mathbf{U} := (\mathbf{u}^T, \|\mathbf{u}\|)^T \in \mathcal{M}^{N+1}(\mathbb{R})$, which generates an inner product on $\mathbb{R}^{N+1}$ in the following form:

$$\langle U, E\rangle = U^T \Pi E = u_1e_1 + \ldots + u_le_l + u_{N+1}e_{N+1}, \tag{20}$$

where

$$\Pi = \begin{bmatrix} I_N & \mathbf{0}_{N\times N}\\ \mathbf{0}_{N\times N} & -1\end{bmatrix}, \tag{21}$$

and

$$U^T = (u_1, \dots, u_N, u_{N+1})^T, \qquad E^T = (e_1, \dots, e_N, e_{N+1})^T.$$

In the Minkowskian construction, the augmented variable $\mathbf{U}$ is a null vector and from the Lorentz inner product, satisfies the cone condition

$$\langle \mathbf{U}, \mathbf{U} \rangle = \mathbf{U}^T \Lambda \mathbf{U} = 0. \tag{22}$$

Using recently added variable, for eq. (19) we can derive

$$\mathbf{U}' = \Upsilon \mathbf{U}, \qquad \mathbf{U} \in \mathcal{H}_{N,1}(0), \tag{23}$$

where

$$\Upsilon := \begin{bmatrix} \mathbf{0}_{N\times N} & \dfrac{\varpi(\mathbf{u},\xi)}{\|\mathbf{u}\|} \\ \dfrac{\varpi^T(\mathbf{u},\xi)}{\|\mathbf{u}\|} & 0 \end{bmatrix}. \tag{24}$$

Definition 1. *Let B be a real square matrix. Then*

$$SN\_Sym_N(\mathcal{M}^N(\mathbb{R})) = \{B : B^T \Pi + \Pi B = 0\}$$

is the definition of the skew symmetric matrices's space in Minkowskian construction.

In eq. (23), we can write $\Upsilon \in SN\_Sym_{N+1}(\mathcal{M}^{N+1}(\mathbb{R}))$.

We can define global linear group for a group of real square matrices:

$$QL_k(\mathbb{R}) = \{Q \in M_{N,N} : \det(Q) \neq 0\}.$$

Indeed, there is a closed subgroup

$$O(N,1) = \{Q \in QL_{N+1}(\mathbb{R}) : Q^T \Pi Q = \Pi\}$$

and

$$Q \in O(N,1) \Longleftrightarrow \langle Q\mathbf{x}, Q\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle, \langle \mathbf{x}, \mathbf{y}\rangle \in \mathbb{R}^{N+1}.$$

So $O(N,1)$ includes the Lorentzian isometries of $\mathbb{R}^{N+1}$. Indeed, for $Q \in O(N,1)$ we can write $\det(Q) = \pm 1$.

There is an applicable subgroup of $O(N,1)$ as

$$SO_0(N,1) = \{Q \in O(N,1) : \det(Q) = 1\},$$

which is known as the proper orthochronous Lorentz group. For $so(N,1)$ as a Lie algebra of $SO_0(N,1)$, an exponential mapping can generate the Lie group as follows:

$$\exp : so(N,1) \to SO_0(N,1). \tag{25}$$

Indeed, we have $so(N,1) = SN\_Sym_{N+1}(\mathcal{M}^{N+1}(\mathbb{R}))$. Therefore, in eq. (23), $\Upsilon \in so(N,1)$ and the related discretized $Q \in SO_0(N,1)$, the obtained eq. (25) satisfies

$$Q^T \Lambda Q = \Lambda, \qquad \det(Q) = 1. \tag{26}$$

Therefore, we can propose our numerical method as follows:

$$\mathbf{U}_{s+1} = Q(s)\mathbf{U}_s, \tag{27}$$

where $\mathbf{U}_s$ describes the numerical amount of $\mathbf{U}$ at a fictitious time ξ_s, and the distincted group component $Q(s)$ is acquired from the Cayley transform:

$$\begin{aligned} Q(s) &= [I_N - \Delta\xi\,\Upsilon(s)]^{-1}[I_N + \Delta\xi\,\Upsilon(s)] \\ &= \begin{bmatrix} I_N + \dfrac{2\Delta\xi^2 \varpi_s \varpi_s^T}{\|\mathbf{u}_s\|^2 - \Delta\xi^2 \|\varpi_s\|^2} & \dfrac{2\Delta\xi\,\|\mathbf{u}_s\|\varpi_s}{\|\mathbf{u}_s\|^2 - \Delta\xi^2\|\varpi_s\|^2} \\ \dfrac{2\Delta\xi\,\|\mathbf{u}_s\|\varpi_s^T}{\|\mathbf{u}_s\|^2 - \Delta\xi^2\|\varpi_s\|^2} & \dfrac{\|\mathbf{u}_s\|^2 + \Delta\xi^2\|\varpi_s\|^2}{\|\mathbf{u}_s\|^2 - \Delta\xi^2\|\varpi_s\|^2} \end{bmatrix}. \end{aligned} \tag{28}$$

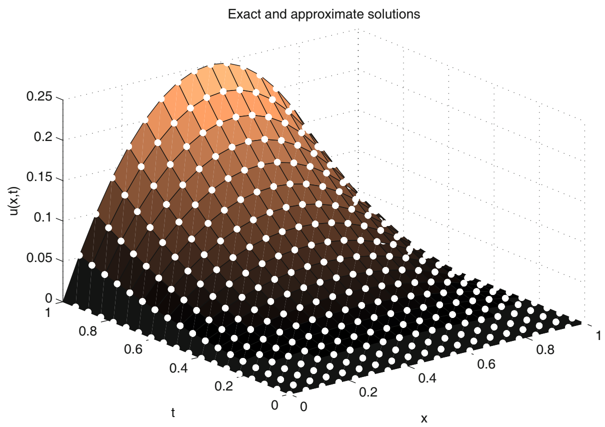

Fig. 1. Plot of the numerical and exact solutions for example 1.

Substituting eq. (28) into eq. (27) and taking its first row, we get

$$\mathbf{u}_{s+1} = \mathbf{u}_s + 2\Delta\xi \frac{\|\mathbf{u}_s\|^2 + \Delta\xi\, \varpi_s \cdot \mathbf{u}_s}{\|\mathbf{u}_s\|^2 - \Delta\xi^2 \|\varpi_s\|^2} \varpi_s = \mathbf{u}_s + \eta_s \varpi_s. \tag{29}$$

Now, the geometric approach GPS can be utilized, by selecting the initial guess of $u_i^j(0)$, to integrate eq. (15) from $\xi = 0$ to ξ_f. We can set a numerical integration stopping criterion as follows:

$$\sqrt{\sum_{i,j=1}^{n_1,n_2} [\mathbf{u}_i^j(s+1) - \mathbf{u}_i^j(s)]^2} \leq \varepsilon, \tag{30}$$

where ε is the convergence criterion. Obviously, the final solution $\mathbf{u}$ can be obtained from

$$\mathbf{u}_i^j = \frac{\mathbf{u}_i^j(\xi_0)}{(1+\xi_0)^l}, \tag{31}$$

where $\xi_0(\leq \xi_f)$ satisfies eq. (31).

4 Numerical examples

Now, we use some numerical examples to display the efficiency of our method by the FTIM to solve the time fractional Diffusion-Wave equations.

Example 1

In order to show the ability of our method we consider the fractional TFDW equation (1) with $\alpha = 1.9$ and

$$\mathcal{P}(x,t) = \frac{2x(1-x)}{\Gamma(3-\alpha)} t^{\alpha} + 2tx(1-x) + 2t^2,$$

the exact solution of eq. (1) for this example is $u(x,t) = t^2 x(1-x)$. During the semi-discretization procedure, we utilize the number of knots $n_1 = 19$ and $n_2 = 19$ in each coordinates of space and time, respectively. The considered domain in this example is $\Omega = [0,1] \times [0,1]$. The boundary conditions in Ω_x are considered to be homogeneous. The initial guess and stepsize for ξ are considered as $u_i^j(0) = 1e-6$ and $\Delta\xi = 1e-8$. In order to control the convergency and stability of FTIM for the current example, we suppose $\rho = 0.21$ and $l = 0.1$. Figure 1 illustrates the exact (surface) and the approximate (points) solutions derived by FTIM which demonstrates the results of our method are extremely close to the exact solution. Numerical errors of the FTIM are plotted in fig. 2. This low error proves that our method is suitable and powerful for solving the time fractional diffusion-wave equation.

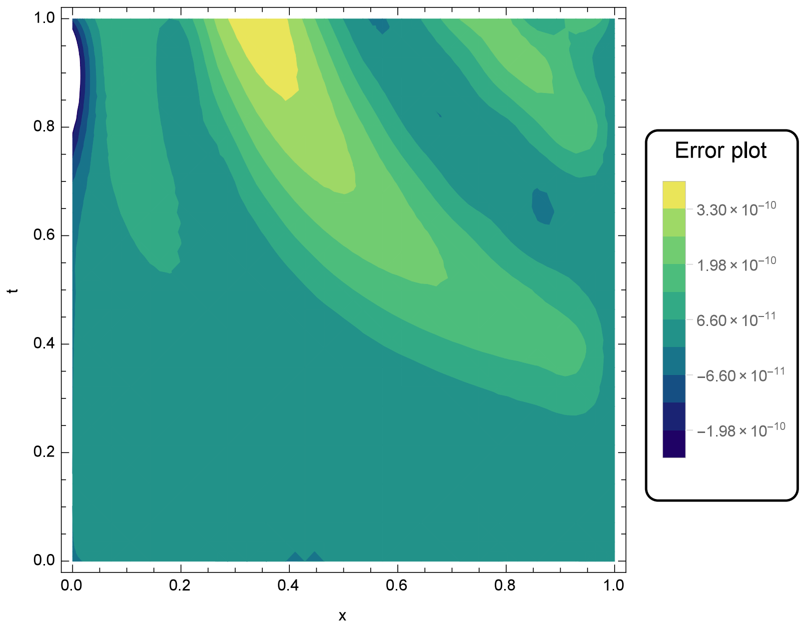

Fig. 2. Error contourplot for example 1.

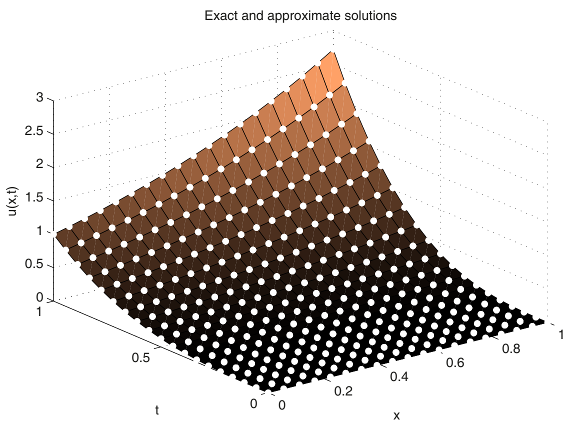

Fig. 3. Plot of the numerical and exact solutions for example 2.

Example 2

As a second example, we consider eq. (1) with $\alpha = 1.5$ and

$$\mathcal{P}(x,t) = \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}e^{x} + 3t^2 e^{-t^3} e^{x}.$$

The number of meshes for this example are assumed as $n_1 = n_2 = 19$. The initial guess is $u_i^j(0) = 0.01$ and the stepsize of GPS is equal to example 1. The domain of the problem and the Dirichlet boundary conditions are supposed as in example 1. Moreover, the auxiliary parameters are chosen as $\rho = 0.2$, $l = 0.11$, and the related stepsize is $\Delta\xi = 1e-8$. The exact solution of the TFDW equation for this example is $u(x,t) = e^x t^3$. The exact solution and the obtained approximate solutions are plotted in fig. 3. As in example 1, the graph is solution and grids are the numerical solution obtained by using the proposed method of eq. (1) for this example. Figure 4 which contains the low error obtained by our method proves the accuracy and power of the proposed method. In fact, the paremeters ρ and l, which are selected, precisely, play an important role.

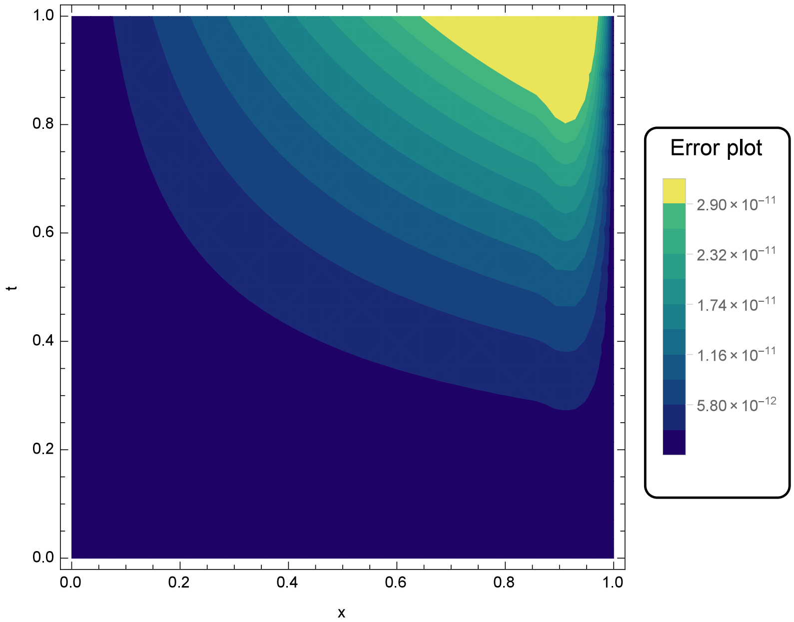

Fig. 4. Error contourplot for example 2.

5 Conclusion

In this work we have converted the time-fractional diffusion-wave equation into a new type of functional partial differential equation in new space with one extra dimension by proposing a fictitious coordinate. After the semi-discretization of the main equation, the group preserving scheme as a geometric approach is applied to solve the system of first-order ordinary differential equations. Some numerical examples were implemented, which demonstrate that the presented scheme is reliable and applicable to the numerical solutions of the time-fractional diffusion-wave equation.

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