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Theory and application for the system of fractional Burger equations with Mittag leffler kernel

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ABSTRACT

In this work, the system of fractional Burger differential equations are presented as a new fractional model for Atangana–Baleanu fractional derivative with Mittag leffler kernel. The approximate consequences are analysed by applying an recurrent process. The existence and uniquenes of solution for this system is discussed. In order to appear the effects of several parameter and variables on the movement, the approximate results are showed in graphics and are compared with obtained solutions for two different derivative in tables.

1. Introduction

In the last few years, there has been considerable interests and significant theoretical developments in fractional calculus used in many fields and in fractional differential equations and its applications [1–13]. Hossein and Eskandar solved variable-order time fractional Burgers equation via a new computational method in [14], F. Gao and X-J. Yang used the local fractional Euler's method to analyse approximate solution of the local fractional heat-relaxation equation in [15], Yang et al. obtained the solutions for local fractional KdV equation in [16], in [17]; is found the solutions of fractional two-dimensional Burgers equations and Zhang et al. used the series expansion method with local fractional derivative to find the solutions of transport equations in [18]. Many more researches related to fractional derivatives can be saw in [19–40].

In this study, we apply the fractional homotopy perturbation transform method (FHPTM) to find series solution for a system of fractional equations. The FHPTM is combination of HPM and Laplace transform process [26–30]. Besides, this method gives the solution in the form of a converging series. An iterative process is composed for the shape of the infinite series solution. In [31], is analysed the numerical solution for fractional RLW equation [31] by using this method and in [32], this method is used to found the series solutions of logaritmik KdV equation.

In this work, we analysed fractional system of Burger equation (FBEs). This system of equation has usuallly performed in different fields of science and engineering such as physics, plasma physics, optics, quantum mechanics and superconductivity [33] and is given by,

$$D_\tau^\alpha p(\varkappa,\tau) - p_{\varkappa\varkappa}(\varkappa,\tau) - 2p(\varkappa,\tau)p_{\varkappa\varkappa}(\varkappa,\tau) + (p(\varkappa,\tau)q(\varkappa,\tau))_\varkappa = 0,$$

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$$D_\tau^\alpha q(\varkappa,\tau) - q_{\varkappa\varkappa}(\varkappa,\tau) + 2q(\varkappa,\tau)q_{\varkappa\varkappa}(\varkappa,\tau) - (p(\varkappa,\tau)q(\varkappa,\tau))_\varkappa = 0,$$
$$\varkappa \in R,\ \ \tau > 0,\ \ 0 < \alpha \le 1, \tag{1.1}$$

Some fractional derivatives contains singular kernels. Two of them are Riemann and Caputo and they have own restriction due to their singular kernels. However, recently some fractional operators such as Atangana–Baleanu (AB) have defeat these restrictions and deficiencies. Especially AB introduced a new fractional derivative with non-local, non-singular and ML kernel and cleared its significant effects [34,35].

We analyse FBEs for the AB fractional derivative with Mittag leffler kernel due to great importance of AB fractional derivative in scientific and engineering field.

The FBEs with AB fractional derivative is gived as

$$\begin{gathered} {}^{ABC}_{a}D_\tau^\alpha p(\varkappa,\tau) - p_{\varkappa\varkappa}(\varkappa,\tau) - 2p(\varkappa,\tau)p_{\varkappa\varkappa}(\varkappa,\tau) + (p(\varkappa,\tau)q(\varkappa,\tau))_\varkappa = 0, \\ {}^{ABC}_{a}D_\tau^\alpha q(\varkappa,\tau) - q_{\varkappa\varkappa}(\varkappa,\tau) + 2q(\varkappa,\tau)q_{\varkappa\varkappa}(\varkappa,\tau) - (p(\varkappa,\tau)q(\varkappa,\tau))_\varkappa = 0, \\ 0 < \alpha \le 1. \end{gathered} \tag{1.2}$$

The main purpose of this article is to analyse FBEs with Mittag leffler kernel and is to research the existence and uniqueness analysis of the solutions by using the fixed-point theorem. Another goal of this work is to compare the numerical results obtained with derivatives other than AB derivative for Eq. (1.1). For this we also obtained the numerical solutions of the FBEs for Caputo–Fabrizio (CF) and Liouville–Caputo (LC) fractional derivative with Mittag leffler kernel by using FHPTM.

In the Section 2 of this study, various basic knowledge concerned to the AB fractional order derivative are defined. In the next section, FBEs with AB fractional derivative are examined and the existence and uniqueness of solutions for these system have been investigated with by using the fixed-point theorem. In the next section, the FHPTM is applied to construct the solutions of the FBEs for AB, CF and LC fractional derivatives with Mittag leffler kernel. Some graphical representations of the solutions are gived to display the accuracy and efficiency of the method, in Section 5. Moreover, some results are pointed out in Section 6.

2. Preliminaries

In this part, we will present the basic definitions and several properties for the AB fractional order derivative [34].

Definition 2.1. When $p \in H^1(\varkappa, y), \alpha \in [0,1],\ y > \varkappa$ and differentiable, the AB fractional order derivative with arbitrary order in case of Caputo is gived as,

$${}^{ABC}_{a}D_\tau^\alpha (p(\tau)) = \frac{B(\alpha)}{1-\alpha}\int_\varkappa^\tau p'(s)E_\alpha\Big[-\frac{\alpha}{1-\alpha}(\tau - s)^\alpha\Big]ds. \tag{2.1}$$

where $B(\alpha)$ provides requirement $B(0) = B(1) = 1$.

Definition 2.2. When $p \in H^1(\varkappa, y), \alpha \in [0,1],\ y > \varkappa$ and non-differentiable, the AB derivative of arbitrary order in case of Riemann–Liouville is gived as,

$${}^{ABR}_{a}D_\tau^\alpha (p(\tau)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\tau}\int_\varkappa^\tau p(s)E_\alpha\Big[-\frac{\alpha}{1-\alpha}(\tau - s)^\alpha\Big]ds. \tag{2.2}$$

Definition 2.3. When $0 < \alpha < 1$, and $p = p(\tau)$, the fractional integral operator of order α is gived as,

$${}^{AB}_{a}I_\tau^\alpha (p(\tau)) = \frac{1-\alpha}{B(\alpha)}p(\tau) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_\varkappa^\tau p(l)(\tau - l)^{\alpha - l}ds. \tag{2.3}$$

3. Analyse of the FBEs with AB fractional derivative

The FBEs are written as: $0 < \alpha < 1$

$$\begin{gathered} {}^{ABC}_{a}D_\tau^\alpha p(\varkappa,\tau) - p_{\varkappa\varkappa}(\varkappa,\tau) - 2p(\varkappa,\tau)p_{\varkappa\varkappa}(\varkappa,\tau) + (p(\varkappa,\tau)q(\varkappa,\tau))_\varkappa = 0, \\ {}^{ABC}_{a}D_\tau^\alpha q(\varkappa,\tau) - q_{\varkappa\varkappa}(\varkappa,\tau) + 2q(\varkappa,\tau)q_{\varkappa\varkappa}(\varkappa,\tau) - (p(\varkappa,\tau)q(\varkappa,\tau))_\varkappa = 0, \end{gathered} \tag{3.1}$$

with the initial condition:

$$p(\varkappa, 0) = \sin\varkappa, \quad q(\varkappa, 0) = \sin\varkappa$$

Using the fractional integral operator produced by AB [34] on Eq. (3.1), we obtain

$$p(\varkappa,\tau)-p(\varkappa,0)=\frac{1-\alpha}{B(\alpha)}K(\varkappa,\tau,p)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}K(\varkappa,l,p)dl,$$
$$q(\varkappa,\tau)-q(\varkappa,0)=\frac{1-\alpha}{B(\alpha)}K(\varkappa,\tau,q)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}K(\varkappa,l,q)dl. \tag{3.2}$$

where

$$K(\varkappa,\tau,p)=p_{\varkappa\varkappa}(\varkappa,\tau)+2p(\varkappa,\tau)p_{\varkappa\varkappa}(\varkappa,\tau)-(p(\varkappa,\tau)q(\varkappa,\tau))_{\varkappa}$$
$$K(\varkappa,\tau,q)=q_{\varkappa\varkappa}(\varkappa,\tau)-2q(\varkappa,\tau)q_{\varkappa\varkappa}(\varkappa,\tau)+(p(\varkappa,\tau)q(\varkappa,\tau))_{\varkappa}.$$

The kernels $K(\varkappa, \tau, p)$ and $K(\varkappa, \tau, q)$ has the Lipschitz state justified that the functions $p(\varkappa, \tau)$ and $q(\varkappa, \tau)$ have upper bound. So,

$$\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\| = \left\|\begin{matrix}(p_{\varkappa\varkappa}-P_{\varkappa\varkappa})+2(p(\varkappa,\tau)p_{\varkappa\varkappa}(\varkappa,\tau)\\ -P(\varkappa,\tau)P_{\varkappa\varkappa}(\varkappa,\tau))-((p(\varkappa,\tau)q(\varkappa,\tau))_{\varkappa}-(P(\varkappa,\tau)Q(\varkappa,\tau))_{\varkappa})\end{matrix}\right\|,$$
$$\|K(\varkappa,\tau,q)-K(\varkappa,\tau,Q)\| = \left\|\begin{matrix}(q_{\varkappa\varkappa}-Q_{\varkappa\varkappa})-2(q(\varkappa,\tau)q_{\varkappa\varkappa}(\varkappa,\tau)\\ -Q(\varkappa,\tau)Q_{\varkappa\varkappa}(\varkappa,\tau))+((p(\varkappa,\tau)q(\varkappa,\tau))_{\varkappa}-(P(\varkappa,\tau)Q(\varkappa,\tau))_{\varkappa})\end{matrix}\right\| \tag{3.3}$$

By apply the triangular inequality of norm on Eq. (3.3),

$$\begin{aligned}\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\| &\le \|p_{\varkappa\varkappa}-P_{\varkappa\varkappa}\|+2\|pp_{\varkappa\varkappa}-PP_{\varkappa\varkappa}\|+\|(pq)_{\varkappa}-(PQ)_{\varkappa}\|,\\ &\le \left\|\frac{\partial^2}{\partial\varkappa^2}(p-P)\right\|+\left\|\frac{\partial^2}{\partial\varkappa^2}(p^2-P^2)\right\|+\left\|\frac{\partial}{\partial\varkappa}(pq-PQ)\right\|\\ &\le \gamma^2\|p-P\|+\delta(a+b)\|p-P\|+\kappa\|p-P\|\\ &\le (\gamma^2+\delta(a+b)+\kappa)\|p-P\|.\\ \|K(\varkappa,\tau,q)-K(\varkappa,\tau,Q)\| &\le \|q_{\varkappa\varkappa}-Q_{\varkappa\varkappa}\|+2\|qq_{\varkappa\varkappa}-QQ_{\varkappa\varkappa}\|+\|(pq)_{\varkappa}-(PQ)_{\varkappa}\|,\\ &\le \left\|\frac{\partial^2}{\partial\varkappa^2}(q-Q)\right\|+\left\|\frac{\partial^2}{\partial\varkappa^2}(q^2-Q^2)\right\|+\left\|\frac{\partial}{\partial\varkappa}(pq-PQ)\right\|\\ &\le \varepsilon^2\|q-Q\|+\zeta(c+d)\|q-Q\|+\eta\|q-Q\|\\ &\le (\varepsilon^2+\zeta(c+d)+\eta)\|q-Q\|.\end{aligned} \tag{3.4}$$

Setting $\Phi=\gamma^2+\delta(a+b)+\kappa$ and $\Psi=\varepsilon^2+\zeta(c+d)+\eta$, where p, q and P, Q are limited functions, therefore we can say $\|p\|\le a$, $\|P\|\le b$, $\|q\|\le c$, $\|Q\|\le d$ and we have

$$\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\| \le \Phi\|p-P\|,$$
$$\|K(\varkappa,\tau,q)-K(\varkappa,\tau,Q)\| \le \Psi\|q-Q\|.$$

Then, the Lipschitz state is justified for the kernels $K(\varkappa, \tau, p)$ and $K(\varkappa, \tau, q)$.

3.1. Existence and uniqueness analysis for solutions

In this part, we will present the existence and uniqueness of the solutions of FBEs for arbitrary order (3.1). From Eq. (3.2), we have

$$p_{n+1}(\varkappa,\tau)=\frac{1-\alpha}{B(\alpha)}K(\varkappa,\tau,p_n)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}K(\varkappa,l,p_n)dl,$$
$$q_{n+1}(\varkappa,\tau)=\frac{1-\alpha}{B(\alpha)}K(\varkappa,\tau,q_n)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}K(\varkappa,l,q_n)dl. \tag{3.5}$$

and $p_0(\varkappa,\tau)=p(\varkappa,0)$, $q_0(\varkappa,\tau)=q(\varkappa,0)$.

The difference of the successive terms is represented as follows

$$\begin{aligned} Y_n(\varkappa,\tau) &= p_n(\varkappa,\tau) - p_{n-1}(\varkappa,\tau) = \frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,p_{n-1}) - K(\varkappa,\tau,p_{n-2})\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\{K(\varkappa,l,p_{n-1}) - K(\varkappa,l,p_{n-2})\}dl, \\ Z_n(\varkappa,\tau) &= q_n(\varkappa,\tau) - q_{n-1}(\varkappa,\tau) = \frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,q_{n-1}) - K(\varkappa,\tau,q_{n-2})\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\{K(\varkappa,l,q_{n-1}) - K(\varkappa,l,q_{n-2})\}dl. \end{aligned} \tag{3.6}$$

where we say that,

$$\begin{aligned} p_n(\varkappa,\tau) &= \sum_{k=0}^{n} Y_k(\varkappa,\tau), \\ q_n(\varkappa,\tau) &= \sum_{k=0}^{n} Z_k(\varkappa,\tau), \end{aligned} \tag{3.7}$$

From Eq. (3.7), we get

$$\begin{aligned} \|Y_n(\varkappa,\tau)\| &= \|p_n(\varkappa,\tau) - p_{n-1}(\varkappa,\tau)\| \\ &= \left\| \begin{array}{c} \frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,p_{n-1}) - K(\varkappa,\tau,p_{n-2})\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\{K(\varkappa,l,p_{n-1}) - K(\varkappa,l,p_{n-2})\}dl \end{array} \right\|, \\ \|Z_n(\varkappa,\tau)\| &= \|q_n(\varkappa,\tau) - q_{n-1}(\varkappa,\tau)\| \\ &= \left\| \begin{array}{c} \frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,q_{n-1}) - K(\varkappa,\tau,q_{n-2})\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\{K(\varkappa,l,q_{n-1}) - K(\varkappa,l,q_{n-2})\}dl \end{array} \right\|. \end{aligned} \tag{3.8}$$

Using the triangular inequality on Eq. (3.8), we have

$$\begin{aligned} \|Y_n(\varkappa,\tau)\| &\le \begin{array}{c} \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,p_{n-1}) - K(\varkappa,\tau,p_{n-2})\| \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\|K(\varkappa,l,p_{n-1}) - K(\varkappa,l,p_{n-2})\|dl \end{array}, \\ \|Z_n(\varkappa,\tau)\| &\le \begin{array}{c} \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,q_{n-1}) - K(\varkappa,\tau,q_{n-2})\| \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\|K(\varkappa,l,q_{n-1}) - K(\varkappa,l,q_{n-2})\|dl \end{array}. \end{aligned} \tag{3.9}$$

As the kernels justify the Lipschitz state, so they give

$$\begin{aligned} \|Y_n(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\Phi\|p_{n-1} - p_{n-2}\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\Phi\|p_{n-1} - p_{n-2}\|dl, \\ \|Z_n(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\Psi\|q_{n-1} - q_{n-2}\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau (\tau-l)^{\alpha-1}\Psi\|q_{n-1} - q_{n-2}\|dl. \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} \|Y_n(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\Phi\|Y_{n-1}(\varkappa,\tau)\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Phi\int_0^\tau (\tau-l)^{\alpha-1}\|Y_{n-1}(\varkappa,\tau)\|dl, \\ \|Z_n(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\Psi\|Z_{n-1}(\varkappa,\tau)\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Psi\int_0^\tau (\tau-l)^{\alpha-1}\|Z_{n-1}(\varkappa,\tau)\|dl. \end{aligned} \tag{3.11}$$

Theorem 1. *The FBEs given as Eq. (3.1), have the solutions provide the following conditions that are found* ξ_0, σ_0,

$$\begin{aligned} &\frac{1-\alpha}{B(\alpha)}\Phi + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Phi\xi_0^\alpha < 1, \\ &\frac{1-\alpha}{B(\alpha)}\Psi + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Psi\sigma_0^\alpha < 1. \end{aligned} \tag{3.12}$$

Proof. Let us consider that the functions $p(\varkappa, \tau)$ and $q(\varkappa, \tau)$ are limited. Additionally, they have been already satisfied that the kernels provides the Lipschitz state, hence from Eqs. (3.11) and (3.12) are written as follows,

$$\begin{aligned} \|Y_n(\varkappa,\tau)\| &\le \left[\frac{1-\alpha}{B(\alpha)}\Phi + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Phi\xi^{\alpha}\right]^n \|p(\varkappa,0)\|, \\ \|Z_n(\varkappa,\tau)\| &\le \left[\frac{1-\alpha}{B(\alpha)}\Psi + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Psi\sigma^{\alpha}\right]^n \|q(\varkappa,0)\|. \end{aligned} \tag{3.13}$$

Therefore, the function

$$\begin{aligned} p_n(\varkappa,\tau) &= \sum_{k=0}^{n} Y_k(\varkappa,\tau), \\ q_n(\varkappa,\tau) &= \sum_{k=0}^{n} Z_k(\varkappa,\tau), \end{aligned} \tag{3.14}$$

exists and smooth. Now, we examine that the function gived with above equations are the solutions of Eq. (3.1). Let us consider

$$\begin{aligned} p(\varkappa,\tau) - p(\varkappa,0) &= p_n(\varkappa,\tau) - D_n(\varkappa,\tau), \\ q(\varkappa,\tau) - q(\varkappa,0) &= q_n(\varkappa,\tau) - E_n(\varkappa,\tau). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|D_n(\varkappa,\tau)\| &= \left\| \begin{matrix} \frac{1-\alpha}{B(\alpha)}[K(\varkappa,\tau,p) - K(\varkappa,\tau,p_{n-1})] \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}[K(\varkappa,l,p) - K(\varkappa,l,p_{n-1})]dl \end{matrix} \right\| \\ &\le \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,p) - K(\varkappa,\tau,p_{n-1})\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}\|K(\varkappa,l,p) - K(\varkappa,l,p_{n-1})\|dl \\ &\le \frac{1-\alpha}{B(\alpha)}\Phi\|p-p_{n-1}\| + \frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\|p-p_{n-1}\|\xi^{\alpha}. \end{aligned} \tag{3.15}$$

$$\begin{aligned} \|E_n(\varkappa,\tau)\| &= \left\| \begin{matrix} \frac{1-\alpha}{B(\alpha)}[K(\varkappa,\tau,q) - K(\varkappa,\tau,q_{n-1})] \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}[K(\varkappa,l,q) - K(\varkappa,l,q_{n-1})]dl \end{matrix} \right\| \\ &\le \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,q) - K(\varkappa,\tau,q_{n-1})\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}\|K(\varkappa,l,q) - K(\varkappa,l,q_{n-1})\|dl \\ &\le \frac{1-\alpha}{B(\alpha)}\Psi\|q-q_{n-1}\| + \frac{1}{B(\alpha)\Gamma(\alpha)}\Psi\|q-q_{n-1}\|\sigma^{\alpha}. \end{aligned} \tag{3.16}$$

By continuing the same process, it gives

$$\begin{aligned} \|D_n(\varkappa,\tau)\| &\le \left(\frac{1-\alpha}{B(\alpha)} + \frac{1}{B(\alpha)\Gamma(\alpha)}\xi^{\alpha}\right)^{n+1}\Phi^{n+1}d, \\ \|E_n(\varkappa,\tau)\| &\le \left(\frac{1-\alpha}{B(\alpha)} + \frac{1}{B(\alpha)\Gamma(\alpha)}\sigma^{\alpha}\right)^{n+1}\Psi^{n+1}e. \end{aligned} \tag{3.17}$$

Then at $\xi = \xi_0$, $\sigma = \sigma_0$ we have

$$\begin{aligned} \|D_n(\varkappa,\tau)\| &\le \left(\frac{1-\alpha}{B(\alpha)} + \frac{1}{B(\alpha)\Gamma(\alpha)}\xi_0^{\alpha}\right)^{n+1}\Phi^{n+1}d, \\ \|E_n(\varkappa,\tau)\| &\le \left(\frac{1-\alpha}{B(\alpha)} + \frac{1}{B(\alpha)\Gamma(\alpha)}\sigma_0^{\alpha}\right)^{n+1}\Psi^{n+1}e. \end{aligned} \tag{3.18}$$

Where when $n \to \infty$, we have

$$\begin{aligned} \|D_n(\varkappa,\tau)\| &\to 0, \\ \|E_n(\varkappa,\tau)\| &\to 0. \end{aligned} \tag{3.19}$$

Then prove of existence is completed. □

Now, we analyse the uniqueness of solutions for FBEs (3.1). Let us think that $p(\varkappa, \tau)$, $q(\varkappa, \tau)$ get an another solution for the Eq. (3.1),

$$\begin{aligned} p(\varkappa,\tau)-P(\varkappa,\tau) &= \frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\} \\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1}\{K(\varkappa,l,p)-K(\varkappa,l,P)\}dl, \\ q(\varkappa,\tau)-Q(\varkappa,\tau) &= \frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,q)-K(\varkappa,\tau,Q)\} \\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1}\{K(\varkappa,l,q)-K(\varkappa,l,Q)\}dl. \end{aligned} \tag{3.20}$$

On taking norm on Eq. (3.20), gives

$$\begin{aligned} \|p(\varkappa,\tau)-P(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\| \\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1}\|K(\varkappa,l,p)-K(\varkappa,l,P)\|dl, \\ \|q(\varkappa,\tau)-Q(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,q)-K(\varkappa,\tau,Q)\| \\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1}\|K(\varkappa,l,q)-K(\varkappa,l,Q)\|dl. \end{aligned} \tag{3.21}$$

Since the kernels justify the Lipschitz states, so we have

$$\begin{aligned} \|p(\varkappa,\tau)-P(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\Phi\|p(\varkappa,\tau)-P(\varkappa,\tau)\| \\ &\quad+\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\|p(\varkappa,\tau)-P(\varkappa,\tau)\|, \\ \|q(\varkappa,\tau)-Q(\varkappa,\tau)\| &\le \frac{1-\alpha}{B(\alpha)}\Psi\|q(\varkappa,\tau)-Q(\varkappa,\tau)\| \\ &\quad+\frac{1}{B(\alpha)\Gamma(\alpha)}\Psi\sigma^\alpha\|q(\varkappa,\tau)-Q(\varkappa,\tau)\|. \end{aligned} \tag{3.22}$$

This gives

$$\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\left(1-\frac{1-\alpha}{B(\alpha)}\Phi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\right)\le 0, \tag{3.23}$$
$$\|q(\varkappa,\tau)-Q(\varkappa,\tau)\|\left(1-\frac{1-\alpha}{B(\alpha)}\Psi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Psi\sigma^\alpha\right)\le 0.$$

Theorem 2. *If the following inequality is provided, there is a unique solution of FBEs (3.1),*

$$\begin{aligned} &\left(1-\frac{1-\alpha}{B(\alpha)}\Phi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\right)>0, \\ &\left(1-\frac{1-\alpha}{B(\alpha)}\Psi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Psi\sigma^\alpha\right)>0. \end{aligned} \tag{3.24}$$

Proof. If the (3.24) condition is satisfied, then

$$\begin{aligned} &\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\left(1-\frac{1-\alpha}{B(\alpha)}\Phi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\right)\le 0, \\ &\|q(\varkappa,\tau)-Q(\varkappa,\tau)\|\left(1-\frac{1-\alpha}{B(\alpha)}\Psi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Psi\sigma^\alpha\right)\le 0. \end{aligned} \tag{3.25}$$

implies that

$$\begin{aligned} &\|p(\varkappa,\tau)-P(\varkappa,\tau)\|=0, \\ &\|q(\varkappa,\tau)-Q(\varkappa,\tau)\|=0. \end{aligned} \tag{3.26}$$

Then, we get

$$\begin{aligned} &p(\varkappa,\tau)=P(\varkappa,\tau), \\ &q(\varkappa,\tau)=Q(\varkappa,\tau). \end{aligned} \tag{3.27}$$

It completes the proof the uniqueness of the solution for Eq. (3.1). □

4. FHPTM For FBEs with AB fractional derivative

In this part, first of all, we consider the Laplace transform for FBEs with AB fractional operator (3.1) by using FHPTM and use the follow initial condition

$$p(\varkappa,0)=\sin\varkappa,\quad q(\varkappa,0)=\sin\varkappa,$$

it yields

$$\begin{aligned}L[p(\varkappa,\tau)]&=\frac{\sin\varkappa}{s}-\left(\frac{s^{\alpha}+\alpha(1-s^{\alpha})}{s^{\alpha}}\right)L[-p_{\varkappa\varkappa}-2pp_{\varkappa\varkappa}+(pq)_{\varkappa}]\\ L[q(\varkappa,\tau)]&=\frac{\sin\varkappa}{s}-\left(\frac{s^{\alpha}+\alpha(1-s^{\alpha})}{s^{\alpha}}\right)L[-q_{\varkappa\varkappa}+2qq_{\varkappa\varkappa}-(pq)_{\varkappa}]\end{aligned}\tag{4.1}$$

By using the inverse of Laplace transform on Eq. (4.1), we have

$$\begin{aligned}p(\varkappa,\tau)&=\sin\varkappa-L^{-1}\left[\left(\frac{s^{\alpha}+\alpha(1-s^{\alpha})}{s^{\alpha}}\right)L[-p_{\varkappa\varkappa}-2pp_{\varkappa\varkappa}+(pq)_{\varkappa}]\right]\\ q(\varkappa,\tau)&=\sin\varkappa-L^{-1}\left[\left(\frac{s^{\alpha}+\alpha(1-s^{\alpha})}{s^{\alpha}}\right)L[-q_{\varkappa\varkappa}+2qq_{\varkappa\varkappa}-(pq)_{\varkappa}]\right]\end{aligned}\tag{4.2}$$

by applying the HPM, we have

$$\begin{aligned}\sum_{n=0}^{\infty}z^{n}p_{n}&=\sin\varkappa-z\Bigg(L^{-1}\Bigg[\left(\frac{s^{\alpha}+\alpha(1-s^{\alpha})}{s^{\alpha}}\right)\\ &\qquad L\Bigg[-\sum_{n=0}^{\infty}z^{n}p_{n\varkappa\varkappa}-2\sum_{n=0}^{\infty}z^{n}H_{n}+\sum_{n=0}^{\infty}z^{n}K_{n}\Bigg]\Bigg]\Bigg)\\ \sum_{n=0}^{\infty}z^{n}q_{n}&=\sin\varkappa-z\Bigg(L^{-1}\Bigg[\left(\frac{s^{\alpha}+\alpha(1-s^{\alpha})}{s^{\alpha}}\right)\\ &\qquad L\Bigg[-\sum_{n=0}^{\infty}z^{n}q_{n\varkappa\varkappa}+2\sum_{n=0}^{\infty}z^{n}T_{n}-\sum_{n=0}^{\infty}z^{n}K_{n}\Bigg]\Bigg]\Bigg)\end{aligned}\tag{4.3}$$

In the Eq. (4.3) $H_n(p)$, $K_n(p)$ and $T_n(p)$ are He's polynomials as follows

$$\sum_{n=0}^{\infty}z^{n}H_{n}(p)=pp_{\varkappa\varkappa},\quad \sum_{n=0}^{\infty}z^{n}K_{n}(p,q)=(pq)_{\varkappa},\quad \sum_{n=0}^{\infty}z^{n}T_{n}(q)=qq_{\varkappa\varkappa}$$

The initial elements of the He's polynomials are described as

$$\begin{aligned}H_0(p)&=p_0p_{0_{\varkappa\varkappa}},\\ H_1(p)&=p_0p_{1_{\varkappa\varkappa}}+p_1p_{0_{\varkappa\varkappa}},\\ &\vdots\\ K_0(p,q)&=p_0q_{0_{\varkappa}}+p_{0_{\varkappa}}q_0,\\ K_1(p,q)&=p_0q_{1_{\varkappa}}+p_1q_{0_{\varkappa}}+p_{0_{\varkappa}}q_1+p_{1_{\varkappa}}q_0,\\ &\vdots\\ T_0(q)&=q_0q_{0_{\varkappa\varkappa}},\\ T_1(q)&=q_0q_{1_{\varkappa\varkappa}}+q_1q_{0_{\varkappa\varkappa}},\\ &\vdots\end{aligned}$$

Comparing the coefficients of the power of z, we obtain

$$\begin{aligned}&z^0:\\ &p_0(\varkappa,\tau)=\sin\varkappa,\quad q_0(\varkappa,\tau)=\sin\varkappa,\\ &z^1:\end{aligned}$$

$$p_1(\varkappa,\tau) = -\Big(1-\alpha+\frac{\tau^\alpha\alpha}{\Gamma(1+\alpha)}\Big)(\sin\varkappa+2\cos\varkappa\sin\varkappa+2\sin^2\varkappa),$$
$$q_1(\varkappa,\tau) = -\Big(1-\alpha+\frac{\tau^\alpha\alpha}{\Gamma(1+\alpha)}\Big)(\sin\varkappa-2\cos\varkappa\sin\varkappa-2\sin^2\varkappa)$$
$$z^2 :$$
$$\begin{aligned} p_2(\varkappa,\tau) = {} & ((-2\tau^\alpha(-1+\alpha)\alpha\Gamma(1+2\alpha)+\Gamma(1+\alpha)(\tau^{2\alpha}\alpha^2+(-1+\alpha)^2 \\ & \times\Gamma(1+2\alpha)))(2+5\cos\varkappa-6\cos2\varkappa-5\cos3\varkappa+8\sin\varkappa \\ & +6\sin2\varkappa-5\sin3\varkappa))/(\Gamma(1+\alpha)\Gamma(1+2\alpha)), \\ q_2(\varkappa,\tau) = {} & ((-2\tau^\alpha(-1+\alpha)\alpha\Gamma(1+2\alpha)+\Gamma(1+\alpha)(\tau^{2\alpha}\alpha^2+(-1+\alpha)^2 \\ & \times\Gamma(1+2\alpha)))(-2+5\cos\varkappa+6\cos2\varkappa-5\cos3\varkappa+8\sin\varkappa \\ & -6\sin2\varkappa-5\sin3\varkappa))/(\Gamma(1+\alpha)\Gamma(1+2\alpha)), \\ & \vdots \end{aligned}$$

Continuing same process, we obtain $p_n(\varkappa, \tau)$ and $q_n(\varkappa, \tau)$. Then, the solutions can be presented as,

$$\begin{aligned} p(\varkappa,\tau) &= p_0(\varkappa,\tau)+p_1(\varkappa,\tau)+p_2(\varkappa,\tau)+\cdots, \\ q(\varkappa,\tau) &= q_0(\varkappa,\tau)+q_1(\varkappa,\tau)+q_2(\varkappa,\tau)+\cdots. \end{aligned} \tag{4.4}$$

Also, by using FHPTM for FBEs with fractional CF operator, we can write

$$\begin{aligned} p(\varkappa,\tau) &= \sin\varkappa - L^{-1}\bigg[\bigg(\frac{s+\alpha(1-s)}{s}\bigg)L[-p_{\varkappa\varkappa}-2pp_{\varkappa\varkappa}+(pq)_\varkappa]\bigg] \\ q(\varkappa,\tau) &= \sin\varkappa - L^{-1}\bigg[\bigg(\frac{s+\alpha(1-s)}{s}\bigg)L[-q_{\varkappa\varkappa}+2qq_{\varkappa\varkappa}-(pq)_\varkappa]\bigg] \end{aligned} \tag{4.5}$$

Then by performing operations similar to AB derivative, we have

$$\begin{aligned} p(\varkappa,\tau) = {} & \sin\varkappa-(1-\alpha+t\alpha)(\sin x+2\cos x\sin x+2\sin^2 x) \\ & +\frac{1}{2}(2+4(-1+t)\alpha+(2-4t+t^2)\alpha^2)(2+5\cos x \\ & -6\cos2x-5\cos3x+8\sin x+6\sin2x-5\sin3x), \\ q(\varkappa,\tau) = {} & \sin\varkappa-(1-\alpha+t\alpha)(\sin x-2\cos x\sin x-2\sin^2 x) \\ & -\frac{1}{2}(2+4(-1+t)\alpha+(2-4t+t^2)\alpha^2)(2-5\cos x \\ & -6\cos2x+5\cos3x-8\sin x+6\sin2x+5\sin3x). \end{aligned} \tag{4.6}$$

Now, by using FHPTM for FBEs with fractional LC operator, we can write

$$\begin{aligned} p(\varkappa,\tau) &= \sin\varkappa - L^{-1}\bigg[\bigg(\frac{1}{s^\alpha}\bigg)L[-p_{\varkappa\varkappa}-2pp_{\varkappa\varkappa}+(pq)_\varkappa]\bigg] \\ q(\varkappa,\tau) &= \sin\varkappa - L^{-1}\bigg[\bigg(\frac{1}{s^\alpha}\bigg)L[-q_{\varkappa\varkappa}+2qq_{\varkappa\varkappa}-(pq)_\varkappa]\bigg] \end{aligned} \tag{4.7}$$

Then by performing operations similar to AB derivative, we have

$$\begin{aligned} p(\varkappa,\tau) = {} & \sin\varkappa-\frac{t^\alpha(\sin x+2\cos x\sin x+2\sin^2 x)}{\Gamma(1+\alpha)} \\ & +\frac{t^{2\alpha}(2+5\cos x-6\cos2x-5\cos3x+8\sin x+6\sin2x-5\sin3x)}{\Gamma(1+2\alpha)}, \\ q(\varkappa,\tau) = {} & \sin\varkappa-\frac{t^\alpha(\sin x-2\cos x\sin x-2\sin^2 x)}{\Gamma(1+\alpha)} \\ & +\frac{t^{2\alpha}(-2+5\cos x+6\cos2x-5\cos3x+8\sin x-6\sin2x-5\sin3x)}{\Gamma(1+2\alpha)}. \end{aligned} \tag{4.8}$$

5. Graphical representations of the solutions

The graphical illustrations of the solutions are given below in the figures and tables with the aid of Mathematica. In Tables 1 and 2, we compared with the results we found in the previous section. These solutions obtained for AB, LC and CF derivative.

Table 1
Comparison of numerical solutions with LC, CF and AB derivative at $\varkappa = 2$ for $p(\varkappa,\ \tau)$.

τ	$\alpha = 1$	*LC*		*CF*		*AB*	
		$\alpha = 0.85$	$\alpha = 0.98$	$\alpha = 0.85$	$\alpha = 0.98$	$\alpha = 0.85$	$\alpha = 0.98$
0.01	0.891395	0.872005	0.889526	0.702571	0.858138	0.694981	0.856437
0.02	0.873809	0.843259	0.870678	0.695648	0.842138	0.684004	0.839291
0.03	0.85654	0.817628	0.852394	0.695648	0.826442	0.674536	0.822678
0.04	0.839589	0.794105	0.834597	0.68249	0.811051	0.666141	0.806524
0.05	0.822954	0.772224	0.817248	0.676254	0.795964	0.658612	0.790797
0.06	0.806637	0.751714	0.800324	0.670248	0.781182	0.651828	0.775475
0.07	0.790636	0.732396	0.783809	0.664471	0.766704	0.645704	0.760542
0.08	0.774953	0.714141	0.767693	0.658923	0.752531	0.640182	0.745989
0.09	0.759587	0.696853	0.751967	0.653604	0.738662	0.635217	0.731808
0.1	0.744538	0.680458	0.736622	0.648514	0.725098	0.630772	0.717993

Table 2
Comparison of numerical solutions with LC, CF and AB derivative at $\varkappa = 2$ for $q(\varkappa,\ \tau)$.

τ	$\alpha = 1$	*LC*		*CF*		*AB*	
		$\alpha = 0.85$	$\alpha = 0.98$	$\alpha = 0.85$	$\alpha = 0.98$	$\alpha = 0.85$	$\alpha = 0.98$
0.01	0.909193	0.90914	0.909185	0.917579	0.90927	0.91868	0.909279
0.02	0.90913	0.909166	0.909125	0.91856	0.909367	0.920437	0.90939
0.03	0.909108	0.909311	0.909113	0.91957	0.909503	0.922148	0.909547
0.04	0.909126	0.909556	0.909147	0.92061	0.909679	0.923852	0.909747
0.05	0.909186	0.90989	0.909228	0.92168	0.909894	0.925564	0.90999
0.06	0.909286	0.910308	0.909354	0.922779	0.910148	0.927292	0.910277
0.07	0.909427	0.910803	0.909526	0.923908	0.910441	0.92904	0.910606
0.08	0.909609	0.911372	0.909742	0.925066	0.910774	0.930812	0.910978
0.09	0.909832	0.912011	0.910002	0.926253	0.911146	0.93261	0.911392
0.1	0.910096	0.912717	0.910307	0.927471	0.911557	0.934434	0.911848

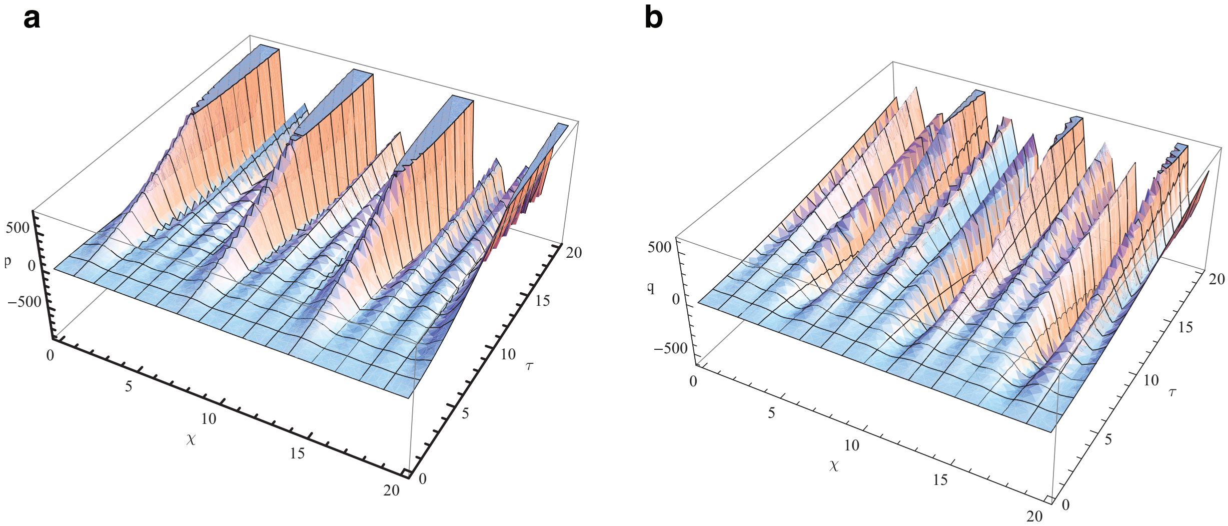

Fig. 1. The 3D graphics for the FBEs with AB derivative when $\alpha = 0.85$.a) $p(\varkappa,\ \tau)$, b) $q(\varkappa,\ \tau)$.

In Tables 1 and 2, we present the comparison between the approximate results for FBEs.These approximate results are obtained fractional AB, CF and LC derivative, (Fig. 1).

In Fig. 2, we plot the approximate solution $p(\varkappa,\ \tau)$ and $q(\varkappa,\ \tau)$ by using FHPTM for $\alpha = 0.75, 0.8, 0.95, 1$. These figures clear that the convergency of the numerical solutions to the exact solution connected to the order of the solution and the exact error is being smaller as the order of the solution is increasing.

6. Final remarks

In this study, the fractional homotopy perturbation transform method are applied to FBEs for CF, LC and AB fractional derivatives. We shown the existence and uniqueness of the obtained solutions for this system in case of AB derivative. Also

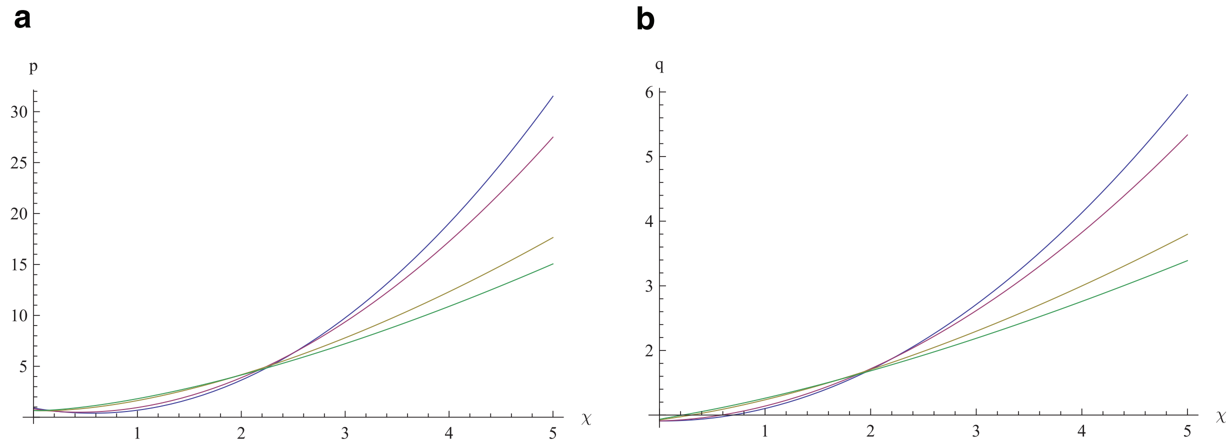

Fig. 2. The 2D graphic of the FBEs with AB derivative for different value of α when $\varkappa = 2$.a) $p(\varkappa, \tau)$, b) $q(\varkappa, \tau)$.

we obtained numerical solutions in case of CF and LC derivative. We compared these approximate solutions with each other by preparing graphics and tables. From these concludes, we say that the presented FBEs with fractional AB derivative are suitable to examine the many problems located in science and engineering.

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