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An algorithm for numerical solution of some nonlinear multi-dimensional parabolic partial differential equations

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ABSTRACT

In this research paper, a numerical method, named the three-step Ultraspherical wavelet collocation method, is presented for solving some nonlinear multi-dimensional parabolic partial differential equations. The method is third-order accurate in time. In this method, the three-step Taylor method is used to get the time derivative, while the Ultraspherical wavelet collocation method is used to get the space derivatives. Ultraspherical wavelets have good properties which make useful to carry out this aim. The presented method is developed for Burgers' equation, Fisher–Kolmogorov–Petrovsky–Piscounov (Fisher–KPP) equation, and quasilinear parabolic equation. Three illustrative numerical problems are solved to demonstrate the efficiency, simplicity, and reliability of the presented method.

1. Introduction

Wavelets analysis is the decomposition of a function onto shifted and scaled versions of the basic wavelet. Due to the ability to accurately represent polynomials to a certain degree and represent functions, it can be efficiently approximate unknown functions. So, it has been applied in many different fields of science and engineering. Wavelet applications have been extensively used to seek a numerical solution of the differentials equations. During the last decades, methods based on different wavelet families have been extensively employed to get numerical solutions of partial differential equations arising in different disciplines. Some of these methods are methods based on Gegenbauer wavelets [1–3], a Müntz wavelets collocation method [4], a Haar wavelet-finite difference hybrid method [5], a three-step Taylor–Galerkin finite element method [6], a Strang splitting method using Chebyshev wavelets [7], Chebyshev wavelet collocation method [8], Legendre wavelets optimization method [9], the Jacobi wavelet collocation method [10], the collocation method using Wilson wavelets [11].

Because partial differential equations have practical importance, many researchers have worked on the numerical solution of different types of partial differential equations. In this paper, we focus on Burgers' Equation, Fisher–Kolmogorov–Petrovsky–Piscounov (Fisher–KPP) and quasilinear parabolic equation, which are parabolic partial differential equations.[12–21]. Many numerical methods have been developed to obtain numerical solutions of these equations, such as finite difference methods [22], Roessler and Hüssner used finite element and Galerkin methods to solve two-dimensional Fisher–KPP equation [23], Macías-Díaz presented implicit finite difference method for the two-dimensional Fisher–KPP equation [24], Oruc studied the Chebyshev wavelet collocation method with two different time discretization schemes to solve two-space dimensional nonlinear Fisher–KPP equation [25],Saleem et al. combined finite-difference scheme with Haar wavelet collocation method for solving Burgers' and quasilinear differential equations [26], Haq and Ghafoor used a composite numerical scheme based upon Haar wavelets and finite differences to solve multi-dimensional time-dependent Burgers' equation [27], Cao et al. suggested a numerical method by combining the discontinuous Galerkin method to spatial variables and a finite difference scheme to temporal variables [28], Khater and Alabdali presented the rigonometric-quantic B-spline scheme to get the numerical solution of two-dimensional Fisher–KPP equation [29].

In this research work, inspiring by the ongoing research, we aim to present the three-step Ultraspherical wavelet collocation method to seek a numerical solution of some nonlinear multi-dimensional parabolic partial differential equations. Up to now, the presented method has not been applied to nonlinear partial differential equations

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and any system of partial differential equations in the literature. We apply the three-step Taylor method for the time discretization before discretizing the spatial variables for the numerical scheme. Then, we use the collocation method with Ultraspherical wavelets for spatial discretization. The presented method transforms the solution of the considered partial differential equation into the solution of a nonlinear system of algebraic equations. The system can be easily solved with a suitable numerical method. This method does not increase the difficulties for higher-dimensional problems and can be easily applied to solve various high dimensions problems. The method's applicability for nonlinear partial differential equations is easy, and the technique provides numerical solutions for nonlinear partial differential equations.

The three-step wavelet Galerkin method was first used in [30], and the time discretization technique was used with Daubechies wavelets in these references. The three-step wavelet collocation method was first used for linear time-dependent PDEs in [31], and the time discretization technique was used with Legendre wavelets in this reference.

2. Ultraspherical (Gegenbauer) polynomials and Ultraspherical (Gegenbauer) wavelets

Ultraspherical polynomials (a special type of Jacobi polynomials) of an order $n \in Z^+$ [32], $U_n^\gamma(x)$ is defined on $[-1,1]$, and the Ultraspherical polynomials can be expressed by the aid of the following recurrence formulae:

$$U_0^\gamma(x) = 1,\ U_1^\gamma(x) = 2\gamma x,$$

$$U_{n+1}^\gamma(x) = \frac{1}{n+1}\left(2(n+\gamma)xU_n^\gamma(x) - (n+2\gamma-1)U_{n-1}^\gamma(x)\right),\quad n=1,2,3,\dots,\quad \gamma > -\frac{1}{2}$$

Ultraspherical polynomials are orthogonal with respect to the L^2 inner product on the interval $[-1,1]$ that is

$$\int_{-1}^{1}(1-x^2)^{\gamma-\frac{1}{2}}U_m^\gamma(x)U_n^\gamma(x)dx = L_n^\gamma\delta_{nm},\quad \gamma > -\frac{1}{2}$$

Herein,

$$L_n^\gamma = \begin{cases} \dfrac{\pi 2^{1-2\gamma}\Gamma(n+2\gamma)}{(n+\gamma)\Gamma(n+1)(\Gamma(\gamma))^2}, & \gamma \neq 0 \\ \dfrac{2\pi}{n^2}, & \gamma = 0 \\ \pi, & \gamma = 0, n = 0 \end{cases}$$

is called the normalizing factor and δ_{nm} is the Kronecker symbol.

Chebyshev polynomials of the first kind $T_n(x)$ for $\gamma = 0$, Legendre polynomials $L_n(x)$ for $\gamma = 1/2$ and Chebyshev polynomials of the second-kind $T_n^*(x)$ for $\gamma = 1$ are all special cases of Ultraspherical polynomials, and these polynomials are shown the following relations:

$$T_n = \frac{n}{2}\lim_{\gamma\to 0}\frac{U_n^\gamma}{\gamma}, T_n = U_n^1, L_n = U_n^{1/2}$$

Ultraspherical wavelets are defined as:

$$\psi_{nm}(t) = \begin{cases} \dfrac{1}{\sqrt{L_m^\gamma}}2^{\frac{k}{2}}U_m^\gamma(2^k x - \tilde{n}) & , \frac{\tilde{n}-1}{2^k} \le x < \frac{\tilde{n}+1}{2^k} \\ 0 & , \text{otherwise} \end{cases}$$

where $k = 1,2,3,\dots$, is the level of resolution, $n = 1,2,3,\dots,2^{k-1}$, $\tilde{n} = 2n-1$ is the translation parameter, and $m = 0,1,2,\dots,M-1$ is the order of the Ultraspherical polynomials, $M > 0$.

Ultraspherical wavelets' weight function is given as:

$$w_n^\gamma(x) = \begin{cases} w(2^k x - 2n + 1) = \left(1 - \left(2^k x - 2n + 1\right)^2\right)^{\gamma-\frac{1}{2}}, \\ x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0 & , \text{otherwise} \end{cases}$$

Similarly, Chebyshev wavelets of the first kind $T_n(x)$ for $\gamma = 0$, Legendre wavelets $L_n(x)$ for $\gamma = 1/2$ and Chebyshev wavelets of the second-kind $T_n^*(x)$ for $\gamma = 1$ are all special cases of Ultraspherical wavelets.

2.1. Function approximation

Any function $v(x)$ which is square-integrable on the interval $[0,1)$ can be expressed in terms of Ultraspherical wavelets as:

$$v(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) \quad (1)$$

where c_{nm} values are wavelet coefficients, and these coefficients can be determined by the following operation:

$$c_{nm} = \langle v(x), \psi_{nm}(x)\rangle_{w_n^\gamma}$$

We approximate $v(x)$ by truncating the infinite series in Eq. (1) as follows:

$$v(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T\Psi(x). \quad (2)$$

For a more compact expression, Eq. (2) can be rewritten as:

$$v(x) = \sum_{j=1}^{\tilde{m}} c_j\psi_j(x)$$

where $\tilde{m} = \left(2^{k-1}M\right)$ and $j = M(n-1)+m+1$.

In a similar way, the expansion of any function $v(x,y) \in [0,1)\times[0,1)$ in terms of Ultraspherical wavelet series is:

$$v(x,y) = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty} c_{n_1m_1n_2m_2}\psi_{n_1m_1}(x)\psi_{n_2m_2}(y) \quad (3)$$

where $c_{n_1m_1n_2m_2}$ values are the wavelet coefficients, and these coefficients can be determined by:

$$c_{n_1m_1n_2m_2} = \left\langle \left\langle v(x,y), \psi_{n_1m_1}(x)\right\rangle_{w_{n_1}^\gamma}, \psi_{n_2m_2}(y)\right\rangle_{w_{n_2}^\gamma}.$$

We approximate $v(x,y)$ by truncating the infinite series in Eq. (3) as:

$$v(x,y) = \sum_{n_1=1}^{2^{k-1}}\sum_{m_1=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\sum_{m_2=0}^{M-1} c_{n_1m_1n_2m_2}\psi_{n_1m_1}(x)\psi_{n_2m_2}(y). \quad (4)$$

For a more compact expression, Eq. (4) can be rewritten as:

$$v(x,y) = \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{m}} c_{ij}\psi_i(x)\psi_j(y) \quad (5)$$

in which $\tilde{m} = \left(2^{k-1}M\right)$, $i = M(n_1-1)+m_1+1$ and $j = M(n_2-1)+m_2+1$.

3. Operational matrix of derivatives

Theorem. *The derivative of $\Psi(x)$ can be approximated by:*

$$\frac{d}{dx}\Psi(x) = D\Psi(x)$$

in which D is called the operational matrix of derivative having order $2^{k-1}M$. D can be calculated from the following operation:

$$D = \begin{bmatrix} \zeta & 0 & \dots & 0 \\ 0 & \zeta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \zeta \end{bmatrix}$$

where ζ is a matrix of order $M \times M$ having (i,j) elements defined as

$$\zeta_{i,j} = \begin{cases} \dfrac{2^{k+1}(j+\gamma-1)}{\sqrt{\frac{(j-1+\gamma)\Gamma(j)\Gamma(i-1+2\gamma)}{(i-1+\gamma)\Gamma(i)\Gamma(j-1+2\gamma)}}} & , i = 2,3,\dots,M, j = 1,2,\dots,i-1, and (i+j) odd \\ 0 & , otherwise \end{cases}$$

[33]. The operational matrix of nth order derivative of $\Psi(x)$ can be expressed as:

$$\frac{d^n}{dx^n}\Psi(x) = D^n\Psi(x)$$

4. Numerical method

In this section, the main structure of the three-step Ultraspherical wavelet collocation method for some nonlinear parabolic partial differential equations is explained.

Firstly, we consider the nonlinear partial differential equation of the following form

$$v_t = Lv + Nf(v) \tag{6}$$

subject to the initial condition

$$v(x,y,0) = v_0(x,y)$$

in which Lv and $Nf(v)$ are the linear and nonlinear parts of Eq. (6), respectively. $f(v)$ is nonlinear function. When $v(t)$ is performed a Taylor series expansion in time, the following equation is obtained:

$$v(t+\Delta t) = v(t) + \Delta t\frac{\partial v(t)}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 v(t)}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 v(t)}{\partial t^3} + O(\Delta t^4) \tag{7}$$

When Eq. (7) is approximated up to third-order accuracy, the three-step method based on a Taylor series expansion is defined as [17]:

$$\begin{aligned} v\left(t+\frac{\Delta t}{3}\right) &= v(t) + \frac{\Delta t}{3}\frac{\partial v(t)}{\partial t} \\ v\left(t+\frac{\Delta t}{2}\right) &= v(t) + \frac{\Delta t}{2}\frac{\partial v(t+\frac{\Delta t}{3})}{\partial t} \\ v(t+\Delta t) &= v(t) + \Delta t\frac{\partial v(t+\frac{\Delta t}{2})}{\partial t} \end{aligned} \tag{8}$$

In the numerical scheme, the three-step method based on a Taylor series expansion for time discretization is used, whereas the Ultraspherical wavelet collocation method for spatial discretization of Eqs. (8) is used.

4.1. Burgers' equation

4.1.1. Time discretization for Burgers' equation

Consider the Burgers' equation

$$\frac{\partial v}{\partial t} + v\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \tag{9}$$

subject to the initial condition

$$v(x,y,0) = v_0(x,y), \quad 0 \le x,y \le 1. \tag{10}$$

Assume that $s \ge 0$ and Δt denote the time step such that $t_s = s\Delta t$, $s = 0,1,2,\ldots,S_t$ and $v(x,y,t_s) = v^s$. Applying the three-step method for time discretization, we get the following system,

$$\begin{aligned} v^{s+1/3} &= v^s + \frac{\Delta t}{3}\left(\frac{\partial^2 v^s}{\partial x^2} + \frac{\partial^2 v^s}{\partial y^2} - v^s\frac{\partial v^s}{\partial x} - v^s\frac{\partial v^s}{\partial y}\right) \\ v^{s+1/2} &= v^s + \frac{\Delta t}{2}\left(\frac{\partial^2 v^{s+1/3}}{\partial x^2} + \frac{\partial^2 v^{s+1/3}}{\partial y^2}\right. \\ &\quad \left. - v^{s+1/3}\frac{\partial v^{s+1/3}}{\partial x} - v^{s+1/3}\frac{\partial v^{s+1/3}}{\partial y}\right) \\ v^{s+1} &= v^s + \Delta t\left(\frac{\partial^2 v^{s+1/2}}{\partial x^2} + \frac{\partial^2 v^{s+1/2}}{\partial y^2}\right. \\ &\quad \left. - v^{s+1/2}\frac{\partial v^{s+1/2}}{\partial x} - v^{s+1/2}\frac{\partial v^{s+1/2}}{\partial y}\right) \end{aligned} \tag{11}$$

Here, $v^{s+1/3}, v^{s+1/2}$ and v^{s+1} correspond the computed solution at time level $\left(t_s+\frac{\Delta t}{3}\right), \left(t_s+\frac{\Delta t}{2}\right)$ and $t_s+\Delta t$,respectively.

4.1.2. Spatial discretization for Burgers' equation

After time discretization, the spatial derivatives of $v(x,y,t)$ are approximated using Ultraspherical wavelets. The Ultraspherical wavelet collocation method is applied in this part. $v(x,y,t_s) \in [0,1)\times[0,1)$ can be expanded in terms of Ultraspherical wavelets as:

$$\begin{aligned} v(x,y,t_s) = v^s &= \sum_{n_1=1}^{2^{k-1}}\sum_{m_1=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\sum_{m_2=0}^{M-1} c^s_{n_1m_1n_2m_2} \\ &\times \psi_{n_1m_1}(x)\psi_{n_2m_2}(y) = \Psi^T(x)C^s\Psi(y). \end{aligned} \tag{12}$$

Eq. (12) can be rewritten as:

$$v(x,y,t_s) = v^s = \sum_{i=1}^{\tilde m}\sum_{j=1}^{\tilde m} c^s_{ij}\psi_i(x)\psi_j(y) = \Psi^T(x)C^s\Psi(y). \tag{13}$$

in which $\tilde m = \left(2^{k-1}M\right), i = M(n_1-1)+m_1+1$ and $j = M(n_2-1)+m_2+1$. Herein, Ultraspherical wavelets are used to obtain the approximate solution. C^s is the coefficient vector at time t_s. As a result, the approximation solution at the time $t_{s+1/3}$ is as follows:

$$v^{s+\frac{1}{3}} = \sum_{i=1}^{\tilde m}\sum_{j=1}^{\tilde m} c^{s+\frac{1}{3}}_{ij}\psi_i(x)\psi_j(y) = \Psi^T(x)C^{s+\frac{1}{3}}\Psi(y). \tag{14}$$

Now, using the operational matrix of derivative for $v_x = \frac{\partial v}{\partial x}, v_y = \frac{\partial v}{\partial y}, v_xx = \frac{\partial^2 v}{\partial x^2}$ and $v_yy = \frac{\partial^2 v}{\partial y^2}$, we obtain the following equations.

$$\frac{\partial v}{\partial x}(x,y,t_s) = \sum_{i=1}^{\tilde m}\sum_{j=1}^{\tilde m} c^s_{ij}\frac{\partial\psi_i(x)}{\partial x}\psi_j(y) = \Psi^T(x)D^TC^s\Psi(y). \tag{15}$$

$$\frac{\partial v}{\partial y}(x,y,t_s) = \sum_{i=1}^{\tilde m}\sum_{j=1}^{\tilde m} c^s_{ij}\psi_i(x)\frac{\partial\psi_j(y)}{\partial y} = \Psi^T(x)C^sD\Psi(y) \tag{16}$$

$$\frac{\partial^2 v}{\partial x^2}(x,y,t_s) = \sum_{i=1}^{\tilde m}\sum_{j=1}^{\tilde m} c^s_{ij}\frac{\partial^2\psi_i(x)}{\partial x^2}\psi_j(y) = \Psi^T(x)\left(D^T\right)^2C^s\Psi(y). \tag{17}$$

and

$$\frac{\partial^2 v}{\partial y^2}(x,y,t_s) = \sum_{i=1}^{\tilde m}\sum_{j=1}^{\tilde m} c^s_{ij}\psi_i(x)\frac{\partial^2\psi_j(y)}{\partial y^2} = \Psi^T(x)C^sD^2\Psi(y). \tag{18}$$

By putting Eqs. (13)–(18) into the first equation of (11), we get

$$\begin{aligned} &\Psi^T(x)C^{s+\frac{1}{3}}\Psi(y) = \Psi^T(x)C^s\Psi(y) + \frac{\Delta t}{3}\left(\Psi^T(x)\left(D^T\right)^2C^s\Psi(y)\right. \\ &+ \Psi^T(x)C^sD^2\Psi(y) - \left(\Psi^T(x)C^s\Psi(y)\right)\left(\Psi^T(x)D^TC^s\Psi(y)\right) \\ &\left.- \left(\Psi^T(x)C^s\Psi(y)\right)\left(\Psi^T(x)C^sD\Psi(y)\right)\right) \end{aligned} \tag{19}$$

Firstly, the initial condition can be expanded in terms of Ultraspherical wavelets as:

$$v_0(x,y) = v(x,y,t_0) = v_0 = (\Psi(x))^T C^0\Psi(y). \tag{20}$$

Taking the inner product from both sides of Eq. (20) with $\{\psi_i\}_{i=1}^{\tilde m}$ and $\{\psi_j\}_{j=1}^{\tilde m}$, respectively, the coefficient vector C^0 at the time t_0 is obtained.

By selecting the collocation points as:

$$x_j = \frac{2j-1}{2\tilde m}, \quad j = 1,2,\ldots,\tilde m$$

and by starting from the solution obtained at the time t_0, when we put the collocation points into Eq. (19), we can get a system of nonlinear algebraic equations with $2^{k-1}M \times 2^{k-1}M$ unknown variables, $c^{s+\frac{1}{3}}_{ij}$. When we solve this system, we can acquire $C^{s+\frac{1}{3}}$. To find the vectors $C^{s+\frac{1}{2}}$ and C^{s+1}, the similar process is applied for the second and third equations in Eq. (11). The solution C^{s+1} is the vector coefficient in the numerical solution of $v(x,y,t_s)$ in each step for $s = 0,1,2,3,\ldots$.

4.2. Fisher–Kolmogorov–Petrovsky–Piscounov (Fisher–KPP) equation

4.2.1. Time discretization for Fisher–KPP equation

Consider the Fisher–KPP equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + v(1-v) \tag{21}$$

subject to the initial condition

$$v(x,y,0) = v_0(x,y), \quad 0 \le x,y \le 1.$$

Assume that $s \ge 0$ and Δt denote the time step such that $t_s = s\Delta t$, $s = 0,1,2,\dots,S_t$ and $v(x,y,t_s) = v^s$. We begin the numerical method with the discretization of the time using the three-step method shown in Eq. (8),

$$\begin{aligned} v^{s+1/3} &= v^s + \frac{\Delta t}{3}\left(\frac{\partial^2 v^s}{\partial x^2} + \frac{\partial^2 v^s}{\partial y^2} + v^s(1-v^s)\right) \\ v^{s+1/2} &= v^s + \frac{\Delta t}{2}\left(\frac{\partial^2 v^{s+1/3}}{\partial x^2} + \frac{\partial^2 v^{s+1/3}}{\partial y^2} + v^{s+1/3}(1-v^{s+1/3})\right) \\ v^{s+1} &= v^s + \Delta t\left(\frac{\partial^2 v^{s+1/2}}{\partial x^2} + \frac{\partial^2 v^{s+1/2}}{\partial y^2} + v^{s+1/2}(1-v^{s+1/2})\right) \end{aligned} \tag{22}$$

Here, $v^{s+1/3}, v^{s+1/2}$ and v^{s+1} correspond the computed solution at time level $\left(t_s + \frac{\Delta t}{3}\right), \left(t_s + \frac{\Delta t}{2}\right)$ and $t_s + \Delta t$,respectively.

4.2.2. Spatial discretization for Fisher–KPP

After time discretization, the spatial derivatives of $v(x,y,t)$ are approximated using the Ultraspherical wavelets. Ultraspherical wavelet collocation method is applied in this part.

$v(x,y,t_s) \in [0,1) \times [0,1)$ can be expanded in terms of Ultraspherical wavelets as:

$$v(x,y,t_s) = v^s = \sum_{n_1=1}^{2^{k-1}} \sum_{m_1=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} c^s_{n_1 m_1 n_2 m_2} \times \psi_{n_1 m_1}(x)\psi_{n_2 m_2}(y) = \Psi^T(x) C^s \Psi(y). \tag{23}$$

Eq. (23) can be rewritten as:

$$v(x,y,t_s) = v^s = \sum_{i=1}^{\tilde m} \sum_{j=1}^{\tilde m} c^s_{ij} \psi_i(x)\psi_j(y) = \Psi^T(x) C^s \Psi(y). \tag{24}$$

in which $\tilde m = \left(2^{k-1}M\right), i = M(n_1-1)+m_1+1$ and $j = M(n_2-1)+m_2+1$. Here, Ultraspherical wavelets are used to obtain the approximate solution. C^s is the coefficient vector at the time t_s. As a result, the approximation solution at time $t_{s+1/3}$ is as follows:

$$v^{s+\frac{1}{3}} = \sum_{i=1}^{\tilde m} \sum_{j=1}^{\tilde m} c^{s+\frac{1}{3}}_{ij} \psi_i(x)\psi_j(y) = \Psi^T(x) C^{s+\frac{1}{3}} \Psi(y). \tag{25}$$

Now, using the operational matrix of derivative for $v_{xx} = \frac{\partial^2 v}{\partial x^2}$ and $v_{yy} = \frac{\partial^2 v}{\partial y^2}$, we obtain the following equations.

$$\frac{\partial^2 v}{\partial x^2}(x,y,t_s) = \sum_{i=1}^{\tilde m} \sum_{j=1}^{\tilde m} c^s_{ij} \frac{\partial^2 \psi_i(x)}{\partial x^2} \psi_j(y) = \Psi^T(x)\left(D^T\right)^2 C^s \Psi(y). \tag{26}$$

and

$$\frac{\partial^2 v}{\partial y^2}(x,y,t_s) = \sum_{i=1}^{\tilde m} \sum_{j=1}^{\tilde m} c^s_{ij} \psi_i(x) \frac{\partial^2 \psi_j(y)}{\partial y^2} = \Psi^T(x) C^s D^2 \Psi(y). \tag{27}$$

By putting Eqs. (24)–(27) into the first equation of (22), we get

$$\begin{aligned} \Psi^T(x) C^{s+\frac{1}{3}} \Psi(y) &= \Psi^T(x) C^s \Psi(y) + \frac{\Delta t}{3}\Big(\Psi^T(x)\left(D^T\right)^2 C^s \Psi(y) \\ &+ \Psi^T(x) C^s D^2 \Psi(y) + \left(\Psi^T(x) C^s \Psi(y)\right) - \left(\Psi^T(x) C^s \Psi(y)\right)^2\Big) \end{aligned} \tag{28}$$

Firstly, the initial condition can be expanded in terms of Ultraspherical wavelets as:

$$v_0(x,y) = v(x,y,t_0) = v_0 = (\Psi(x))^T C^0 \Psi(y). \tag{29}$$

Table 1
Comparison of the absolute errors for $\Delta t = 0.01$ and $\gamma = 1/2$.

$\lvert v_{exact} - v_{\gamma=1/2} \rvert$				
(x,y)	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
(0.1, 0.1)	7.388×10^{-7}	5.256×10^{-7}	6.9260×10^{-6}	1.1127×10^{-4}
(0.2, 0.2)	5.143×10^{-7}	1.522×10^{-7}	7.8531×10^{-6}	7.8900×10^{-5}
(0.3, 0.3)	2.088×10^{-7}	1.316×10^{-6}	1.1799×10^{-5}	9.0113×10^{-5}
(0.4, 0.4)	1.701×10^{-7}	2.916×10^{-6}	1.8389×10^{-5}	1.3795×10^{-4}
(0.5, 0.5)	1.785×10^{-7}	1.110×10^{-6}	4.3000×10^{-6}	2.3228×10^{-5}
(0.6, 0.6)	1.830×10^{-7}	4.827×10^{-7}	3.4491×10^{-6}	6.1673×10^{-5}
(0.7, 0.7)	5.227×10^{-7}	1.998×10^{-6}	1.0881×10^{-5}	1.4301×10^{-4}
(0.8, 0.8)	8.395×10^{-7}	3.415×10^{-6}	1.7893×10^{-5}	2.1970×10^{-4}
(0.9, 0.9)	1.113×10^{-6}	4.714×10^{-6}	2.4390×10^{-5}	2.9068×10^{-4}

Table 2
Comparison of the absolute errors for $\Delta t = 0.01$ and $\gamma = 5/2$.

$\lvert v_{exact} - v_{\gamma=5/2} \rvert$				
(x,y)	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
(0.1, 0.1)	1.1827×10^{-6}	4.4166×10^{-6}	1.7165×10^{-5}	1.5816×10^{-4}
(0.2, 0.2)	8.7570×10^{-7}	3.1333×10^{-6}	1.1879×10^{-5}	1.2121×10^{-4}
(0.3, 0.3)	5.4420×10^{-7}	1.6680×10^{-6}	5.3078×10^{-6}	6.4626×10^{-5}
(0.4, 0.4)	1.9010×10^{-7}	4.0800×10^{-8}	2.3837×10^{-6}	8.3603×10^{-6}
(0.5, 0.5)	1.4561×10^{-6}	1.0066×10^{-5}	5.1227×10^{-5}	3.9124×10^{-4}
(0.6, 0.6)	1.1504×10^{-6}	9.0264×10^{-6}	4.7921×10^{-5}	3.7355×10^{-4}
(0.7, 0.7)	1.0250×10^{-6}	8.8658×10^{-6}	4.9401×10^{-5}	4.1767×10^{-4}
(0.8, 0.8)	1.1122×10^{-6}	9.7613×10^{-6}	5.6684×10^{-5}	5.3923×10^{-4}
(0.9, 0.9)	1.5117×10^{-6}	1.1952×10^{-5}	7.0996×10^{-5}	7.5555×10^{-4}

Taking the inner product from both sides of Eq. (29) with $\{\psi_i\}_{i=1}^{\tilde m}$ and $\{\psi_j\}_{j=1}^{\tilde m}$, respectively, the coefficient vector C^0 at the time t_0 is obtained.

By selecting the collocation points as:

$$x_j = \frac{2j-1}{2\tilde m}, \quad j = 1,2,\dots,\tilde m$$

and by starting from the solution obtained at the time t_0, when we put the collocation points into Eq. (28), we can get a system of nonlinear algebraic equations with $2^{k-1}M \times 2^{k-1}M$ unknown variables, $c^{s+\frac{1}{3}}_{ij}$. When we solve this system, we can acquire $C^{s+\frac{1}{3}}$. To find the vectors $C^{s+\frac{1}{2}}$ and C^{s+1}, the similar process is applied for the second and third equations in Eq. (22). The solution C^{s+1} is the vector coefficient in the numerical solution of $v(x,y,t_s)$ in each step for $s = 0,1,2,3,\dots$.

5. Test problems

For measuring efficiency of the presented method, the absolute error and the maximum error L_∞ at some points are used [33–35].

Test Problem 1. As a first case, we consider the Burgers' equation :

$$\frac{\partial v}{\partial t} + v\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \tag{30}$$

The analytical solution of this problem is given in [26]:

$$v(x,y,t) = \frac{1}{1-t}\left(\frac{1-x-y}{2}\right).$$

For $k = 2, M = 3$ and different γ values, applying the presented solution procedure to Eq. (30), we obtain the numerical results in the Tables 1, 2, and 3. In Table 3, the comparison of the error norms using different γ values is shown according to [26] for $\Delta t = 0.01$. According to Table 3, it is obvious that the presented method is more accurate and has faster convergence than the Haar wavelet collocation method in [26]. For $k = 2, M = 3, \Delta t = 0.01$ and $\gamma = 1/2$, graphs of the obtained numerical and exact solutions are shown at the final time $t = 0.9$ in Fig. 1.

Test Problem 2. Let us consider the Fisher–KPP equation :

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + v(1-v). \tag{31}$$

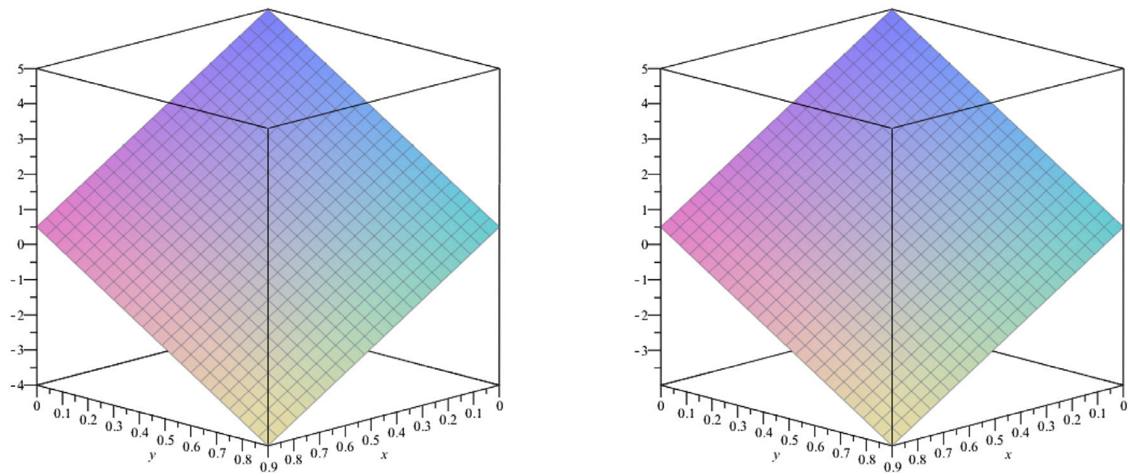

Fig. 1. Comparison of the exact and approximate solutions with $\gamma = 1/2$ and $\Delta t = 0.01$ at the time $t = 0.9$ for Test Problem 1.

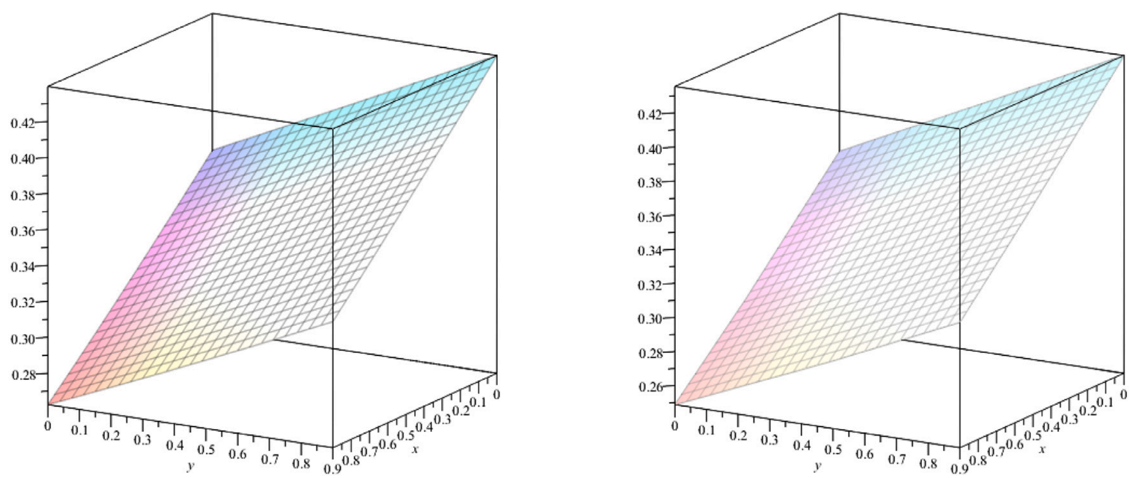

Fig. 2. Comparison of the exact and approximate solutions with $\gamma = 1/2$ and $\Delta t = 0.005$ at the time $t = 0.5$ for Test Problem 2.

Table 3
Comparison of the three-step Ultraspherical wavelet collocation method with the Haar wavelet collocation method.

$\Delta t = 0.01$				
t	Suggested method ($\gamma = 1/2$)	Suggested method ($\gamma = 3/2$)	Suggested method ($\gamma = 5/2$)	[26]
0.1	4.7220×10^{-7}	5.0090×10^{-7}	5.5380×10^{-7}	7.5480×10^{-5}
0.2	1.4315×10^{-6}	1.5623×10^{-6}	1.5117×10^{-6}	1.0783×10^{-4}
0.3	3.3271×10^{-6}	3.7422×10^{-6}	4.6844×10^{-6}	1.5907×10^{-4}
0.4	7.2238×10^{-6}	8.3827×10^{-6}	1.1952×10^{-5}	2.4867×10^{-4}
0.5	1.5964×10^{-5}	1.9213×10^{-5}	2.8731×10^{-5}	4.2063×10^{-4}
0.6	3.8330×10^{-5}	4.7730×10^{-5}	7.0996×10^{-5}	7.9658×10^{-4}
0.7	1.0825×10^{-4}	1.3921×10^{-4}	1.9868×10^{-4}	1.7984×10^{-3}
0.8	4.2752×10^{-4}	5.6479×10^{-4}	7.5555×10^{-4}	5.5532×10^{-3}
0.9	4.0702×10^{-3}	5.5108×10^{-3}	6.8810×10^{-3}	3.5729×10^{-2}

Table 4
Comparison of the absolute errors for $\Delta t = 0.01$ and $\gamma = 1/2$.

$\left\lvert v_{exact} - v_{\gamma=1/2} \right\rvert$				
(x, y)	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
(0.1, 0.1)	4.0324×10^{-3}	7.3954×10^{-3}	9.7839×10^{-3}	1.0966×10^{-2}
(0.2, 0.2)	3.8092×10^{-3}	7.3159×10^{-3}	9.8855×10^{-3}	1.1272×10^{-2}
(0.3, 0.3)	3.8675×10^{-3}	7.4579×10^{-3}	1.0128×10^{-2}	1.1624×10^{-2}
(0.4, 0.4)	4.2075×10^{-3}	7.8218×10^{-3}	1.0513×10^{-2}	1.2020×10^{-2}
(0.5, 0.5)	4.8875×10^{-3}	8.4836×10^{-3}	1.1131×10^{-2}	1.2564×10^{-2}
(0.6, 0.6)	4.3295×10^{-3}	8.1106×10^{-3}	1.1006×10^{-2}	1.2735×10^{-2}
(0.7, 0.7)	4.0767×10^{-3}	7.9901×10^{-3}	1.1059×10^{-2}	1.2992×10^{-2}
(0.8, 0.8)	4.1294×10^{-3}	8.1226×10^{-3}	1.1292×10^{-2}	1.3336×10^{-2}
(0.9, 0.9)	4.4877×10^{-3}	8.5083×10^{-3}	1.1705×10^{-2}	1.3768×10^{-2}

The exact solution of the problem is given by the following [25]:

$$v(x, y, t) = \left(1 + exp\left(\frac{\left(x - y/\sqrt{2}\right) - \left(5/\sqrt{6}\right)t}{\sqrt{6}}\right)\right)^{-2}.$$

For $k = 2, M = 2$ and $\gamma = 1/2$, we apply the presented solution procedure to Eq. (31). For $k = 2, M = 2, \Delta t = 0.005$ and $\gamma = 1/2$, graphs of the obtained numerical and exact solutions are shown at $t = 0.5$ in Fig. 2. Applying the solution procedure on Section 4 to Eq. (31) for $k = 2, M = 2, \Delta t = 0.01$ and $\gamma = 1/2$, we get the results in Table 4.

Test Problem 3. We consider the following Quasilinear Parabolic equation :

$$\frac{\partial v}{\partial t} = v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{32}$$

The analytical solution of this problem is given in [26]:

$$v(x, y, t) = \frac{1}{1-t}\left(\frac{1 + x^2 + y^2}{4} + xy + x + y\right).$$

Table 5
Maximum errors for $\Delta t = 0.001$ and different values of γ.

t	Suggested method ($\gamma = 1/2$)	Suggested method ($\gamma = 3/2$)
0.01	8.0000×10^{-8}	2.5000×10^{-8}
0.02	3.1600×10^{-7}	7.3000×10^{-8}
0.03	7.6200×10^{-7}	1.5300×10^{-7}
0.04	1.5070×10^{-6}	2.2600×10^{-7}
0.05	2.5730×10^{-6}	3.4700×10^{-7}
0.06	4.1040×10^{-6}	5.3000×10^{-7}
0.07	6.2490×10^{-6}	8.4200×10^{-7}
0.08	9.1300×10^{-6}	1.3200×10^{-6}
0.09	1.2979×10^{-5}	1.9880×10^{-6}

Table 6
Errors norms for $k = 2, M = 3, \gamma = 1/2, \gamma = 3/2$ and different values of Δt at $t = 0.1$.

Δt	L_∞ ($\gamma = 1/2$)	L_∞ ($\gamma = 3/2$)
0.001	4.215×10^{-6}	2.836×10^{-6}
0.01	6.621×10^{-6}	1.062×10^{-5}
0.1	4.833×10^{-4}	4.844×10^{-4}

When Table 5 is examined, it can be said that there is not much change between the obtained errors for $\gamma = 1/2$ and $\gamma = 3/2$. According to Table 6, the error norm decreases when the time step Δt is reduced from 0.1 to 0.001.

6. Conclusion

In this research paper, for the solution of some nonlinear multi-dimensional parabolic partial differential equations, a numerical method using a combination of the three-step Taylor method with the Ultraspherical wavelet collocation method is presented. Since there is no complex methodology in this method, the application of the presented method is quite simple. Moreover, this method is flexible, reliable, fast, and convenient alternative method for partial differential equations. To show the achievement of the presented method, we applied the presented method to three problems. All of the calculations of the presented method were executed successfully by MAPLE.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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