

Article

On Leonardo Pisano Hybrinomials

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Abstract: A generalization of complex, dual, and hyperbolic numbers has recently been defined as hybrid numbers. In this study, using the Leonardo Pisano numbers and hybrid numbers we investigate Leonardo Pisano polynomials and hybrinomials. Furthermore, we also describe the basic algebraic properties and some identities of the Leonardo Pisano polynomials and hybrinomials.

Keywords: hybrid numbers; Leonardo Pisano numbers; hybrinomials



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1. Introduction

Fibonacci numbers, for $n \geq 2$, are given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0, F_1 = 1$. There are many works regarding Fibonacci numbers, such as [1–3]. Furthermore, they have also been studied on different number systems such as quaternions and hybrid numbers [4,5].

When the literature is examined, it is seen that Leonardo Pisano numbers are only called Leonardo numbers. However, the name Leonardo in these studies refers to Leonardo Pisano, who is also called Fibonacci, from the Latin Filius Bonacci. It is well known that Leonardo commonly stands for Leonardo Da Vinci, so in this study to avoid confusion we prefer to use the name “Leonardo Pisano”.

Leonardo Pisano numbers have recently been studied in detail by Catarino and Borges in [6] and they have demonstrated some properties of this sequence. For $n \geq 2$, Leonardo Pisano numbers are defined by the following recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \quad (1)$$

where the initial conditions are $Le_0 = Le_1 = 1$. Moreover, another recurrence relation for Leonardo Pisano's numbers is

$$Le_{n+1} = 2Le_n - Le_{n-2}. \quad (2)$$

The relationship between Leonardo Pisano numbers and Fibonacci numbers can be given as

$$Le_n = 2F_{n+1} - 1. \quad (3)$$

Binet's formula and the characteristic equation of Leonardo Pisano numbers are

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta} \quad \text{and} \quad \lambda^3 - 2\lambda^2 + 1 = 0 \quad (4)$$

respectively. Cassini's, Catalan and d'Ocagne's identities were defined in [6]. In [7] generalized Leonardo Pisano numbers have been defined. In [8], incomplete Leonardo Pisano numbers and matrix representation of Leonardo Pisano numbers have also been

given. Additionally, in [9] the authors give some important properties of Leonardo Pisano numbers and in [10] hybrid Leonardo Pisano numbers are defined.

Hybrid numbers are a generalization of the complex, dual and hyperbolic numbers. A hybrid number can be denoted by $w = r_0 + r_1\mathbf{i} + r_2\boldsymbol{\varepsilon} + r_3\mathbf{h}$, where r_0, r_1, r_2, r_3 are real numbers [11]. Hybrid numbers, \mathbb{K} , are defined as

$$\mathbb{K} = \left\{ w = r_0 + r_1\mathbf{i} + r_2\boldsymbol{\varepsilon} + r_3\mathbf{h} : r_0, r_1, r_2, r_3 \in \mathbb{R}, \begin{matrix} \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1 \\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i} \end{matrix} \right\}. \tag{5}$$

Equality, addition and multiplication are defined by

- (i) $w_1 = w_2 \iff r_0 = s_0, r_1 = s_1, r_2 = s_2, r_3 = s_3,$
- (ii) $w_1 + w_2 = (r_0 + s_0) + (r_1 + s_1)\mathbf{i} + (r_2 + s_2)\boldsymbol{\varepsilon} + (r_3 + s_3)\mathbf{h},$
- (iii) $w_1 \cdot w_2 = (r_0 + r_1\mathbf{i} + r_2\boldsymbol{\varepsilon} + r_3\mathbf{h}) \cdot (s_0 + s_1\mathbf{i} + s_2\boldsymbol{\varepsilon} + s_3\mathbf{h})$
 $= (r_0s_0 - r_1s_1 + r_2s_1 - r_1s_2 + r_3s_3) + (r_0s_2 + r_1s_0 + r_1s_3 - r_3s_1)\mathbf{i}$
 $+ (r_0s_2 + r_2s_0 - r_2s_3 + r_3s_2 + r_1s_3 - r_3s_1)\boldsymbol{\varepsilon} + (r_0s_3 + r_3s_0 + r_1s_2 + r_2s_1)\mathbf{h},$

for hybrid numbers $w_1 = r_0 + r_1\mathbf{i} + r_2\boldsymbol{\varepsilon} + r_3\mathbf{h}$ and $w_2 = s_0 + s_1\mathbf{i} + s_2\boldsymbol{\varepsilon} + s_3\mathbf{h}$. More details on hybrid numbers can be found in [11].

Fibonacci polynomials were first studied by Bicknell [12]. Since then, many authors have published works on the subject of Fibonacci-type polynomials [13–17]. These types of polynomials have some considerable applications in number theory, geometry and algebra. Moreover, hybrid sequences have become an increasing area of attention. For instance, the authors studied Fibonacci and Lucas hybrid numbers [18], Petroudi and Pirouz studied Van der Laan hybrid numbers [19] and Petroudi et al. studied Narayana polynomials and Narayana hybrid sequences [20]. Moreover, Horadam hybrid numbers were studied by Kızılateş in [21].

In this work, in Section 2, we define the Leonardo Pisano polynomials and hybrid numbers. Then, in Section 3, we present some identities of Leonardo Pisano polynomial and hybrid sequences such as Catalan-like identities and summing formulas.

2. Leonardo Pisano Polynomials and Hybrid Numbers

In this section, we introduce Leonardo Pisano polynomials and hybrid numbers.

Definition 1. For a non-negative integer n , Leonardo Pisano polynomials $Le_n(x)$ are defined by

$$Le_n(x) = \begin{cases} 1 & , n = 0, 1 \\ x + 2 & , n = 2 \\ 2xLe_{n-1}(x) - Le_{n-3}(x) & , n \geq 3 \end{cases} .$$

The first few Leonardo Pisano polynomials are $1, 1, x + 2, 2x^2 + 4x - 1, 4x^3 + 8x^2 - 2x - 1, 8x^4 + 16x^3 - 4x^2 - 3x - 2, 16x^5 + 32x^4 - 8x^3 - 8x^2 - 8x + 1$. Using Leonardo Pisano polynomials, we can define Leonardo Pisano hybrid numbers.

Definition 2. The n th Leonardo Pisano hybrid number is defined as follows:

$$Le_n^{[H]}(x) = Le_n(x) + \mathbf{i}Le_{n+1}(x) + \boldsymbol{\varepsilon}Le_{n+2}(x) + \mathbf{h}Le_{n+3}(x). \tag{6}$$

The first four Leonardo Pisano hybrid numbers are

$$Le_0^{[H]}(x) = 1 + \mathbf{i} + \varepsilon(x + 2) + \mathbf{h}(2x^2 + 4x - 1) \tag{7}$$

$$Le_1^{[H]}(x) = 1 + \mathbf{i}(x + 2) + \varepsilon(2x^2 + 4x - 1) + \mathbf{h}(4x^3 + 8x^2 - 2x - 1) \tag{8}$$

$$Le_2^{[H]}(x) = (x + 2) + \mathbf{i}(2x^2 + 4x - 1) + \varepsilon(4x^3 + 8x^2 - 2x - 1) + \mathbf{h}(8x^4 + 16x^3 - 4x^2 - 3x - 2) \tag{9}$$

$$Le_3^{[H]}(x) = (2x^2 + 4x - 1) + \mathbf{i}(4x^3 + 8x^2 - 2x - 1) + \varepsilon(8x^4 + 16x^3 - 4x^2 - 3x - 2) + \mathbf{h}(16x^5 + 32x^4 - 8x^3 - 8x^2 - 8x + 1). \tag{10}$$

If we put $x = 1$, then we obtain

$$Le_0^{[H]}(x) = 1 + \mathbf{i} + 3\varepsilon + 5\mathbf{h} = LeH_0$$

$$Le_1^{[H]}(x) = 1 + 3\mathbf{i} + 5\varepsilon + 9\mathbf{h} = LeH_1$$

$$Le_2^{[H]}(x) = 3 + 5\mathbf{i} + 9\varepsilon + 15\mathbf{h} = LeH_2$$

$$Le_3^{[H]}(x) = 5 + 9\mathbf{i} + 15\varepsilon + 25\mathbf{h} = LeH_3$$

where LeH_n stand for Leonardo Pisano hybrid numbers. Therefore, it can be easily seen that Leonardo Pisano hybrid numbers are a generalization of Leonardo Pisano hybrid numbers.

We can define the character of Leonardo Pisano hybrid numbers by making use of the character of hybrid numbers that Özdemiş defined in his article.

Definition 3. The character of Leonardo Pisano hybrid numbers is

$$C(Le_n^{[H]}(x)) = Le_n^2(x) + Le_{n+1}^2(x) - 2Le_{n+1}(x)Le_{n+2}(x) - Le_{n+3}^2(x).$$

Using the character of the Leonardo Pisano hybrid numbers, we can set the norm.

Theorem 1. The norm of the Leonardo Pisano hybrid numbers $Le_n^{[H]}(x)$ is given by

$$\|Le_n^{[H]}(x)\| = \sqrt{|Le_{n+1}^2(x) - 2Le_{n+1}(x)Le_{n+2}(x) - 4x^2Le_{n+2}^2(x) + 4xLe_{n+2}(x)Le_n(x)|}.$$

Proof. From the character of Leonardo Pisano hybrid numbers which is given above,

$$\begin{aligned} \|Le_n^{[H]}(x)\|^2 &= |Le_n^2(x) + Le_{n+1}^2(x) - 2Le_{n+1}(x)Le_{n+2}(x) - Le_{n+3}^2(x)| \\ &= |Le_n^2(x) + Le_{n+1}(x) - 2Le_{n+1}(x)Le_{n+2}(x) - (2xLe_{n+2}(x) - Le_n(x))^2| \\ &= |Le_{n+1}^2(x) - 2Le_{n+1}(x)Le_{n+2}(x) - 4x^2Le_{n+2}^2(x) + 4xLe_{n+2}(x)Le_n(x)|. \end{aligned}$$

□

In the following theorem using the matrix representation of hybrid numbers:

$$M(a + b\mathbf{i} + c\varepsilon + d\mathbf{h}) = \begin{bmatrix} a + c & b - c + d \\ c - b + d & a - c \end{bmatrix},$$

we give the matrix representation for Leonardo Pisano hybrid numbers.

Theorem 2. Every Leonardo Pisano hybrid numbers can be written in the following matrix form, for $Le_m^{[H]}(x)$

$$M(Le_m^{[H]}(x)) = \begin{bmatrix} Le_m(x) + Le_{m+2}(x) & Le_{m+1}(x) - Le_{m+2}(x) + Le_{m+3}(x) \\ Le_{m+2}(x) + Le_{m+3}(x) - Le_{m+1}(x) & Le_m(x) - Le_{m+2}(x) \end{bmatrix}.$$

Proof. Using the matrix equivalent of hybrid units $1, \mathbf{i}, \varepsilon$ and \mathbf{h} , we have

$$\begin{aligned}
 Le_m^{[H]}(x) &= Le_m(x) + \mathbf{i}Le_{m+1}(x) + \varepsilon Le_{m+2}(x) + \mathbf{h}Le_{m+3}(x) \\
 M(Le_m^{[H]}(x)) &= Le_m(x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Le_{m+1}(x) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + Le_{m+2}(x) \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\
 &\quad + Le_{m+3}(x) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
 \end{aligned}$$

then we can write

$$M(Le_m^{[H]}(x)) = \begin{bmatrix} Le_m(x) + Le_{m+2}(x) & Le_{m+1}(x) - Le_{m+2}(x) + Le_{m+3}(x) \\ Le_{m+2}(x) + Le_{m+3}(x) - Le_{m+1}(x) & Le_m(x) - Le_{m+2}(x) \end{bmatrix}$$

□

Lemma 1. Let $Le_n^{[H]}(x)$ be the n th Leonardo Pisano hybridnomial. The recurrence relation of $Le_n^{[H]}(x)$ is

$$Le_n^{[H]}(x) = 2xLe_{n-1}^{[H]}(x) - Le_{n-3}^{[H]}(x).$$

Proof. Using equation (6);

$$\begin{aligned}
 Le_n^{[H]}(x) &= Le_n(x) + \mathbf{i}Le_{n+1}(x) + \varepsilon Le_{n+2}(x) + \mathbf{h}Le_{n+3}(x) \\
 &= 2xLe_{n-1}(x) - Le_{n-3}(x) + \mathbf{i}\{2xLe_n(x) - Le_{n-2}(x)\} \\
 &\quad + \varepsilon\{2xLe_{n+1}(x) - Le_{n-1}(x)\} + \mathbf{h}\{2xLe_{n+2}(x) - Le_n(x)\} \\
 &= 2xLe_{n-1}^{[H]}(x) - Le_{n-3}^{[H]}(x).
 \end{aligned}$$

□

Theorem 3. Let $Le_n(x)$ be the n th Leonardo Pisano polynomial. The generating function of $Le_n(x)$ is defined as

$$g_{Le_n(x)}(\lambda) = \sum_{n=0}^{\infty} Le_n(x)\lambda^n = \frac{1 + (1 - 2x)\lambda + (2 - x)\lambda^2}{1 - 2x\lambda + \lambda^3}.$$

Proof. Assume that the generating function of Leonardo Pisano polynomial series $\{Le_n(x)\}$ is,

$$g_{Le_n(x)}(\lambda) = \sum_{n=0}^{\infty} Le_n(x)\lambda^n = Le_0(x) + Le_1(x)\lambda + Le_2(x)\lambda^2 + \dots \tag{11}$$

Using this form, we obtain

$$-2x\lambda g_{Le_n(x)}(\lambda) = -2x\lambda Le_0(x) - 2x\lambda^2 Le_1(x) - 2x\lambda^3 Le_2(x) - \dots, \tag{12}$$

$$\lambda^3 g_{Le_n(x)}(\lambda) = \lambda^3 Le_0(x) + \lambda^4 Le_1(x) + \lambda^5 Le_2(x) + \dots. \tag{13}$$

If we sum these Equations (11)–(13) side by side, then we obtain

$$\begin{aligned}
 [1 - 2x\lambda + \lambda^3]g_{Le_n(x)}(\lambda) &= Le_0(x) + Le_1(x)\lambda + Le_2(x)\lambda^2 - 2xLe_0(x)\lambda - 2xLe_1(x)\lambda^2 \\
 &= Le_0(x) + \lambda(Le_1(x) - 2xLe_0(x)) + \lambda^2(Le_2(x) - 2xLe_1(x)) \\
 &= 1 + (1 - 2x)\lambda + (2 - x)\lambda^2
 \end{aligned} \tag{14}$$

Then the proof is completed. □

Corollary 1. Let $Le_n^{[H]}(x)$ be the n th Leonardo Pisano hybrinomial. The generating function of $Le_n^{[H]}(x)$ is

$$G_{Le_n^{[H]}(x)}(\lambda) = \frac{Le_0^{[H]}(x) + \lambda[Le_1^{[H]}(x) - 2xLe_0^{[H]}(x)] + \lambda^2[Le_2^{[H]}(x) - 2xLe_1^{[H]}(x)]}{1 - 2x\lambda^2 + \lambda^3}.$$

Theorem 4. Let $Le_n(x)$ be the n th Leonardo Pisano polynomial. The Binet-like formula for the sequence $\{Le_n(x)\}$ is

$$Le_n(x) = A\alpha^n + B\beta^n + C\gamma^n$$

where α, β and γ are the roots of the characteristic equation $\lambda^3 - 2x\lambda^2 + 1 = 0$ and A, B, C are as follows:

$$A = \frac{\alpha^2 + (2 - x)\alpha + (x + 2)}{(\alpha - \beta)(\alpha - \gamma)}, \tag{15}$$

$$B = \frac{\beta^2 + (2 - x)\beta + (x + 2)}{(\beta - \alpha)(\beta - \gamma)}, \tag{16}$$

$$C = \frac{\gamma^2 + (2 - x)\gamma + (x + 2)}{(\gamma - \alpha)(\gamma - \beta)}. \tag{17}$$

Proof. Let $f(\lambda) = \lambda^3 - 2x\lambda^2 + 1 = 0$ be the characteristic equation of the recurrence relation $Le_n(x) = 2xLe_{n-1}(x) - Le_{n-3}(x)$. It is obvious that $f(\lambda)$ should have three distinct roots α, β and γ . Then, $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of the equation

$$f(\lambda) = \lambda^3 - 2x\lambda^2 + 1 = 0 \tag{18}$$

$$f\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^3} - \frac{2x}{\lambda^2} + 1 = \frac{\lambda^3 - 2x\lambda + 1}{\lambda^3} = 0 \tag{19}$$

Since $\lambda^3 \neq 0$, then $h(\lambda) = \lambda^3 - 2x\lambda + 1$.

From the generating function of Leonardo Pisano polynomials and from the sum of geometric series, we have

$$\begin{aligned} g_{Le_n(x)}(\lambda) &= \frac{1 + (1 - 2x)\lambda + (2 - x)\lambda^2}{1 - 2x\lambda + \lambda^3} = \frac{A}{1 - \alpha\lambda} + \frac{B}{1 - \beta\lambda} + \frac{C}{1 - \gamma\lambda} \\ &= A \sum_{n=0}^{\infty} (\alpha\lambda)^n + B \sum_{n=0}^{\infty} (\beta\lambda)^n + C \sum_{n=0}^{\infty} (\gamma\lambda)^n \\ &= \frac{A(1 - \beta\alpha)(1 - \gamma\lambda) + B(1 - \alpha\lambda)(1 - \gamma\lambda) + C(1 - \alpha\lambda)(1 - \beta\lambda)}{(1 - \alpha\lambda)(1 - \beta\lambda)(1 - \gamma\lambda)}. \end{aligned} \tag{20}$$

Then we have the following equality

$$1 + (1 - 2x)\lambda + (2 - x)\lambda^2 = A(1 - \beta\alpha)(1 - \gamma\lambda) + B(1 - \alpha\lambda)(1 - \gamma\lambda) + C(1 - \alpha\lambda)(1 - \beta\lambda).$$

If we substitute λ by $\frac{1}{\alpha}$, we obtain

$$1 + (1 - 2x)\frac{1}{\alpha} + (2 - x)\frac{1}{\alpha^2} = A\left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\gamma}{\alpha}\right) \tag{21}$$

$$\alpha^2 + (1 - 2x)\alpha + (2 - x) = A(\alpha - \beta)(\alpha - \gamma) \tag{22}$$

then it is easy to obtain

$$A = \frac{\alpha^2 + (1 - 2x)\alpha + (2 - x)}{(\alpha - \beta)(\alpha - \gamma)}. \tag{23}$$

In the same way, we can obtain B and C as follows:

$$B = \frac{\beta^2 + (1 - 2x)\beta + (2 - x)}{(\beta - \alpha)(\beta - \gamma)}, \quad C = \frac{\gamma^2 + (1 - 2x)\gamma + (2 - x)}{(\gamma - \alpha)(\gamma - \beta)}.$$

If we substitute A, B, C into (20), we obtain the following equation

$$\begin{aligned} g_{Le_n(x)}(\lambda) &= A \sum_{n=0}^{\infty} (\alpha\lambda)^n + B \sum_{n=0}^{\infty} (\beta\lambda)^n + C \sum_{n=0}^{\infty} (\gamma\lambda)^n \\ &= \sum_{n=0}^{\infty} [A\alpha^n + B\beta^n + C\gamma^n] \lambda^n \end{aligned}$$

Consequently, we get

$$Le_n(x) = A\alpha^n + B\beta^n + C\gamma^n. \tag{24}$$

□

Theorem 5. The Binet-like formula for the Leonardo Pisano hybrinomial sequence is

$$Le_n^{[H]}(x) = A\underline{\alpha}\alpha^n + B\underline{\beta}\beta^n + C\underline{\gamma}\gamma^n$$

where $\underline{\alpha} = 1 + \alpha i + \alpha^2 \varepsilon + \alpha^3 \mathbf{h}$, $\underline{\beta} = 1 + \beta i + \beta^2 \varepsilon + \beta^3 \mathbf{h}$ and $\underline{\gamma} = 1 + \gamma i + \gamma^2 \varepsilon + \gamma^3 \mathbf{h}$. Additionally, A, B and C are as in (17).

Proof. We have $Le_n^{[H]}(x) = Le_n(x) + iLe_{n+1}(x) + \varepsilon Le_{n+2}(x) + \mathbf{h}Le_{n+3}(x)$ from (6) and $Le_n(x) = A\alpha^n + B\beta^n + C\gamma^n$ from the previous theorem.

$$\begin{aligned} Le_n^{[H]}(x) &= (A\alpha^n + B\beta^n + C\gamma^n) + (A\alpha^{n+1} + B\beta^{n+1} + C\gamma^{n+1})i \\ &\quad + (A\alpha^{n+2} + B\beta^{n+2} + C\gamma^{n+2})\varepsilon + (A\alpha^{n+3} + B\beta^{n+3} + C\gamma^{n+3})\mathbf{h} \\ &= A\underline{\alpha}\alpha^n + B\underline{\beta}\beta^n + C\underline{\gamma}\gamma^n. \end{aligned}$$

□

3. Some Identities for Leonardo Pisano Polynomial and Hybrinomial Sequences

In this section we state some identities, such as Cassini-like, d’Ocagne-like identities, where Leonardo Pisano polynomials and hybrinomials are involved.

Theorem 6 (Sum identities). Let $n \geq 1$ be an integer, $\{Le_k(x)\}$ is the Leonardo Pisano polynomial sequence and $\{Le_k^{[H]}(x)\}$ is the Leonardo Pisano hybrinomial sequence, respectively. Then

- i. $\sum_{k=0}^n Le_k(x) = \frac{1}{2-2x} (4 - 3x + Le_{n-1}(x) + Le_n(x) - Le_{n+1}(x)),$
- ii. $\sum_{k=0}^n Le_k^{[H]}(x) = \frac{1}{2-2x} (Le_0^{[H]}(x) + Le_1^{[H]}(x) + Le_2^{[H]}(x) - 2xLe_0^{[H]}(x) - 2xLe_1^{[H]}(x) + Le_{n-1}^{[H]}(x) + Le_n^{[H]}(x) - Le_{n+1}^{[H]}(x)).$

Proof.

- i. From the definition of Leonardo Pisano polynomial sequence

$$Le_n(x) = 2xLe_{n-1}(x) - Le_{n-3}(x)$$

we have the following equations

$$\begin{aligned} Le_3(x) &= 2xLe_2(x) - Le_0(x) \\ Le_4(x) &= 2xLe_3(x) - Le_1(x) \\ Le_5(x) &= 2xLe_4(x) - Le_2(x) \\ &\vdots \quad \quad \quad \vdots \\ Le_n(x) &= 2xLe_{n-1}(x) - Le_{n-3}(x). \end{aligned}$$

when we sum up these equations side by side we get

$$\sum_{k=3}^n Le_k(x) = 2x \sum_{k=2}^{n-1} Le_k(x) - \sum_{k=0}^{n-3} Le_k(x) \tag{25}$$

In order to equalize the indexes, we add $P_1 = Le_0(x) + Le_1(x) + Le_2(x)$, $P_2 = 2x(Le_0(x) + Le_1(x) + Le_n(x))$, and $P_3 = -(Le_{n-2}(x) + Le_{n-1}(x) + Le_n(x))$ to both sides. Therefore we can write

$$P_2 + P_3 + \sum_{k=0}^n Le_k(x) = P_1 + 2x \sum_{k=0}^n Le_k(x) - \sum_{k=0}^n Le_k(x) \tag{26}$$

Eventually, we obtain the desired result.

- ii. The proof can be demonstrated the same way. □

We can verify the previous theorem with the following example.

Example 1. The sum of the first four Leonardo Pisano polynomials $Le_0 = 1, Le_1 = 1, Le_2 = x + 2, Le_3 = 2x^2 + 4x - 1$ is $2x^2 + 5x + 3$. Indeed, we can verify this result using the previous theorem as follows:

$$\begin{aligned} \sum_{k=0}^3 Le_k(x) &= \frac{4 - 3x + x + 2 + 2x^2 + 4x - 1 - 4x^3 - 8x^2 + 2x + 1}{2 - 2x} \\ &= \frac{-4x^3 - 6x^2 + 4x + 6}{2 - 2x} = 2x^2 + 5x + 3. \end{aligned}$$

Theorem 7 (Catalan-like Identities). Let n and r be two non-negative integers and $n \geq r$.

- i. The Catalan-like identity for Leonardo Pisano polynomials is

$$\begin{aligned} Le_{n+r}(x)Le_{n-r}(x) - Le_n^2(x) &= AB\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 \\ &\quad + AC\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 \\ &\quad + BC\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2. \end{aligned}$$

- ii. The Catalan-like identity for Leonardo Pisano hybrinomials is

$$\begin{aligned} Le_{n+r}^{[H]}(x)Le_{n-r}^{[H]}(x) - (Le_n^{[H]}(x))^2 &= A^2\alpha^{2n}\underline{\alpha}^2 + B^2\beta^{2n}\underline{\beta}^2 + C^2\gamma^{2n}\underline{\gamma}^2 \\ &\quad + AB\alpha^{n-r}\beta^{n-r}(\alpha^{2r}\underline{\alpha}\underline{\beta} + \beta^{2r}\underline{\beta}\underline{\alpha}) \\ &\quad + AC\alpha^{n-r}\gamma^{n-r}(\alpha^{2r}\underline{\alpha}\underline{\gamma} + \gamma^{2r}\underline{\gamma}\underline{\alpha}) \\ &\quad + BC\beta^{n-r}\gamma^{n-r}(\beta^{2r}\underline{\beta}\underline{\gamma} + \gamma^{2r}\underline{\gamma}\underline{\beta}). \end{aligned}$$

Proof. Proofs can be easily seen if the Binet-like formula is used for Leonardo Pisano polynomials and Leonardo Pisano hybrinomials. □

Corollary 2 (Cassini-like identities). *Let n be a positive integer.*

i. The Cassini-like identity for Leonardo Pisano polynomials is

$$\begin{aligned} Le_{n+1}(x)Le_{n-1}(x) - Le_n^2(x) &= AB\alpha^{n-1}\beta^{n-1}(\alpha - \beta)^2 \\ &+ AC\alpha^{n-1}\gamma^{n-1}(\alpha - \gamma)^2 \\ &+ BC\beta^{n-1}\gamma^{n-1}(\beta - \gamma)^2. \end{aligned}$$

ii. The Cassini-like identity for Leonardo Pisano hybrinomials is

$$\begin{aligned} Le_{n+1}^{[H]}(x)Le_{n-1}^{[H]}(x) - (Le_n^{[H]}(x))^2 &= A^2\alpha^{2n}\underline{\alpha}^2 + B^2\beta^{2n}\underline{\beta}^2 + C^2\gamma^{2n}\underline{\gamma}^2 \\ &+ AB\alpha^{n-1}\beta^{n-1}(\alpha^2\underline{\alpha}\underline{\beta} + \beta^2\underline{\beta}\underline{\alpha}) \\ &+ AC\alpha^{n-1}\gamma^{n-1}(\alpha^2\underline{\alpha}\underline{\gamma} + \gamma^2\underline{\gamma}\underline{\alpha}) \\ &+ BC\beta^{n-1}\gamma^{n-1}(\beta^2\underline{\beta}\underline{\gamma} + \gamma^2\underline{\gamma}\underline{\beta}). \end{aligned}$$

Theorem 8 (d’Ocagne-like identities). *Let m and n be two non-negative integers.*

i. The d’Ocagne-like identity for Leonardo Pisano polynomials is

$$\begin{aligned} Le_m(x)Le_{n+1}(x) - Le_{m+1}(x)Le_n(x) &= AB(\alpha - \beta)(\alpha^n\beta^m - \alpha^m\beta^n) \\ &+ AC(\alpha - \gamma)(\alpha^n\gamma^m - \alpha^m\gamma^n) \\ &+ BC(\beta - \gamma)(\beta^n\gamma^m - \beta^m\gamma^n). \end{aligned}$$

ii. The d’Ocagne-like identity for Leonardo Pisano hybrinomials is

$$\begin{aligned} Le_m^{[H]}(x)Le_{n+1}^{[H]}(x) - Le_{m+1}^{[H]}(x)Le_n^{[H]}(x) &= AB(\alpha - \beta)(\alpha^n\beta^m\underline{\alpha}\underline{\beta} - \alpha^m\beta^n\underline{\beta}\underline{\alpha}) \\ &+ AC(\alpha - \gamma)(\alpha^n\gamma^m\underline{\alpha}\underline{\gamma} - \alpha^m\gamma^n\underline{\gamma}\underline{\alpha}) \\ &+ BC(\beta - \gamma)(\beta^n\gamma^m\underline{\beta}\underline{\gamma} - \beta^m\gamma^n\underline{\gamma}\underline{\beta}). \end{aligned}$$

Proof. i. Let us use the Binet-like formula for Leonardo Pisano polynomials.

$$\begin{aligned} &Le_m(x)Le_{n+1}(x) - Le_{m+1}(x)Le_n(x) \\ &= AB\alpha^m\beta^n(\beta - \alpha) + AC\alpha^m\gamma^n(\gamma - \alpha) + AB\alpha^n\beta^m(\alpha - \beta) + BC\beta^m\gamma^n(\gamma - \beta) \\ &+ AC\alpha^n\gamma^m(\alpha - \gamma) + BC\beta^n\gamma^m(\beta - \gamma) \\ &= AB(\alpha - \beta)(\alpha^n\beta^m - \alpha^m\beta^n) + AC(\alpha - \gamma)(\alpha^n\gamma^m - \alpha^m\gamma^n) \\ &+ BC(\beta - \gamma)(\beta^n\gamma^m - \beta^m\gamma^n). \end{aligned}$$

ii. This can be proven similarly to (i) using the Binet-like formula for Leonardo Pisano hybrinomials.

□

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