Flexible Time-Varying Betas in a Novel Mixture Innovation Factor Model with Latent Threshold

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Abstract: This paper introduces a new methodology to estimate time-varying alphas and betas in conditional factor models, which allows substantial flexibility in a time-varying framework. To circumvent problems associated with the previous approaches, we introduce a Bayesian time-varying parameter model where innovations of the state equation have a spike-and-slab mixture distribution. The mixture distribution specifies two states with a specific probability. In the first state, the innovation variance is set close to zero with a certain probability and parameters stay relatively constant. In the second state, the innovation variance is large and the change in parameters is normally distributed with mean zero and a given variance. The latent state is specified with a threshold that governs the state change. We allow a separate threshold for each parameter; thus, the parameters may shift in an unsynchronized manner such that the model moves from one state to another when the change in the parameter exceeds the threshold and vice versa. This approach offers great flexibility and nests a plethora of other time-varying model specifications, allowing us to assess whether the betas of conditional factor models evolve gradually over time or display infrequent, but large, shifts. We apply the proposed methodology to industry portfolios within a five-factor model setting and show that the threshold Capital Asset Pricing Model (CAPM) provides robust beta estimates coupled with smaller pricing errors compared to the alternative approaches. The results have significant implications for the implementation of smart beta strategies that rely heavily on the accuracy and stability of factor betas and yields.

Keywords: time-varying beta; risk premium; asset pricing; bayesian estimation; thresholds

JEL Classification: C11; C32; G11; G12; G14

1. Introduction

Factor investing or smart beta strategies that involve portfolio allocations based on factor exposures have been increasingly utilized in the finance industry with the total value of smart beta assets under management expanding by 257% from $280 billion in 2012 to over $999 billion at the end of 2017 [1]. The factor industry is estimated to grow to $3.4 trillion by 2022 [2]. These investment strategies exploit return predictability patterns driven by exposures to a given set of systematic risk factors and design portfolio allocations based on the risk exposures to those factors [3–5]. At the heart of these strategies lies the determination of the beta values of the target portfolio relative to each factor. A major challenge in this regard is the instability of the factor loadings (e.g., [6–8]), which in turn leads to incorrect portfolio allocations, exposing the strategy to unnecessary risks, thus resulting in inferior risk-adjusted returns. This paper presents a novel perspective to estimate time-varying alphas and betas in conditional factor models via the threshold

CAPM (Capital Asset Pricing Model) methodology by introducing the threshold time-varying parameter with stochastic volatility (TTVP-SV) model. Using a fundamental economic variable called a threshold variable to model factor exposures, the model accounts for distinct beta regimes in which the risk exposures remain stable. This, in turn, allows us to assess whether factor betas evolve gradually over time or display infrequent, but large, shifts. In an application to industry portfolios, we show that the TTVP-SV model indeed yields significantly lower pricing errors compared to the static and rolling regression-based alternatives.

The relationship between risk and expected return in financial markets has been examined in a large strand of the literature (e.g., [9–11]). Starting with the pioneering works of Sharpe [12] and Lintner [13] the literature has produced a plethora of multivariate models that incorporate a wide variety of firm-level, financial and macroeconomic factors as determinants of expected stock market returns. The traditional approach to these models, however, has generally adopted a static specification which ignores the time-variation in factor exposures as well as factor yields. This assumption, however, has critical implications for factor investing (or smart beta) strategies as factor yields vary over time, and thus the beta of the portfolio with regard to the factor needs to be revised accordingly [14]. To that end, many studies make use of overt or implicit short-term memory mechanisms to simulate beta dynamics. An example is the autoregressive (AR) mechanism employed in studies by Ang and Chen [15] and Levi and Welch [16] as well as the autoregressive moving average (ARMA) framework of Pagan [17]. Extending the research to non-linear specifications, Chen et al. [6] proposed a two-stage procedure to detect breaks in factor loadings, while Han and Inoue [7] tested for structural breaks in factor exposures based on the second moments of the estimated factors. Nonetheless, the evidence indicating time-varying patterns in factor exposures is overwhelming, which is of high importance in the implementation of smart beta strategies that rely on the accuracy and stability of factor exposures associated with target portfolios.

Recent studies provide fresh support to the time-variation in beta estimates in factor models. Adopting a single factor framework and examining the role of uncertainty on industry betas, Yu et al. [18] established a link between market stress conditions, proxied by economic policy uncertainty, and industry betas, arguing that industry betas are driven by market uncertainty. More recently, utilizing a smooth transition regime-switching framework, Aslanidis and Hartigan [19] showed that the instability in factor loadings is mainly concentrated on financial variables, while Becker et al. [20] argued that accounting for long memory in beta dynamics not only provides superior beta estimates but also generates payoffs for portfolio formation. The economic value of accounting for the time-variation in beta patterns was further supported by Wang [21] who showed that a regime-based beta model conditional on market volatility can help improve diversification strategies. Extending the analysis to global stock markets, Hollstein [22] further showed that the accuracy of the estimated global and local beta values depends on the historical window size used in the estimation as well as the particular estimator employed in the analysis. Coupled with the earlier evidence by Arshanapalli et al. [23] that risk factors exhibit heterogeneous patterns during normal and distressed market conditions, one can thus argue that failing to account for the dynamic nature of factor exposures can lead to significant underperformance of smart beta strategies that have experienced remarkable growth over the past decade.

The extant literature has proposed several time-varying extensions of factor models in order to consider the time-variation in betas. Several studies explicitly model time variation via parametric models, relating betas to time-varying risk [24–27]. Others have accounted for time-variation by allowing betas to vary through a dynamic stochastic specification such as an autoregressive (AR) process, random walk or a time varying parameter (TVP) model [28–32]. Analogous to time varying parameter models, several studies estimate time-varying betas using the rolling window regression approach (see [33–37]). Another, more common, approach is based on the conditional covariance matrix wherein time-varying be-

tas are estimated via multivariate generalized autoregressive conditional heteroskedasticity (GARCH) models [38–40]. Despite the multitude of studies that address the time-variation in betas, the literature does not offer conclusive proof or evidence on the choice of the conditional models over the multivariate GARCH models [41–43].

As Bauwens et al. [44] pointed out, multivariate GARCH model are either not flexible or suffer from overparameterization. Moreover, most multivariate GARCH models do not involve time-varying correlations and those that involve dynamic correlations, such as the dynamic conditional correlation GARCH, are shown to be inconsistent and do not directly yield conditional correlations, as shown by Caporin and McAleer [45].

This study contributes to the literature by proposing a novel, threshold time-varying parameter (TTVP) model with slab and spike mixture distribution to model dynamic factor exposures. Our model is motivated by the argument by Ghysels [46] that betas change steadily and linearly over time and linear models such as the conditional CAPM tend to overestimate the pace and frequency of changes in the estimated betas, thus resulting in significant pricing errors. Building on the threshold regression framework [47–49], our model improves the conditional CAPM framework by introducing the threshold CAPM that can reliably measure the shifts in business cycles and the time-variation in betas in response to business cycle fluctuations. To that end, we use a fundamental economic variable called a threshold variable to model market risk. If the threshold variable gets to a certain level, the model is designed to allow for two distinct beta regimes during which the beta value remains stable. Given the evidence in prior works of a non-linear relationship between factor exposures and expected returns and the lack of a time-varying asset pricing model that is shown to provide accurate results, the threshold approach to modeling betas and allowing for a slowly shifting continuous beta pattern within a conditional CAPM framework presents a favorable approach to simulate beta dynamics and address the time-variation in betas within the standard asset pricing model setting.

The TTVP model we introduce remedies some of the issues in the previous studies. The conditional beta approach to estimating time-varying betas essentially employs three techniques: (i) modeling beta as a function of various relevant variables by including polynomial time functions; (ii) specifying beta as a stochastic process; and (iii) rolling regression approaches. Modeling betas as a function of various seemingly related variables is prone to misspecification errors and there is no guidance on the selection of the set of relevant variables that should be employed in the model. The rolling regression method avoids this misspecification issue; however, it has the inherent difficulty of choosing the right window size, as also highlighted recently by Hollstein [22]. The stochastic beta specification offers an alternative approach that avoids the problems associated with the other conditional beta estimation methods and one such stochastic beta specification is the time-varying parameter (TVP) model that is also adopted in our approach. The Kalman filter method for estimating the TVP model is used in a number of studies to estimate time-varying betas [28–32].

The TTVP model we advocate allows substantial flexibility in a time-varying framework. To circumvent problems associated with the previous approaches, we introduce a Bayesian time-varying parameter model where innovations of the state equation have a spike-and-slab mixture distribution. The mixture distribution specifies two states with each state having a specific probability. In the first state, the innovation variance is set close to zero with a certain probability and parameters stay relatively constant. In the second state, the innovation variance is large and the change in parameters is normally distributed with mean zero and a given larger variance value. The latent state is specified with a threshold that governs the state change. We allow a separate threshold for each parameter so that the parameters may shift in an unsynchronized manner and so the model moves from one state to another when the change in the parameter exceeds the threshold and vice versa. This approach has great flexibility and nests a plethora of other time-varying model specifications. For instance, the parameters may tend to show strong variation over a certain passage of time and stay constant over some horizons. Accordingly, the TTVP

model allows us to assess whether the betas of conditional factor models evolve gradually over time or display infrequent, but large, shifts.

In our empirical application, we implement the TTVP model to examine the dynamic factor exposures for selected U.S. industries that account for the consumption and production sides of the economy. In particular, we examine industry portfolio returns for Consumer Durables (DURBL), Manufacturing (MANUF), Energy (ENRGY) and Technology (HITEC) and estimate their time-varying factor exposures to the well-established risk factors associated with market, size, value, momentum and the recently proposed profitability and investment factors based on the five-factor specification of Fama and French [46]. To that end, we conduct a scenario analysis similar to that of Ghysels [46] and measure the pricing errors associated with the CAPM specification augmented with threshold risk. Our results show that, while the estimated factor exposures exhibit significant time variation for most industries, there is also a great deal of heterogeneity in the factor exposures across the industries. Risk exposures to the investment factor in particular are found to exhibit the greatest variability, consistently across all industries, while market risk exposure displays a relatively more stable pattern. More importantly, we observe significantly lower pricing errors obtained from the TTVP-SV model compared to the static alternative as well as the model that ignores the stochastic volatility component. Overall, the findings suggest that the threshold CAPM model can reasonably address the time-variation in factor exposures and minimize pricing errors in multi-factor asset pricing models, which is an important consideration for the implementation of smart beta strategies that rely on the accuracy and stability of factor exposures in the determination of investment positions in the target portfolio.

The rest of the paper is structured as follows. Section 2 presents a brief review of the extant literature on the pricing models and the popularly utilized factors suggested in the literature. Section 3 describes the data and the econometric methodology behind the TTVP-SV model. Section 4 presents the empirical findings. Section 5 concludes with a discussion of investment implications.

2. Literature Review

The pioneering works of Sharpe [12] and Lintner [13] serve as baseline to investigate the risk and return nexus in stock markets. Recognizing the shortcomings of the single-factor Capital Asset Pricing Model (CAPM), numerous works have proposed a wide range of firm-level, financial and macroeconomic variables that serve as a determinant of expected stock returns. As Kumar et al. [50] noted, starting with the pioneering studies by Fama and French [51–53], the evidence in the asset pricing literature generally points to an economically significant risk premium embedded in the cross-section of stock returns associated with market risk, firm size and book-to-market factors. Later studies including those by Jegadeesh and Titman [54] and Carhart [55] further document the presence of a momentum effect in which the past performance of an asset drives excess returns in subsequent periods. The dominant benchmark model in most asset pricing studies today is generally specified in terms of the four-factor model that includes the market, size, value and the momentum factors, as proposed by Fama and French [51,56], followed by a multitude of subsequent studies.

In a more recent extension of the multi-factor model to describe the cross-section of stock returns, Fama and French [57] introduced a five-factor model (FF5) that integrates profitability and investment factors to their popularly employed three-factor model. The five-factor model has since been employed in a number of applications, while the explanatory power of the factors is generally found to be specific to the market examined. For example, examining a large number of anomalies, Hou et al. [58] showed that about half of the anomalies are insignificant in the broad cross-section of stock returns, while a model that includes the market, size, investment, and profitability factors performs at least comparable to, and in many cases better than, the Fama–French [57] three-factor and the Carhart [55] four-factor models in capturing the remaining significant anomalies. In

an application to the Australian market, Chiah [59] showed that the five-factor model of Fama and French [57] describes more observed anomalies than other competitive asset pricing models such as the Fama–French three-factor and Carhart four-factor models. Despite the evidence that supports the five-factor pricing model, however, there is also growing evidence in the opposite direction, suggesting that the FF5 model is not suited to every market context. For example, in an examination of global stock markets, Fama and French [60] showed that the investment factor does not necessarily add explanatory power for European and Asian Pacific stocks, while Kubota and Takehara [61] found evidence against the profitability and investment factors in Japan. Similarly, in an application to the Chinese stock market, Guo et al. [62] observed that the investment factor is redundant, while strong profitability patterns are found in average returns. Accordingly, the evidence on a robust set of factors that can explain the cross-section of stock market returns is weak at best with a number of other factors proposed in the literature including asset growth [63], accrual [64] and profitability [65], among others.

One can argue that the inconclusive evidence regarding the explanatory power of systematic risk factors in the cross-section of stock returns could be due to the assumption of static factor exposures that fails to consider the time-variation in betas with respect to changing market conditions. Indeed, examining the risk premia associated with market, size, value and momentum factors under various economic scenarios, Arshanapalli et al. [20] showed that factor exposures exhibit heterogeneous patterns during normal and distressed market conditions with the market and size factors performing as risk factors offering investors high (low) returns during good (bad) economic conditions, while the value factor does not. This finding is then supported by the finding of Lustig and Verdelhan [66] of higher risk-adjusted cost of capital estimates during economic expansions. The instability of betas was further supported in an application to U.S. industry portfolios by Yu et al. [18], who showed that industry betas display regime specific patterns conditional on market uncertainty. In more recent studies, utilizing a wide range of time series forecasting methods on U.S. stock returns, Becker et al. [20] showed that betas display consistent long-term memory features, while Hollstein et al. [67] extended this evidence to 48 global stock markets, rejecting the short-term memory and the random walk features in the evolution of betas. Overall, while the evidence presents inconclusive evidence regarding the validity of risk factors as determinants of the cross-section of stock market returns, the literature presents overwhelming evidence for the time-variation in factor exposures as well as factor returns. To that end, our study presents a methodological innovation by introducing a flexible time-varying parameter model to capture factor exposures and comparing the performance of this model against the standard, linear specification with respect to the pricing errors obtained from the models

3. Data and Methodology

3.1. Data

We utilize monthly returns for U.S. industries including Consumer Durables (DURBL), Manufacturing (MANUF), Energy (ENRGY) and Technology (HITEC) over the period July 1963 to December 2020, obtained from Kenneth French's data library [68]. The choice of these particular industries is motivated by the preference to cover the production and consumption sides of the economy as one would expect a great deal of association between firm-level uncertainty in these industries and business cycles captured by the threshold framework adopted in the TTVP-SV model. Likewise, the data on monthly factor returns for Market (MKT), Size (SMB), Book-to-Market (HML), Profitability (RMW), and Investment (CMA) factors, based on the five-factor specification of Fama and French [57], are obtained from the same database maintained by Kenneth French.

Table 1 presents the descriptive statistics for monthly return series. Not surprisingly, the Technology sector experiences the greatest return volatility coupled with the highest mean returns, likely due to the high growth experienced in this innovation driven industry. Durables and Energy exhibit high kurtosis indicating the occurrence of extreme fluctuations

in the time series of these industry returns. Expectedly, the Jarque–Bera statistic rejects
the null hypothesis of normality in all series, implying the presence of possible non-linear
patterns in the time-variation of monthly return series. The effects of the 2008 global credit
crunch and the COVID-19 pandemic are evident in the time series plots for the monthly
industry returns, as presented in Figure 1. Clearly, consumer durables and manufacturing
have experienced a great deal of return variability during these two periods of increased
market uncertainty, while Energy firms have been particularly hit by the recent COVID-
19 pandemic. Not surprisingly, we observe the greatest level of return volatility in the
Technology sector during the infamous dot-com period in early 2000s when the new
economy bubble finally burst following a steady bull run in the late 1990s.

Table 1. Descriptive statistics.

Statistic	DURBL	MANUF	ENRGY	HITEC	MKT	SMB	HML	RMW	CMA
Mean	0.624	0.610	0.537	0.711	0.568	0.230	0.251	0.249	0.256
S.D.	6.726	4.955	5.901	6.394	4.468	3.026	2.867	2.167	1.988
Min	−32.710	−27.930	−34.730	−26.530	−23.240	−14.890	−13.960	−18.480	−6.860
Max	42.620	17.500	32.330	20.320	16.100	18.080	12.580	13.380	9.560
Skewness	0.602	−0.504	0.005	−0.241	−0.507	0.334	0.013	−0.327	0.316
Kurtosis	5.975	2.506	4.032	1.290	1.878	2.947	2.358	12.238	1.603
Jarque–Bera (JB)	1077.237 *	212.099 *	472.145 *	55.471 *	132.675 *	265.584 *	161.942 *	4349.274 *	86.602 *
Q(1)	8.152 *	1.676	0.717	2.408	2.652	2.947 *	21.859 *	15.273 *	9.901 *
Q(6)	23.302 *	7.253	4.158	4.806	7.417	10.664 *	30.461 *	20.396 *	19.610 *
ARCH(1)	14.192 *	10.178 *	83.967 *	55.253 *	18.336 *	58.167 *	40.465 *	122.214 *	69.473 *

Note: This table presents the descriptive statistics for monthly industry portfolio excess returns for Consumer Durables (DURBL),
Manufacturing (MANUF), Energy (ENRGY) and Technology (HITEC) firms over July 1963–December 2020. The last five columns are the
Fama–French factors including Market (MKT), Size (SMB), Book-to-Market (HML), Profitability (RMW) and Investment (CMA) factors
based on the five-factor specification of [57]. The Jarque–Bera normality test (JB), the first- [Q(1)] and sixth-order [Q(6)] serial correlation
test and first- [ARCH(1)] and sixth-order [ARCH(6)] autoregressive conditional heteroskedasticity test. * denotes significance at 1% level.
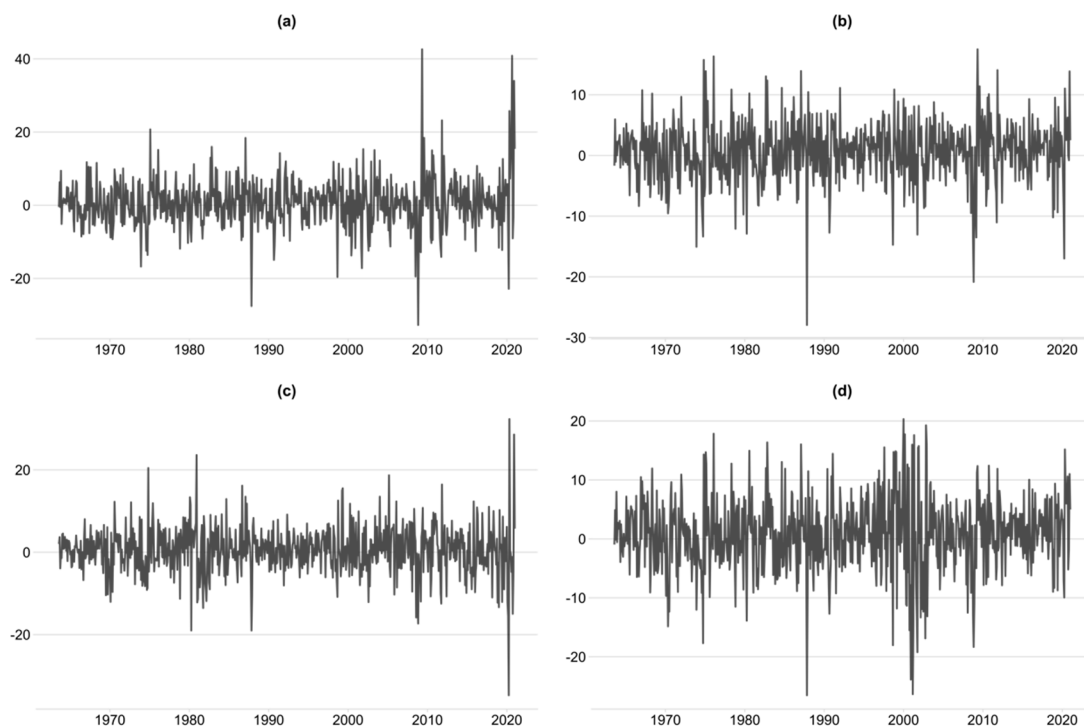
Figure 1. Monthly industry portfolio excess returns: (**a**) Consumer durables; (**b**) Manufacturing; (**c**) Energy; (**d**) High tech.
Note: The figure presents the plots for monthly excess portfolio returns over the sample period of July 1963–December 2020.

3.2. Methodology

In this paper, we extend the FF5 model of Fama and French [57] by modeling factor betas as a flexible threshold time-varying process. Specifically, the excess return, r_{it}, on industry portfolio i in period t is defined as:

$$r_{it} = \beta_{i0t} + \beta_{i1t}MKT_t + \beta_{i2t}SMB_t + \beta_{i3t}HML_t + \beta_{i4t}RMW_t + \beta_{i5t}CMA_t + \varepsilon_{it} \quad (1)$$

where MKT_t is the excess return on the market portfolio; SMB_t, HML_t, RMW_t and CMA_t are the returns on a zero-cost portfolio of small minus big stocks, high minus low book-to-market value stocks, high minus low operating profitability stocks and low minus high investment stocks, respectively; and ε_{it} is independently and normally distributed pricing error at time t with potentially heteroskedastic (time-varying) variance σ_{it}^2, that is $\varepsilon_{it} \sim N(0, \sigma_{it}^2)$. In this setting, the intercept β_{i0t} represents the alpha of the portfolio, while the other parameters, β_{i1t}, β_{i2t} β_{i3t}, β_{i4t} and β_{i5t}, represent factor exposures associated with the market, size, value, profitability and investment factors, respectively.

For ease of exposition, we introduce a compact notation for the time-varying FF5 model. Defining a (6×1) vector at time t as $z_t = (1, MKT_t, SMB_t, HML_t, RMW_t, CMA_t)'$ and $(n \times 1)$ portfolio excess returns as $r_t = (r_{1t}, r_{2t}, \ldots, r_{nt})'$, we represent the system of equations in Equation (1) with the compact notation as:

$$r_t = x_t'\beta_t + \varepsilon_t \quad (2)$$

where $x_t = (z_t' \otimes I_n)'$, with I_n as an $(n \times n)$ identity matrix. Time varying alphas (β_{i0t}) and betas for all portfolios are collected into the $k = (5n \times 1)$ vector

$$\beta_t = (\beta_{10,t}, \beta_{20,t}, \ldots, \beta_{n0,t}, \beta_{12,t}, \beta_{21,t}, \ldots, \beta_{n1,t}, \ldots, \beta_{15,t}, \beta_{25,t}, \ldots, \beta_{n5,t})'$$

and the $(n \times 1)$ vector of white noise pricing errors (shocks) $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{nt})'$ is distributed as

$$\varepsilon_t \sim N(0_n, \Sigma_t) \quad (3)$$

In Equation (3), 0_n denotes a $(n \times 1)$ zero vector and $\Sigma_t = V_t H_t V_t'$ is a time-varying variance-covariance matrix. Here, V_t is a lower triangular matrix with unit diagonal and $H_t = \text{diag}\left(e^{h_{1t}}, e^{h_{2t}}, \ldots, e^{h_{nt}}\right)$ is a diagonal matrix with time-varying variances $\sigma_{it}^2 = e^{h_{it}}$ on the diagonal; thus, h_{it} is the logarithm of conditional variance of the ith portfolio.

We introduce time-variation into the five-factor model by assuming that the relationship between the factors x_t and returns r_t is not constant through time and the time-variation is governed by a law of motion that defines how β_t evolves over time. As mentioned above, the most typical of such specifications is the random walk TVP model, which assumes that the (ij)th element of β_t, $\beta_{ij,t}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, 5$ is given by

$$\beta_{ij,t} = \beta_{ij,t-1} + v_{ij,t}, \qquad v_{ij,t} \sim N\left(0, \xi_{ij}^2\right) \quad (4)$$

where $v_{ij,t}$ is a white noise innovation with variance ξ_{ij}^2. Thus, Equations (1)–(4) together define a latent state TVP model where the latent states are defined by Equation (4) with the latent state innovation variance given by ξ_{ij}^2. As can be seen, the model subsumes the constant parameter case when $\xi_{ij}^2 \approx 0$, which would imply $\beta_{ij,t} \approx \beta_{ij,t-1}$ for all t.

Although the FF5-TVP model defined in Equations (1)–(4) is conceptually flexible and looks appealing for modeling factor betas in the presence of structural breaks, i.e., time-varying or conditional betas, it suffers a serious shortcoming in that the model may generate spurious beta changes that can significantly reduce the empirical performance of the model, thus leading to large pricing errors (see, e.g., [69]). In TVP models wherein a random walk process governs parameter changes, the parameters are programmed to evolve gradually over time, which may rule out adaptation to abrupt changes. Indeed, a random walk parameter process has unit root memory, and it will not forget the past in the

absence of any new shock. Accordingly, the FF5-TVP model that imposes a pre-specified random walk parameter structure on the time variation process may be restrictive for modeling asset returns, since financial markets often experience sudden changes driven by market shocks. Loosely speaking, the coefficients in TVP models are assumed to change every time period, whereas the innovation variance ξ_{ij}^2 in the state Equation (4) is generally estimated to be small, thus $\beta_{ij,t}$ will be close to $\beta_{ij,t-1}$. Thus, one can think of TVP models as models of "many small breaks" which is usually not consistent with return dynamic in financial markets since regime changes occur less frequently and coefficient changes are usually large. In contrast with TVP models, several studies consider structural change models with fewer breaks in parameters, bu.t when parameters do change, the size of the change is unrestricted and is allowed to be large (see, e.g., [70–77]).

Against this backdrop, following studies by, e.g., [70,71,74,75], who built on the previous literature that considers fewer structural breaks with unrestricted coefficient changes, we specify a spike-and-slab mixture distribution for the innovations $v_{ij,t}$ of the state Equation (1). Specifically, we specify the state equation innovation as follows:

$$v_{ij,t} \sim N\left(0, \xi_{ij,t}^2\right) \tag{5}$$

where $\xi_{ij,t}^2$ is the time-varying innovation variance given by

$$\xi_{ij,t}^2 = \left(1 - s_{ij,t}\right)\xi_{ij,0}^2 + s_{ij,t}\xi_{ij,1}^2 \tag{6}$$

In Equation (6), $\xi_{ij,0}^2$ and $\xi_{ij,1}^2$ are two distinct state innovation variances. Specifically, $\xi_{ij,0}^2$ is the spike variance which is set close to zero and $\xi_{ij,1}^2 \gg \xi_{ij,0}^2$ is the slab variance. The state innovation variances are endogenously determined from the data using indicator variables $s_{ij,t}$, which are independent sequence of Bernoulli distributed variables, defined as:

$$s_{ij,t} = \begin{cases} 1 & \text{with probability } \pi_{ij} \\ 0 & \text{with probability } 1-\pi_{ij} \end{cases} \tag{7}$$

Equations (1)–(7) thus form a mixture innovation model, which is relatively standard in the literature [70–72,75]. This specification also relates to multiple structural change models Pesaran et al. [77] and Koop and Potter [75], which build on the former work of Chib [70]. The mixture innovation model with Bernoulli distributed indicator variables $s_{ij,t}$ states that when $s_{ij,t}$ equals one, the change in the parameters $\beta_{ij,t}$ is normally distributed with zero mean and variance $\xi_{ij,1}^2$. In contrast, when $s_{ij,t}$ equals zero, the innovation variance is equal to $\xi_{ij,0}^2$, which implies that the parameters have almost no variability, effectively making $\beta_{ij,t}$ constant, that is $\beta_{ij,t} \approx \beta_{ij,t-1}$. This last case occurs because the innovation variance $\xi_{ij,0}^2$ is close to zero when $s_{ij,t} = 0$. Therefore, the coefficients in the mixture innovation model alters between constant coefficient state and changing coefficient state with unrestricted magnitude of change in the latter case.

The mixture innovation TVP model has quite appealing features for modeling time varying betas. The difficulty arises from the stochastic nature of indicators $s_{ij,t}$ and their joint simulation along with latent states, which in turn can make the model computationally cumbersome. Huber et al. [78] and Cuaresma et al. [79] circumvented the computational issue by introducing a deterministic threshold switching mechanism to identify the indicator sequences $s_{ij,t}$, leading to a TTVP model. Analogous to mixture innovation models, the TTVP approach also conditions on the states in order to simulate the indicator sequences $s_{ij,t}$ during the Monte Carlo Markov Chain simulation (MCMC), but the $s_{ij,t}$ are not directly simulated from their full conditional distribution, and, instead, they are identified from a threshold parameter d_{ij} defined for each of the $\beta_{ij,t}$ process. Specifically, let $\left\{\beta_{ij,t}^{(l)}\right\}_{t=1}^{T}$ be the draws of $\beta_{ij,t}$ in the lth iteration of the MCMC simulation, conditional on draws of $s_{ij,t}$

in the $(l-1)$th, that is $\left\{ s_{ij,t}^{(l-1)} \right\}_{t=1}^{T}$ iteration and the remaining parameters. Then, $s_{ij,t}$ in the lth iteration is defined as

$$
s_{ij,t} = \begin{cases} 1 \text{ if } |\Delta\beta_{ij,t}^{(l)}| > d_{ij}^{(l-1)} \\ 0 \text{ if } |\Delta\beta_{ij,t}^{(l)}| \leq d_{ij}^{(l-1)} \end{cases} \tag{8}
$$

where $d_{ij}^{(l-1)}$ denotes the $(l-1)$th draw of the threshold parameter d_{ij} specific to the coefficient $\beta_{ij,t}$ and $\Delta\beta_{ij,t}^{(l)} = \beta_{ij,t}^{(l)} - \beta_{ij,t-1}^{(l)}$. The threshold identification specification for $s_{ij,t}$ in Equation (8) makes it conditionally deterministic, freeing the MCMC algorithm from simulating $s_{ij,t}$ from its full conditional distribution. According to Equation (8), $s_{ij,t}$ is set equal to 1 if the absolute period-on-period change of the lth draw of $\beta_{ij,t}$ exceeds the $(l-1)$th draw of the corresponding threshold d_{ij}. In this case, the innovation variance is set to a large value for $\xi_{ij,1}^2$ and the motion of betas is in the change state. In contrast, if the absolute period-on-period change of the lth draw of $\beta_{ij,t}$ is smaller than the $(l-1)$th draw of the threshold d_{ij}, then $s_{ij,t} = 0$ and the innovation variance is set close to zero, i.e., $\xi_{ij,0}^2$, effectively making $\beta_{ij,t}$ constant with $\beta_{ij,t} = \beta_{ij,t-1}$. The model with the threshold specification in Equation (8) can thus be called the threshold mixture innovation model wherein regime shifts are governed by a deterministic law of motion. Compared to the standard threshold models, this model allows each parameter to switch across regimes independently of the state of other parameters. That is, the model does not require all parameters to simultaneously move into or out of a certain regime which allows some parameters to be constant during a given period while others might be changing.

Given that most financial return series display conditional heteroskedasticity, our model can easily incorporate this statistical feature into the modeling framework. Indeed, the specification for ε_t in Equation (3) is quite general with a time-varying variance feature that allows for conditional heteroskedasticity commonly observed for financial returns. Additionally, the model also allows time varying covariances via the matrix V_t. Conditional heteroskedasticity is commonly modeled using GARCH models. In this paper, we model the conditional volatility using a stochastic volatility (SV) model due to its several attractive features and modeling flexibility in our framework (see, e.g., [80]). Hence, we specify the logarithm of the variances to follow the first-order stationary SV process:

$$
h_{it} = \mu_i + \rho_i(h_{i,t-1} - \mu_i) + u_{it}, i = 1, 2, \ldots, n \tag{9}
$$

where white noise error u_{it} is normally distributed with zero mean and variance ω_i^2, i.e., $u_{it} \sim N(0, \omega_i^2)$, μ_i is the mean parameter and ρ_i is the persistence (autoregressive) parameter satisfying the stationarity condition $|\rho_i| < 1$. For the covariances V_t, we specify a random walk process with spike-and-slab error variances [70–72,75], as explained above. With the introduction of the conditional heteroskedasticity through the SV process in Equation (9), we label the extended model as TTVP with stochastic volatility or TTVP-SV.

4. Empirical Results

In our empirical application, we compare the performance of the TTVP-SV model with the TTVP and two additional FF5 factor model specifications. One of the specifications is the standard, static FF5 factor model and the other is a rolling regression specification (see, e.g., [33,36]. We use prior specifications and MCMC posterior simulation algorithm of Huber et al. [78] and Cuaresma et al. [79] for the estimation of the TTVP-SV and TTVP models. The rolling regression estimation is performed with a window size of 120 months.

Tables 2–5 present the descriptive statistics for the time-varying beta estimates for the industry portfolios obtained from the threshold TVP with stochastic volatility (Panel A) and rolling regression models (Panel B). Static beta estimates obtained using ordinary least squares are also reported in Panel C for comparison purposes. The profitability and investment factor exposure estimates generally display the greatest variability compared to the market, size and value factors, consistently for all industry portfolios. The uncertainty in factor exposures is particularly large for the Energy industry relative to the other industries, while factor exposures for the Manufacturing industry display the least variability. Clearly, considering that betas are not directly observable and uncertainty in betas potentially indicates greater possibility of disagreement about beta among investors, the relatively large variability of factor betas for the Energy industry could be a manifestation of the disagreement among market participants regarding energy market fundamentals. Nevertheless, given the recent finding by Hollstein et al. [67] that beta uncertainty carries an economically large and statistically significant negative premium in the cross-section of stock returns, the variability in factor exposures of industry portfolios implies possible trading strategies based on factor exposures, which could be exploited in smart beta strategies.

Table 2. Time-varying, rolling and static factor betas—Consumer Durables.

Variable	Mean	Median	S.D.	Min	Max	5th Percentile	95th Percentile
			Panel A: TTVP-SV Model				
Intercept	−0.338	−0.381	0.093	−0.436	−0.037	−0.431	−0.153
MKT	1.165	1.162	0.010	1.153	1.180	1.154	1.180
SMB	0.224	0.378	0.192	0.000	0.415	0.002	0.414
HML	0.287	0.267	0.073	0.162	0.421	0.171	0.416
RMW	0.157	0.097	0.110	0.047	0.356	0.050	0.350
CMA	0.094	0.057	0.098	−0.023	0.269	−0.018	0.266
			Panel B: Rolling Model				
Intercept	−0.443	−0.506	0.288	−0.978	0.208	−0.827	0.105
MKT	1.221	1.181	0.182	0.954	1.623	0.991	1.583
SMB	0.303	0.256	0.246	−0.149	0.838	−0.106	0.762
HML	0.376	0.406	0.333	−0.408	1.030	−0.277	0.887
RMW	0.264	0.198	0.424	−0.382	1.150	−0.287	1.034
CMA	0.117	0.071	0.300	−0.575	0.632	−0.340	0.583
			Panel C: Static Model				
Intercept	−0.340		0.150			−0.586	−0.094
MKT	1.267		0.037			1.207	1.327
SMB	0.214		0.052			0.128	0.300
HML	0.377		0.069			0.263	0.491
RMW	0.254		0.072			0.136	0.373
CMA	0.147		0.106			−0.027	0.320

Note: Panels A and B present the descriptive statistics for the time-varying beta estimates from the threshold TVP with stochastic volatility (TTVP-SV) and rolling regression models, respectively. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of [57]. Static beta estimates are also reported in Panel C for comparison purposes. The window size for the rolling beta estimates is 120 months. Estimation is carried out over the July 1963–December 2020 period with 690 months (covering July 1963–December 2020 period) and 570 months (covering July 1973–December 2020 period) for the TTVP-SV and rolling methods, respectively. The static model is estimated using ordinary least squares for which the Median, Min and Max estimates are not available.

Table 3. Time varying, rolling and static factor betas—Manufacturing.

Variable	Mean	Median	S.D.	Min	Max	5th Percentile	95th Percentile
			Panel A: TTVP-SV Model				
Intercept	−0.086	−0.083	0.033	−0.135	−0.001	−0.133	−0.025
MKT	1.067	1.067	0.000	1.066	1.067	1.066	1.067
SMB	0.102	0.102	0.001	0.101	0.104	0.101	0.104
HML	0.085	0.085	0.001	0.084	0.086	0.085	0.086
RMW	0.240	0.240	0.003	0.235	0.245	0.236	0.245
CMA	0.053	0.075	0.036	0.002	0.108	0.006	0.107
			Panel B: Rolling Model				
Intercept	−0.118	−0.095	0.148	−0.471	0.120	−0.369	0.078
MKT	1.105	1.091	0.058	0.980	1.265	1.022	1.200
SMB	0.103	0.115	0.062	−0.024	0.258	0.001	0.198
HML	0.014	0.018	0.106	−0.253	0.263	−0.204	0.161
RMW	0.222	0.281	0.230	−0.337	0.606	−0.225	0.478
CMA	0.116	0.127	0.134	−0.175	0.406	−0.105	0.336
			Panel C: Static Model				
Intercept	−0.164		0.061			−0.265	−0.063
MKT	1.084		0.015			1.059	1.108
SMB	0.110		0.021			0.074	0.145
HML	0.088		0.028			0.041	0.134
RMW	0.320		0.030			0.271	0.369
CMA	0.126		0.043			0.055	0.197

Note: Panels A and B present the descriptive statistics for the time-varying beta estimates from the threshold TVP with stochastic volatility (TTVP-SV) and rolling regression models, respectively. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of [57]. Static beta estimates are also reported in Panel C for comparison purposes. The window size for the rolling beta estimates is 120 months. Estimation is carried out over the July 1963–December 2020 period with 690 months (covering July 1963–December 2020 period) and 570 months (covering July 1973–December 2020 period) for the TTVP-SV and rolling methods, respectively. The static model is estimated using ordinary least squares for which the Median, Min and Max estimates are not available.

Table 4. Time varying, rolling and static factor betas—Energy.

Variable	Mean	Median	S.D.	Min	Max	5th Percentile	95th Percentile
			Panel A: TTVP-SV Model				
Intercept	−0.087	−0.117	0.129	−0.302	0.099	−0.292	0.094
MKT	0.963	0.954	0.017	0.951	1.007	0.952	1.003
SMB	−0.118	−0.181	0.143	−0.255	0.335	−0.250	0.261
HML	0.096	0.026	0.172	−0.018	0.547	−0.011	0.527
RMW	−0.086	−0.103	0.184	−0.404	0.195	−0.400	0.192
CMA	0.415	0.168	0.403	−0.084	1.261	−0.047	1.256
			Panel B: Rolling Model				
Intercept	0.022	0.131	0.529	−1.181	1.010	−0.823	0.815
MKT	0.967	0.973	0.135	0.680	1.348	0.739	1.153
SMB	−0.161	−0.196	0.218	−0.577	0.518	−0.487	0.172
HML	0.031	−0.063	0.261	−0.362	0.733	−0.279	0.563
RMW	0.042	0.052	0.545	−0.991	0.921	−0.910	0.773
CMA	0.304	0.145	0.605	−0.689	1.748	−0.510	1.675
			Panel C: Static Model				
Intercept	−0.224		0.168			−0.499	0.052
MKT	1.008		0.041			0.941	1.075
SMB	−0.063		0.058			−0.159	0.033
HML	0.230		0.078			0.102	0.357
RMW	0.209		0.081			0.076	0.342
CMA	0.365		0.118			0.170	0.559

Note: Panels A and B present the descriptive statistics for the time-varying beta estimates from the threshold TVP with stochastic volatility (TTVP-SV) and rolling regression models, respectively. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of [57]. Static beta estimates are also reported in Panel C for comparison purposes. The window size for the rolling beta estimates is 120 months. Estimation is carried out over the July 1963–December 2020 period with 690 months (covering July 1963–December 2020 period) and 570 months (covering July 1973–December 2020 period) for the TTVP-SV and rolling methods, respectively. The static model is estimated using ordinary least squares for which the Median, Min and Max estimates are not available.

**Table 5.** Time varying, rolling and static factor betas—High Tech.

Variable	Mean	Median	S.D.	Min	Max	5th Percentile	95th Percentile
			Panel A: TTVP-SV Model				
Intercept	0.227	0.247	0.057	0.105	0.318	0.151	0.315
MKT	1.054	1.053	0.002	1.051	1.056	1.051	1.056
SMB	0.075	0.084	0.021	0.025	0.088	0.026	0.087
HML	−0.234	−0.278	0.119	−0.362	−0.010	−0.359	−0.028
RMW	−0.149	−0.163	0.219	−0.646	0.181	−0.619	0.179
CMA	−0.466	−0.462	0.054	−0.561	−0.387	−0.556	−0.392
			Panel B: Rolling Model				
Intercept	0.326	0.178	0.390	−0.150	1.452	−0.104	1.181
MKT	1.043	1.032	0.124	0.835	1.376	0.873	1.266
SMB	0.125	0.107	0.127	−0.222	0.398	−0.035	0.379
HML	−0.290	−0.263	0.306	−0.860	0.318	−0.768	0.202
RMW	−0.131	−0.246	0.320	−0.718	0.573	−0.508	0.493
CMA	−0.398	−0.419	0.407	−1.357	0.365	−1.088	0.242
			Panel C: Static Model				
Intercept	−0.224		0.168			−0.499	0.052
MKT	1.008		0.041			0.941	1.075
SMB	−0.063		0.058			−0.159	0.033
HML	0.230		0.078			0.102	0.357
RMW	0.209		0.081			0.076	0.342
CMA	0.365		0.118			0.170	0.559

Note: Panels A and B present the descriptive statistics for the time-varying beta estimates from the threshold TVP with stochastic volatility (TTVP-SV) and rolling regression models, respectively. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of [57]. Static beta estimates are also reported in Panel C for comparison purposes. The window size for the rolling beta estimates is 120 months. Estimation is carried out over the July 1963–December 2020 period with 690 months (covering July 1963–December 2020 period) and 570 months (covering July 1973–December 2020 period) for the TTVP-SV and rolling methods, respectively. The static model is estimated using ordinary least squares for which the Median, Min and Max estimates are not available.

Interestingly, we observe that the threshold CAPM model yields the smallest model alphas (the intercept term), consistently for all industries, compared to the rolling and static regression models. The largest alphas (in absolute value) are observed for Consumer Durables followed by Technology, suggesting the presence of industry specific factors driving stock returns in these industries. The lower alpha estimates obtained from the TTVP-SV model are coupled with the lowest variability in alpha estimates, consistently for all sectors, suggesting that the threshold CAPM model performs favorably against the popularly employed rolling window and static alternatives in explaining return patterns in industry portfolios. Overall, the analysis of estimated beta statistics provides robust support for the reliability of factor exposures obtained from the TTVP-SV model, which is an important consideration for factor investing strategies that utilize factor exposures to determine asset allocations in target portfolios.

Figures 2–5 present the plots for the posterior medians (solid line) with 68% confidence bands (gray shaded region) of the time varying beta estimates from the TTVP model with stochastic volatility. The figures present a visual confirmation of the inferences obtained from the results in Tables 2–5 that the profitability and investment factor betas exhibit relatively greater variability over time. While market beta (MKT) exhibits a relatively stable pattern over time compared to the rest of the factors, we observe significant pattern changes in the other factor betas. For example, Consumer Durables industry experiences a jump in its exposure to the size and value factors in 1990s, while a similar jump in betas is observed for Energy during the post-global financial crisis period. Interestingly, Manufacturing stands out from the rest of the industries with quite stable factor exposures to the market, size, value and profitability factors, suggesting that smart beta strategies would be less costly for target portfolios that concentrate on stocks in this industry. At the

same time, Energy and Technology industries show relatively more variable exposures to the profitability and investment factors, while, in some cases, the sign of the exposure alternates between positive and negative, suggesting that positions in these industries will require more frequent rebalancing between long and short positions, which in turn would lead to great transaction costs associated with the investment strategy.
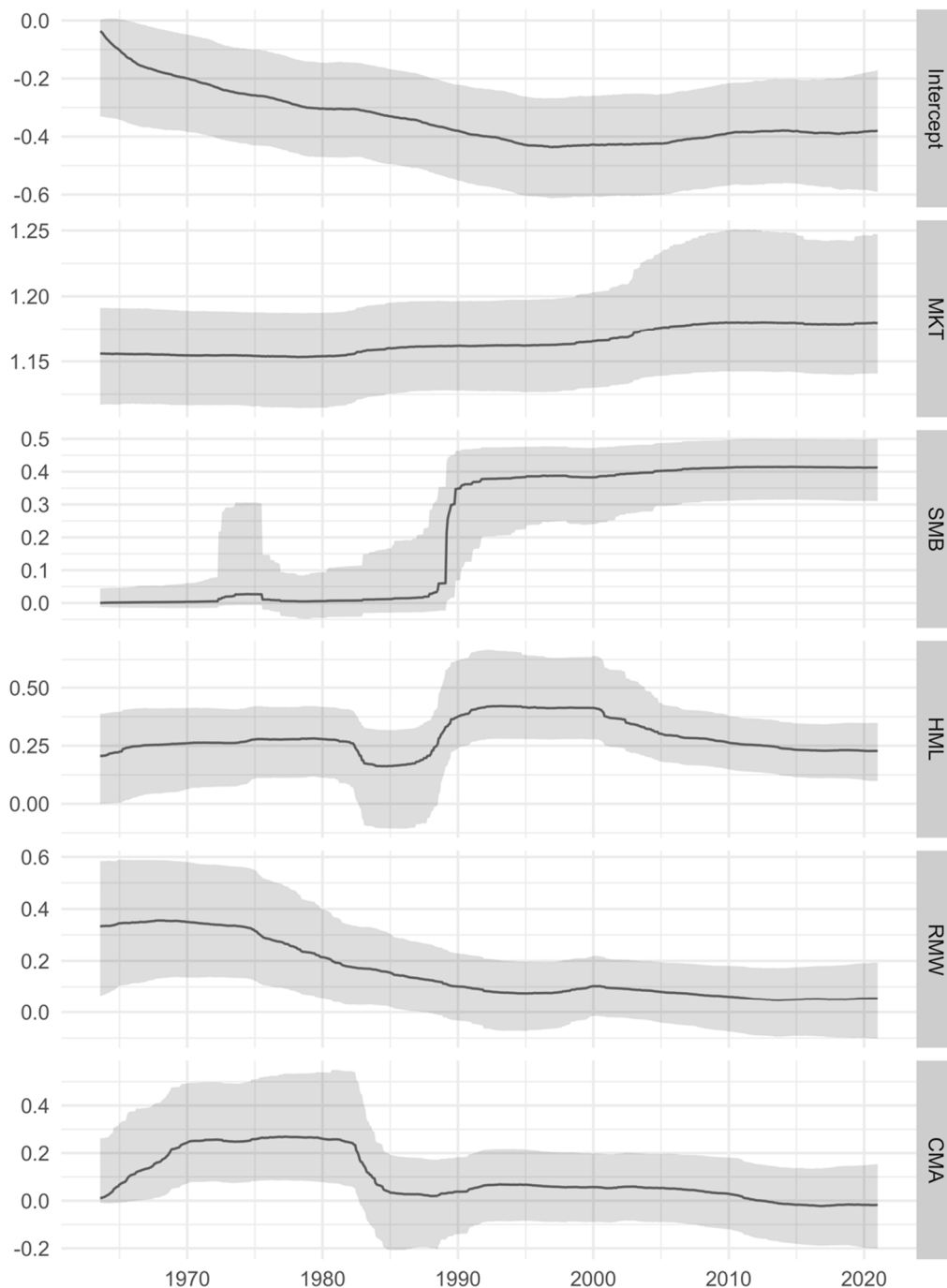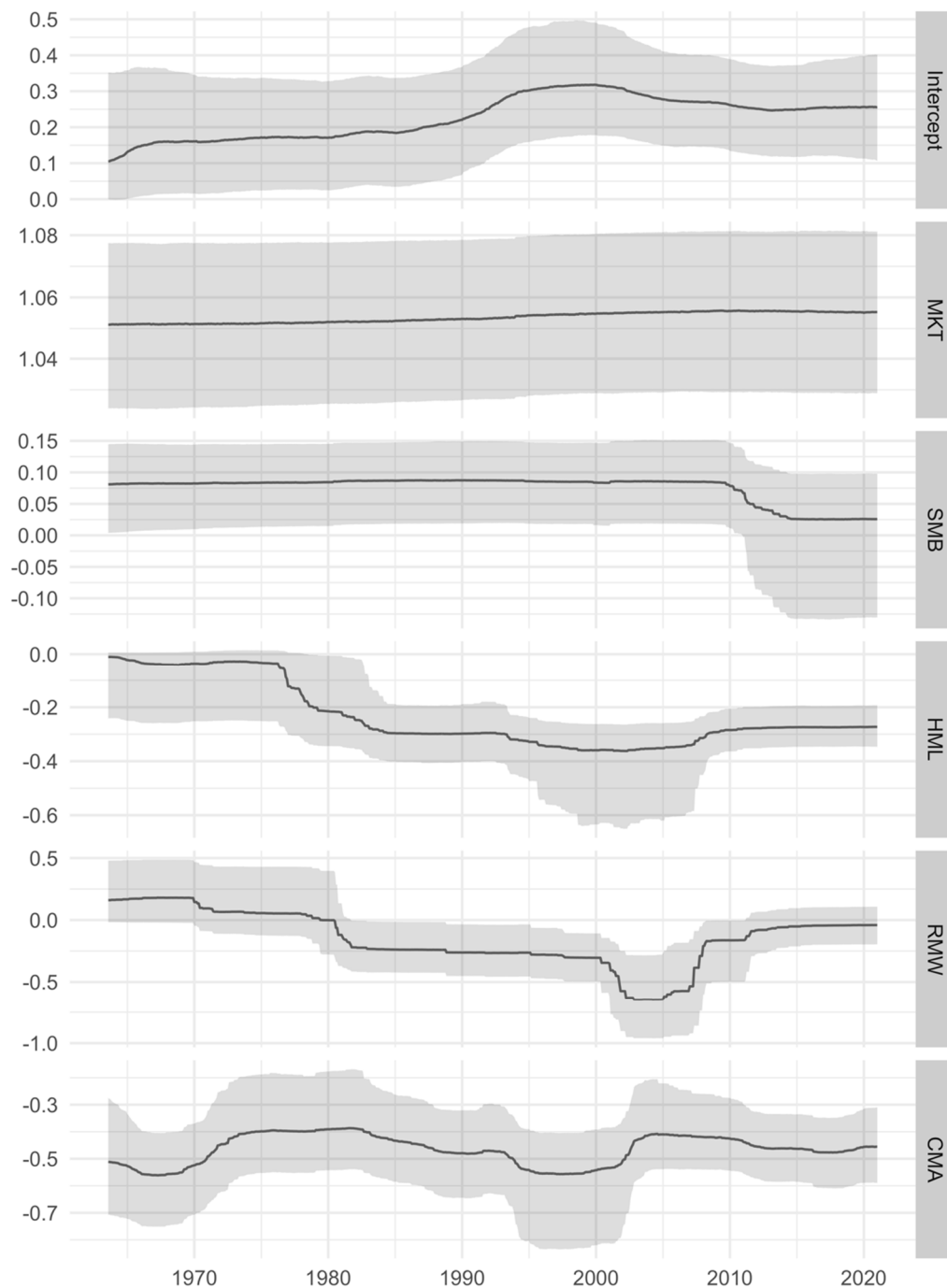

Figure 2. Threshold time-varying parameter beta estimates—Consumer Durables. Note: The figure presents the plots for the posterior medians (solid line) with 68% confidence bands (gray shaded region) of the time varying beta estimates from the TTVP model with stochastic volatility. Gibbs sampling is used with 10,000 posterior and 10,000 burn-in draws. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of Fama and French [57].
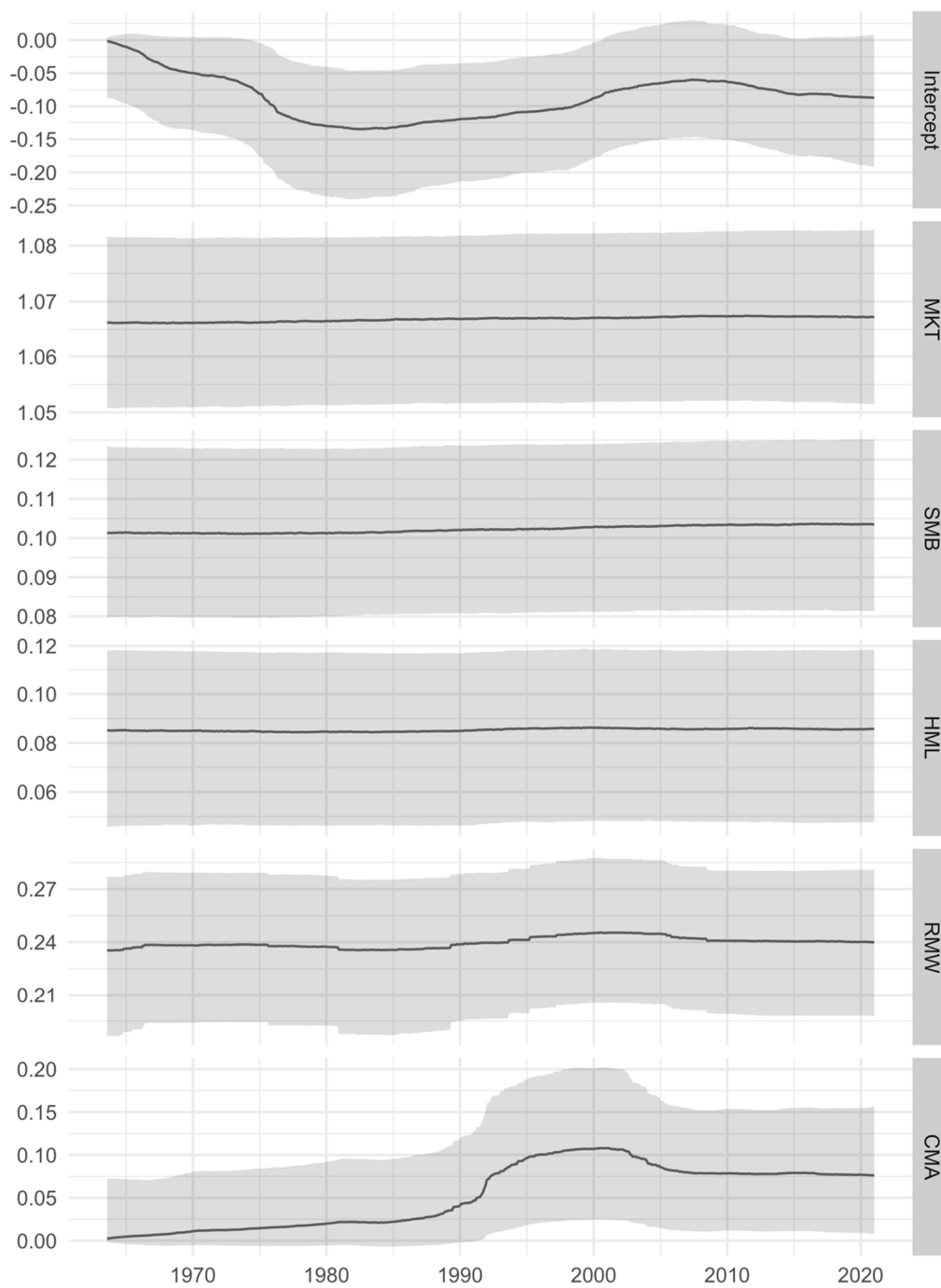
Figure 3. Threshold time-varying parameter beta estimates—Manufacturing. Note: The figure presents the plots for the posterior medians (solid line) with 68% confidence bands (gray shaded region) of the time varying beta estimates from the TTVP model with stochastic volatility. Gibbs sampling is used with 10,000 posterior and 10,000 burn-in draws. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of Fama and French [57].
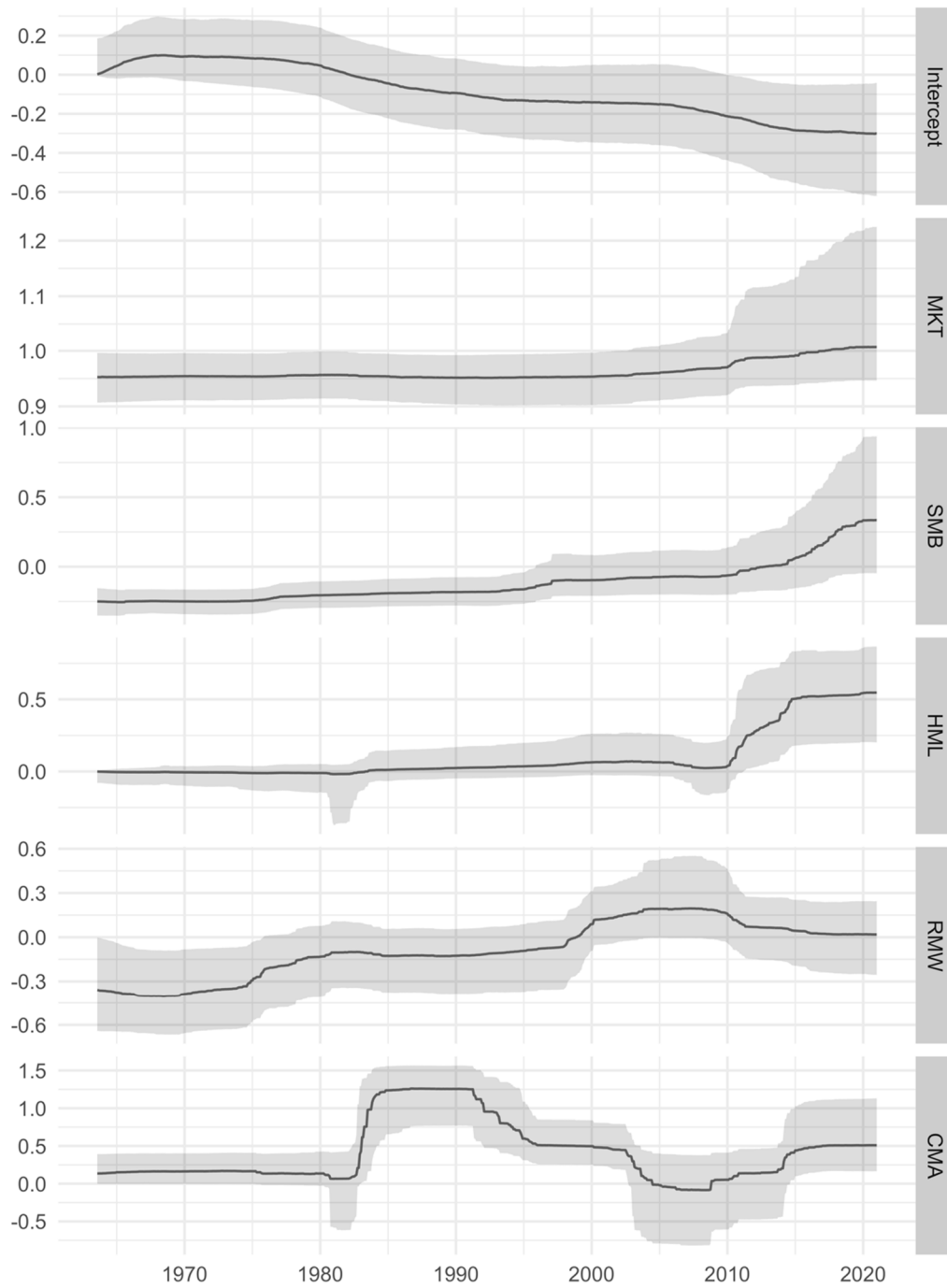
Figure 4. Threshold time-varying parameter beta estimates—Energy. Note: The figure presents the plots for the posterior medians (solid line) with 68% confidence bands (gray shaded region) of the time varying beta estimates from the TTVP model with stochastic volatility. Gibbs sampling is used with 10,000 posterior and 10,000 burn-in draws. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of Fama and French [57].

Figure 5. Threshold time-varying parameter beta estimates—High Tech. Note: The figure presents the plots for the posterior medians (solid line) with 68% confidence bands (gray shaded region) of the time varying beta estimates from the TTVP model with stochastic volatility. Gibbs sampling is used with 10,000 posterior and 10,000 burn-in draws. MKT, SMB, HML, RMW and CMA represent the Market, Size, Book-to-Market, Profitability and Investment factors based on the five-factor specification of Fama and French [57].

Finally, these findings are further confirmed by the pricing errors reported in Table 6. We observe that the threshold TVP model (both variations) yields smaller pricing errors, measured by the in-sample root mean square errors, consistently for all four industries, compared to the rolling and static regression alternatives. Overall, the findings present robust evidence that betas indeed exhibit time-varying patterns and the threshold CAPM model successfully captures the time-variation in factor exposures, which in turn leads to

smaller pricing errors. From an investment perspective, the findings suggest that threshold-based pricing models can offer a great advantage for smart beta strategies by reducing beta uncertainty, which in turn helps to mitigate the risk of over/under exposing the target portfolio to selected factors.

Table 6. In-sample Pricing Errors.

Model	DURBL	MANUF	ENRGY	HITEC
TTVP-SV	3.670	1.531	3.796	2.403
TTVP	3.525	1.520	3.696	2.372
Rolling	3.708	1.544	4.163	2.484
Static	3.747	1.533	4.195	2.575

Note: The table reports in-sample root mean square error (pricing error) estimates from the threshold TVP with stochastic volatility (TTVP-SV), the threshold TVP without stochastic volatility (TTVP), rolling regression and static regression models. The window size for the rolling beta estimates is 120 months. Estimation is carried out over the July 1963–December 2020 period with 690 months (covering July 1963–December 2020 period) and 570 months (covering July 1973–December 2020 period) for the TVP and rolling methods, respectively. The static model is estimated using ordinary least squares.

5. Conclusions

Factor exposures play a central role in the design and implementation of factor investing or smart beta strategies that have gained increasing interest among institutional investors with a projected value of assets under management of $3.4 trillion by 2022 [2]. Successful implementation of these strategies, however, relies heavily on the accuracy of models that can explain the dynamics of factor betas and yields. A major challenge in this regard is that betas are unobservable and highly unstable, driven by market uncertainty and/or business cycles. This study contributes to the literature by proposing a novel, threshold time-varying parameter (TTVP) model that can reliably measure the shifts in business cycles and the time-variation in betas in response to business cycle fluctuations. Our findings suggest that factor betas indeed display significant variation over time, particularly in the case of the profitability and investment factors. The threshold CAPM model is found to reduce beta uncertainty for all industries in the sample compared to the rolling and static regression alternatives. The lower beta uncertainty offered by the threshold CAPM model is coupled with smaller pricing errors achieved by this model, suggesting that the proposed approach not only allows for a more accurate representation of return dynamics but also has the potential to minimize transaction costs and the potential for over/under exposure to factors in factor investing strategies. It will be interesting for future research to examine the economic implications of implementing the threshold CAPM model in factor-based portfolio strategies and compare with static strategies that ignore the time-variation in factor betas.

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