LEGENDRE WAVELET OPERATIONAL MATRIX METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS IN SOME SPECIAL CONDITIONS

by

Aydin SECER[a*], Selvi ALTUN[a], and Mustafa BAYRAM[b]

[a] Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey
[b] Faculty of Engineering and Architecture, Istanbul Gelisim University, Istanbul, Turkey

Original scientific paper
https://doi.org/10.2298/TSCI180920034S

This paper proposes a new technique which rests upon Legendre wavelets for solving linear and non-linear forms of fractional order initial and boundary value problems. In some particular circumstances, a new operational matrix of fractional derivative is generated by utilizing some significant properties of wavelets and orthogonal polynomials. We approached the solution in a finite series with respect to Legendre wavelets and then by using these operational matrices, we reduced the fractional differential equations into a system of algebraic equations. Finally, the introduced technique is tested on several illustrative examples. The obtained results demonstrate that this technique is a very impressive and applicable mathematical tool for solving fractional differential equations.

Key words: *operational matrix, Legendre wavelets, Caputo fractional derivative, fractional differential equations*

Introduction

Fractional calculus is the study of any real-order or complex-order derivative and integral composed of combining and extending the concepts of multiple integral and integer order derivative. Let $\mathrm{D} = \mathrm{d}/\mathrm{d}t$ be a differential operator and n be a positive integer. It is well known that, the meaning of the $\mathrm{D}^n x(t)$ is the n^{th} derivative of the function $x(t)$. But if n is not a positive integer, it is difficult to comment the meaning of the $\mathrm{D}^{-\alpha}$ or D^{α} for $\mathrm{Re}(\alpha) > 0$. A variety of definitions that satisfy the idea of fractional derivative have been found by several great mathematician. But Riemann-Liouville and Caputo fractional derivatives are most commonly utilized refinements in the world of fractional calculus [1-5].

For three centuries, analysis of fractional calculus has been restraint to the discipline of pure theoretical mathematics. Over the past several decades, fractional differential equations (FDE) have considerable attention, because of their suitability for the explanation of a great many scientific phenomena such as thermal science especially bio-heat transfer, fluid mechanics, image processing, fluid-dynamic traffic model, earthquake engineering, damping laws, biomedical engineering, diffusion process, solid mechanics, colored noise, control theory, viscoelasticity, electromagnetic, *etc.* We can list some well-established techniques for the numerical solutions of FDE as operational matrix methods improved in [6-14] by using Bernstein, Genocchi, Legendre, Jacobi polynomials and Genocchi, Legendre, Chebyshev wavelets, per-

* Corresponding author, e-mail: aydinsecer@gmail.com

turbation-iteration method [15], sinc-Galerkin method [16], computational matrix method [17], homotopy analysis method [18], variational iteration method, a hybrid technique [19], Adomian decomposition method [20].

The orthogonal functions and polynomial series are very important field in science and engineering. They are basis of several numerical methods developed for the solution of differential equations. The reason is that the use of orthogonal polynomials is easy. Because they have good convergence properties and they properly represent the weight distribution of a function on a definite network. Mentioned equations are solved by truncating series of orthogonal basis functions. Block-pulse functions, Legendre, Hermite, Jacobi, Laguerre and Chebyshev polynomials are the most commonly utilized among these functions. What makes these functions important is that they permit the undertaking problem to be reduced to system of algebraic equations and the approximation of analytic functions.

Wavelet theory is very significant in science, engineering and technology and in recent years wavelets have achieved to attract an enormous attention in many fields of investigation, such as spectroscopy, signal analysis, feature detection in earth science, time-frequency analysis, image manipulation, *etc.* Many scholars have contributed to the development of wavelets. Especially, Daubechies, Belkin, Meyer and Mallat are some of them. Thanks to their contribution, there has been a substantial increment in the number of studies on wavelets. There are a wide variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shannon, Lagrange, Hermitian and Chebyshev wavelets available. Among them, we chose Legendre wavelets in this study because of their orthonormality and explicatory. Many applications of Legendre wavelets can be viewed from [21-26]. In addition, in these papers [27, 28], applications of Legendre wavelets related to thermal sciences can be reached.

Preliminaries and notations

It is necessary to know some mathematical definitions to understand the applications of fractional analysis, shifted Legendre polynomials and wavelets required for this paper. Some of these definitions and theorems are presented.

The fractional derivative in the Caputo sense

Definition 1. The Caputo fractional derivative is defined [1]:

$$ {}^{C}\mathrm{D}^{\beta}x(t)=\frac{1}{\Gamma(i-\beta)}\int_0^t\frac{x^{(i)}(\zeta)}{(t-\zeta)^{\beta+1-i}}\mathrm{d}\zeta,\quad i-1<\beta\le i,\ i\in\mathrm{N} \tag{1}$$

This derivative has the following characteristics:

$$ {}^{C}\mathrm{D}^{\alpha}S=0 \tag{2}$$

where S is a constant.

$$\mathrm{D}^{\beta}t^{\omega}=\begin{cases}0, & \omega\in\mathrm{N}_0 \text{ and } \omega<\lceil\beta\rceil\\ \dfrac{\Gamma(\omega+1)}{\Gamma(\omega+1-\beta)}t^{\omega-\beta}, & \omega\in\mathrm{N}_0 \text{ and } \omega\ge\lceil\beta\rceil \text{ or } \omega\notin\mathrm{N} \text{ and } \omega>\lfloor\beta\rfloor\end{cases} \tag{3}$$

Here, $\lfloor\beta\rfloor$ implies largest integer less than or equal to β and $\lceil\beta\rceil$ indicates the smallest integer greater than or equal to β.

Legendre wavelets

There are a wide variety of wavelet functions such as Daubechies, Haar, Laguerre, Legendre, Shannon, Lagrange, Hermitian and Chebyshev wavelets available. Among them, we chose Legendre wavelets in this study because of their orthonormality and explicity. This wavelet function can be defined.

Define:

$$\psi_{nm}(t)=\begin{cases}2^{\frac{k+1}{2}}\sqrt{m+\dfrac{1}{2}}P_m\left(2^k t-n\right), & \dfrac{n}{2^k}\le t\le\dfrac{n+1}{2^k}\\ 0, & \text{otherwise}\end{cases}\tag{4}$$

where $n=0,1,...,(2^k-1);\ \ m=0,1,...,M$. The coefficient $[(m+1)/2]^{1/2}$ is for orthonormality.

Shifted Legendre polynomials $P_m(t)$ are defined [21]:

$$P_m(t)=\sum_{k=0}^{m}\left(-1\right)^{m+k}\frac{\left(m+k\right)!}{\left(m-k\right)!}\frac{t^k}{\left(k!\right)^2}\qquad 0\le t\le 1\tag{5}$$

Function approximation

A function $x(t)$ defined over $[0,1]$ can be expanded in the terms of Legendre wavelet:

$$x(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}w_{nm}\psi_{nm}\left(t\right)\tag{6}$$

where $w_{nm}=(x(t),\psi_{nm}(t))$ in which $(.,.)$ indicates the inner product. Truncating infinite series in eq. (6), we get:

$$x(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}w_{nm}\psi_{nm}(t)=W^T\psi(t)\tag{7}$$

where W and $\psi(t)$ are matrices given by [21]:

$$\begin{aligned}W&=\left[w_{0,0},w_{0,1},...,w_{0,M},...,w_{2,M},...,w_{2^k-1,0},w_{2^k-1,1},...,w_{2^k-1,M}\right]^T\\ \psi&=\left[\psi_{0,0},\psi_{0,1},...,\psi_{0,M},...,\psi_{2,M},...,\psi_{2^k-1,0},\psi_{2^k-1,1},...,\psi_{2^k-1,M}\right]^T\end{aligned}\tag{8}$$

The operational matrix of derivative and fractional derivative

Theorem 1. Let $\psi(t)$ be the Legendre wavelets vector presented in eq. (4), then $\psi(t)$ can be written [21]:

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}\cong D\psi(t)\tag{9}$$

where D is the $2^k(M+1)$ operational matrix of derivative expressed:

$$D=\begin{bmatrix}X & O & \cdots & O\\ O & X & \cdots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \cdots & X\end{bmatrix}\tag{10}$$

where X is an $(M+1)(M+1)$ matrix and its $(i,j)^{\text{th}}$ element is expressed [21]:

$$X_{i,j}=\begin{cases}2^{k+1}\sqrt{(2i-1)(2j-1)}, & i=2,...,(M+1),\ j=1,...,i-1 \text{ and } (i+j) \text{ odd}\\ 0, & \text{otherwise}\end{cases} \tag{11}$$

Corollary 1. If we use eq. (9), then the operational matrix for the n^{th} derivative can be defined:

$$\frac{\mathrm{d}^n\psi(t)}{\mathrm{d}t^n}\cong D^n\psi(t) \tag{12}$$

where D^n is the n^{th} power of matrix D [8].

Lemma 1. Let $\psi(t)$ be the Legendre wavelets vector given in eq. (4) and assume that $k=0$ then:

$$D^\beta\psi_r(t)=0,\ \ r=0,1,...,\lceil\beta\rceil-1,\ \ \beta>0 \tag{13}$$

Proof. Please see [29].

Theorem 2. Let $\psi(t)$ be the Legendre wavelets vector given in eq. (5). Assume that $k=0$ and $\beta>0$, then:

$$D^\beta\psi(t)\cong D^{(\beta)}\psi(t) \tag{14}$$

where $D^{(\beta)}$ is the (M+1)x(M+1) the operational matrix of fractional derivative of the order $\beta>0$, $N-1<\beta\le N$ in the Caputo sense and is expressed:

$$D^{(\beta)}=\begin{bmatrix} 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\\ \sum\limits_{h=\lceil\beta\rceil}^{\lceil\beta\rceil}\varpi_{\lceil\beta\rceil,0,h} & \sum\limits_{h=\lceil\beta\rceil}^{\lceil\beta\rceil}\varpi_{\lceil\beta\rceil,1,h} & \cdots & \sum\limits_{h=\lceil\beta\rceil}^{\lceil\beta\rceil}\varpi_{\lceil\beta\rceil,m,h}\\ \vdots & \vdots & \cdots & \vdots\\ \sum\limits_{h=\lceil\beta\rceil}^{r}\varpi_{r,0,h} & \sum\limits_{h=\lceil\beta\rceil}^{r}\varpi_{r,1,h} & \cdots & \sum\limits_{h=\lceil\beta\rceil}^{r}\varpi_{r,m,h}\\ \vdots & \vdots & \cdots & \vdots\\ \sum\limits_{h=\lceil\beta\rceil}^{m}\varpi_{m,0,h} & \sum\limits_{h=\lceil\beta\rceil}^{m}\varpi_{m,1,h} & \cdots & \sum\limits_{h=\lceil\beta\rceil}^{m}\varpi_{m,m,h} \end{bmatrix} \tag{15}$$

where $\varpi_{r,s,h}$ is presented by:

$$\varpi_{r,s,h}=\sqrt{2r+1}\sqrt{2s+1}\sum_{p=0}^{s}\frac{(-1)^{r+s+h+p}(r+h)!(s+p)!}{(r-h)!h!\Gamma(h-\beta+1)(s-p)!(p!)^2(h+p-\beta+1)} \tag{16}$$

Take in consideration in $D^{(\beta)}$, the first $\lceil\beta\rceil$ rows are all zero.

Proof. Please see [29].

Solving linear fractional differential equations

This section introduces the application of Legendre wavelets optimization matrix (LWOM) of fractional derivative for solving linear FDE. Consider the following equation:

$$D^{\beta}x(t)=v_o(t)D^{\zeta_0}x(t)+\ldots+v_{k-1}(t)D^{\zeta_{k-1}}x(t)+v_k(t)x(t)+f(t) \tag{17}$$

with these initial conditions:

$$x(t_0)=x_0,\ \frac{\mathrm{d}x}{\mathrm{d}t}(t_0)=x_1,\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t_0)=x_2,\ldots,\frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}(t_0)=x_{n-1} \tag{18}$$

where $v_0(t),v_1(t),\ldots,v_k(t)$ can be any function and $0<\zeta_0<\zeta_1<\ldots<\zeta_{k-1}<\beta,\ n<\beta\le n+1$, and D^{β} indicates the Caputo fractional derivative of order β.

First approximating $x(t), f(t)$ and $v_0(t),v_1(t),\ldots,v_k(t)$ by the Legendre wavelets, then we obtain:

$$x(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}w_{n,m}\psi_{n,m}=W^T\psi(t) \tag{19}$$

$$f(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}f_{n,m}\psi_{n,m}=F^T\psi(t) \tag{20}$$

$$v_0(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}v_{0n,m}\psi_{n,m}={V_0}^T\psi(t),\ldots,v_k(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}v_{kn,m}\psi_{n,m}={V_k}^T\psi(t) \tag{21}$$

where F and V_i $(i=0,1,\ldots,k)$ are known vectors but W is an unknown vector and $\psi(t)$ is the vector given in eq. (4). By utilizing eqs. (14) and (19), we get:

$$D^{\beta}x(t)\cong W^TD^{\beta}\psi(t)\cong W^TD^{(\beta)}\psi(t) \tag{22}$$

$$D^{\zeta_i}x(t)\cong W^TD^{\zeta_i}\psi(t)\cong W^TD^{(\zeta_i)}\psi(t),\ i=0,\ldots,(k-1). \tag{23}$$

Substituting eqs. (19)-(23) in eq. (17) the residual $R(t)$ can be expressed:

$$\begin{aligned}R(t)\cong{} & W^TD^{(\beta)}\psi(t)-\left({V_0}^T\psi(t)\right)\left(W^TD^{(\zeta_0)}\psi(t)\right)-\ldots-\left({V_{k-1}}^T\psi(t)\right)\left(W^TD^{(\zeta_{k-1})}\psi(t)\right)-\\ & -\left({V_k}^T\psi(t)\right)\left(W^TD^{(\zeta_k)}\psi(t)\right)-\left({V_{k+1}}^T\psi(t)\right)\left(F^T\psi(t)\right)\end{aligned} \tag{24}$$

We get $2^k(M+1)-n$ linear equations by employing:

$$\left\langle R(t),\psi_r(t)\right\rangle=\int_0^1\psi_r(t)R(t)\,\mathrm{d}t=0,\ \ r=0,\ldots,2^k(M+1)-n \tag{25}$$

If we substitute eq. (19) in eq. (18) then we have:

$$\begin{gathered}x(t_0)=W^T\psi(t_0)=x_0,\quad \frac{\mathrm{d}x}{\mathrm{d}t}(t_0)=W^TD\psi(t_0)=x_1\\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t_0)=W^TD^2\psi(t_0)=x_2,\ldots,\frac{\mathrm{d}^nx}{\mathrm{d}t^n}(t_0)=W^TD^{(n)}\psi(t_0)=x_n\end{gathered} \tag{26}$$

The $2^k(M+1)$ set of linear equations are obtained eqs. (25) and (26). We can solve these linear equations for unknown coefficients of the vector W. Accordingly, $x(t)$ presented in eq. (18) can be computed.

Solving non-linear fractional differential equations

This section introduces the application of LWOM of fractional derivative for solving non-linear FDE. Consider the following equation:

$$D^{\beta} x(t) = F\left[t, x(t), D^{\zeta_1} x(t), ..., D^{\zeta_k} x(t)\right] \tag{27}$$

with these initial conditions:

$$x(t_0) = x_0,\ \frac{\mathrm{d}x}{\mathrm{d}t}(t_0) = x_1,\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t_0) = x_2, ..., \frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}}(t_0) = x_{n-1} \tag{28}$$

where $n < \beta \le n+1,\ 0 < \zeta_1 < \zeta_2 < ... < \zeta_k < \beta$ and D^{β} indicates the Caputo fractional derivative of order β.

First approximating $x(t)$, $D^{\beta} x(t)$ and for $i = 1, 2, ..., k$ $D^{\zeta_i} x(t)$ by the Legendre wavelets as eqs. (19), (22), and (23), respectively, and substituting these equations in eq. (26), then we obtain:

$$W^T D^{(\beta)} \psi(t) \cong F\left[t, W^T \psi(t), W^T D^{(\zeta_1)} \psi(t), ..., W^T D^{(\zeta_k)} \psi(t)\right] \tag{29}$$

Also, if we substitute eq. (20) in eq. (29) then we have:

$$\begin{gathered} x(t_0) = W^T \psi(t_0) = x_0, \quad \frac{\mathrm{d}x}{\mathrm{d}t}(t_0) = W^T D \psi(t_0) = x_1 \\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t_0) = W^T D^2 \psi(t_0) = x_2, ..., \frac{\mathrm{d}^n x}{\mathrm{d}t^n}(t_0) = W^T D^{(n)} \psi(t_0) = x_n \end{gathered} \tag{30}$$

First collocating eq. (29) at $2^k (M+1) - n$ points, then we can obtain the solution $x(t)$. We should use the first $2^k (M+1) - n$ roots of shifted Legendre $P_{k 2^k (M+1)}(t)$ to get a better result. Utilizing these equations together with eq. (30), then we have $2^k (M+1)$ non-linear equations. We can solve these non-linear equations for unknown coefficients of the vector W. Accordingly, $x(t)$ presented in eq. (27) can be computed.

Illustrative examples

Example 1. We first consider the following FDE of the linear form [12]:

$$4(t+1) D^{\frac{5}{2}} x(t) + 4 D^{\frac{3}{2}} x(t) + \frac{1}{\sqrt{t+1}} x(t) = \sqrt{t} + \sqrt{\pi}$$

subject to:

$$x(0) = \sqrt{\pi},\ \frac{\mathrm{d}x}{\mathrm{d}t}(0) = \frac{\sqrt{\pi}}{2},\ x(1) = \sqrt{2\pi}$$

The exact solution of the previous system is:

$$x(t) = \sqrt{\pi(t+1)}$$

If we implement the method illustrated in section *Solving linear fractional differential equations* with $M = 3,\ k = 0$ then we get:

$$D=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 0 & 0 & 2\sqrt{35} & 0 \end{bmatrix},\quad D^{\left(\frac{3}{2}\right)}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{16\sqrt{5}}{\sqrt{\pi}} & \frac{16\sqrt{5}\sqrt{3}}{5\sqrt{\pi}} & -\frac{16}{7\sqrt{\pi}} & \frac{16\sqrt{7}\sqrt{5}}{105\sqrt{\pi}} \\ -\frac{16\sqrt{7}}{\sqrt{\pi}} & \frac{80\sqrt{7}\sqrt{3}}{7\sqrt{\pi}} & \frac{16\sqrt{7}\sqrt{5}}{3\sqrt{\pi}} & \frac{-80}{11\sqrt{\pi}} \end{bmatrix},$$

$$D^{\left(\frac{5}{2}\right)}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160\sqrt{7}}{\sqrt{\pi}} & \frac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}} & -\frac{32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{bmatrix}$$

The approximate solution obtained by using Legendre wavelets operational matrix method (LWOMM) and the exact solution are displayed in tab. 1.

Table 1. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for *Example 1*

t	$x(t)$	$x_{LWOMM}(t)$	Absolute error
0.0	1.772455923	1.772453851	$0.2072\cdot10^{-5}$
0.1	1.858967455	1.859556019	–0.000588564
0.2	1.941628183	1.942759272	–0.001131089
0.3	2.020910686	2.022385308	–0.001474622
0.4	2.097198131	2.098755828	–0.001557697
0.5	2.170806302	2.172192531	–0.001386229
0.6	2.241999108	2.243017118	–0.001018010
0.7	2.310999784	2.311551286	–0.000551502
0.8	2.377999159	2.378116734	–0.000117575
0.9	2.443161886	2.443035165	0.000126721
1.0	2.506631205	2.506628274	$0.2931\cdot10^{-5}$

Example 2. Consider the following fractional Bagley-Torvik differential equation of the linear form with the initial conditions [7]:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+D^{\frac{3}{2}}x(t)+x(t)=1+t$$

$$x(0)=0,\ \frac{\mathrm{d}x}{\mathrm{d}t}(0)=1$$

The exact solution of this problem is known as:

$$x(t)=1+t$$

If we apply the method introduced in section *Solving linear fractional differential equations* with $M = 2$, $k = 0$ then we get:

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4\sqrt{3}\sqrt{15} & 0 & 0 \end{bmatrix},$$

$$D^{\left(\frac{3}{2}\right)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{16\sqrt{5}}{\sqrt{\pi}} & \dfrac{16\sqrt{3}\sqrt{5}}{5\sqrt{\pi}} & -\dfrac{16}{7\sqrt{\pi}} \end{bmatrix}$$

The numerical results which are attained by utilizing the LWOMM are shown in tab. 2.

Table 2. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for *Example 2*

t	$x(t)$	$x_{LWOMM}(t)$	Absolute error
0.0	1.0000000000	0.999999999	$0.1 \cdot 10^{-8}$
0.1	1.1000000000	1.099999999	$0.1 \cdot 10^{-8}$
0.2	1.2000000000	1.199999999	$0.1 \cdot 10^{-8}$
0.3	1.3000000000	1.299999999	$0.1 \cdot 10^{-8}$
0.4	1.4000000000	1.399999999	$0.1 \cdot 10^{-8}$
0.5	1.5000000000	1.499999999	$0.1 \cdot 10^{-8}$
0.6	1.6000000000	1.599999999	$0.1 \cdot 10^{-8}$
0.7	1.7000000000	1.699999999	$0.1 \cdot 10^{-8}$
0.8	1.8000000000	1.799999999	$0.1 \cdot 10^{-8}$
0.9	1.9000000000	1.899999999	$0.1 \cdot 10^{-8}$
1.0	2.0000000000	1.999999999	$0.1 \cdot 10^{-8}$

Example 3. Consider the following FDE of the non-linear form with the initial conditions [13]:

$$\frac{d^3 x(t)}{dt^3} + D^{\frac{5}{2}} x(t) + x^2(t) = t^4$$

$$x(0) = 0, \quad \frac{dx}{dt}(0) = 0, \quad \frac{d^2 x}{dt^2}(0) = 2$$

The exact solution of the previous system is:

$$x(t) = t^2$$

If we apply the method introduced in section *Solving non-linear fractional differential equations* with $M = 3$, $k = 0$ then we get:

$$D=\begin{bmatrix}0&0&0&0\\2\sqrt{3}&0&0&0\\0&2\sqrt{15}&0&0\\0&0&2\sqrt{35}&0\end{bmatrix},\quad D^2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\4\sqrt{3}\sqrt{15}&0&0&0\\0&4\sqrt{15}\sqrt{35}&0&0\end{bmatrix}$$

$$D^3=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\8\sqrt{3}\sqrt{15}\sqrt{35}&0&0&0\end{bmatrix},\quad D^{\left(\frac{5}{2}\right)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\\dfrac{160\sqrt{7}}{\sqrt{\pi}}&\dfrac{32\sqrt{7}\sqrt{3}}{\sqrt{\pi}}&-\dfrac{32\sqrt{7}\sqrt{5}}{7\sqrt{\pi}}&\dfrac{32}{3\sqrt{\pi}}\end{bmatrix}$$

The numerical results of the exact solution and the approximate solution are displayed in tab. 3.

Table 3. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for *Example 3*

t	$x(t)$	$x_{LWOMM}(t)$	Absolute error
0.0	0.00	$0.8410661107\cdot10^{-12}$	$-0.8410661107\cdot10^{-12}$
0.1	0.01	0.01000000006	$-0.6\cdot10^{-9}$
0.2	0.04	0.04000000011	$-0.11\cdot10^{-9}$
0.3	0.09	0.09000000014	$-0.14\cdot10^{-9}$
0.4	0.16	0.16000000020	$-0.2\cdot10^{-9}$
0.5	0.25	0.25000000020	$-0.2\cdot10^{-9}$
0.6	0.36	0.36000000010	$-0.1\cdot10^{-9}$
0.7	0.49	0.49000000010	$-0.1\cdot10^{-9}$
0.8	0.64	0.64000000010	$-0.1\cdot10^{-9}$
0.9	0.81	0.81000000000	0
1.0	1.00	0.99999999999	$0.1\cdot10^{-9}$

Example 4. Consider the following FDE of the non-linear form with the boundary conditions [11]:

$$\frac{d^2x(t)}{dt^2}+\Gamma\left(\frac{4}{5}\right)t^{\frac{6}{5}}D^{\frac{6}{5}}x(t)+\frac{11}{9}\Gamma\left(\frac{5}{6}\right)t^{\frac{1}{6}}D^{\frac{1}{6}}x(t)-\left[\frac{dx(t)}{dt}\right]^2=2+\frac{1}{10}t^2$$

subject to:

$$x(0)=1,\quad x(1)=2$$

The exact solution of the previous system is:

$$x(t)=1+t^2$$

If we apply the method illustrated in section *Solving non-linear fractional differential equations* with $M=2$, $k=0$ then we get:

$$D=\begin{bmatrix}0 & 0 & 0\\ 2\sqrt{3} & 0 & 0\\ 0 & 2\sqrt{15} & 0\end{bmatrix},\quad D^{\left(\frac{1}{6}\right)}=\begin{bmatrix}0 & 0 & 0\\ 2.008717540 & 1.023294363 & -0.05743771053\\ -2.288145774 & 0.5858729181 & 1.235012279\end{bmatrix}$$

$$D^{\left(\frac{6}{5}\right)}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 16.00534636 & 7.920592311 & -0.5381810900\end{bmatrix}$$

The approximate solution obtained by using introduced method and the exact solution are shown in tab. 4.

Table 4. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for *Example 4*

t	$x(t)$	$x_{LWOMM}(t)$	Absolute error
0.0	1.00	0.9999999998	$0.2\cdot10^{-9}$
0.1	1.01	1.009995734	$0.4266\cdot10^{-5}$
0.2	1.04	1.039992417	$0.7583\cdot10^{-5}$
0.3	1.09	1.089990047	$0.9953\cdot10^{-5}$
0.4	1.16	1.159988625	0.000011375
0.5	1.25	1.249988151	0.000011849
0.6	1.36	1.359988625	0.000011375
0.7	1.49	1.489990047	$0.9953\cdot10^{-5}$
0.8	1.64	1.639992416	$0.7584\cdot10^{-5}$
0.9	1.81	1.809995734	$0.4266\cdot10^{-5}$
1.0	2.00	2.000000000	0.000000000

Example 5. Consider the following FDE of the non-linear form with the initial conditions:

$$D^{1.3}x(t)+x^2(t)=\frac{20}{7}\frac{t^{0.7}}{\Gamma(0.7)}+t^4$$

subject to:

$$x(0)=0,\quad \frac{\mathrm{d}x}{\mathrm{d}t}(0)=0$$

The exact solution of the previous system is:

$$x(t)=t^2$$

If we apply the method illustrated in section *Solving non-linear fractional differential equations* with $M=2$, $k=0$ then we get:

$$D=\begin{bmatrix}0 & 0 & 0\\ 2\sqrt{3} & 0 & 0\\ 0 & 2\sqrt{15} & 0\end{bmatrix},\quad D^{(1.3)}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 17.37179094 & 7.800806362 & -0.8165511655\end{bmatrix}$$

The numerical results of the approximate solution and the exact solution are shown in tab. 5.

Table 5. Comparison of our solution $x_{LWOMM}(t)$ and the exact solution $x(t)$ for *Example 5*

t	$x(t)$	$x_{LWOMM}(t)$	Absolute error
0.0	0.00	$-0.1\cdot 10^{-9}$	$0.1\cdot 10^{-9}$
0.1	0.01	0.01009304147	0.00009304147
0.2	0.04	0.04037216618	0.00037216618
0.3	0.09	0.09083737403	0.00083737403
0.4	0.16	0.16148866500	0.00148866500
0.5	0.25	0.25232603910	0.00232603910
0.6	0.36	0.36334949640	0.00334949640
0.7	0.49	0.49455903680	0.00455903680
0.8	0.64	0.64595466040	0.00595466040
0.9	0.81	0.81753636710	0.00753636710
1.0	1.00	1.00930415700	0.00930415700

Conclusion

In this study, we first introduced LWOMM shortly, and then utilized it to attain approximate solutions of FDE. These approximate solutions were clearly written and were matched against the exact solutions of the presented problems. A very effective algorithm is also formulated to obtain the numerical solution of these problems on the Maple. We produced all graphical representations and numerical results via Maple. The results illustrate that the LWOMM can solve FDE efficaciously and simply.

References

[1] Miller, K. S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, USA, 1993
[2] Oldham, K. B., Spanier, J., *The Fractional Calculus*, Academic Press, New York, USA, 1974
[3] Jafari, H., *et al*., Application of Legendre Wavelets for Solving Fractional Differential Equations, *Computers and Mathematics with Applications, 62* (2011), 3, pp. 1038-1045
[4] Balaji, S., Legendre Wavelet Operational Matrix Method for Solution of Fractional Order Riccati Differential Equation, *Journal of the Eqyptian Mathematical Society, 23* (2015), 2, pp. 263-270
[5] Chen, Y.-M., *et al.*, Numerical Solution for a Class of Nonlinear Variable Order Fractional Differential Equations with Legendre Wavelets, *Applied Mathematics Letters*, *46* (2015), Aug., pp. 83-88
[6] Rehman, M., Khan, R. A., The Legendre Wavelet Method for Solving Fractional Differential Equations, *Commun Nonlinear Sci Numer Simulat, 16* (2011), 11, pp. 4163-4173
[7] Saadatmandi, A., Dehghan, M., A New Operational Matrix for Solving Fractional-Order Differential Equations, *Computers and Mathematics with Applications, 59* (2010), 3, pp. 1326-1336
[8] Mohammadi, F., *et al*., A New Operational Matrix for Legendre Wavelets and its Applications for Solving Fractional Order Boundary Value Problems, *International Journal of Systems Science, 6,* (2011), 32, pp. 7371-7378
[9] Alshbool, M. H. T., *et al*., Solution of Fractional-Order Differential Equations Based on the Operational Matrices of New Fractional Bernstein Functions, *Journal of King Saud University-Science, 29* (2017), 1, pp. 1-18
[10] Mohammadi, F., Numerical Solution of Bagley-Torvik Equation Using Chebyshev Wavelet Operational Matrix of Fractional Derivative, *International Journal of Advances in Applied Mathematics and Mechanics, 2*, (2014), 1, pp. 83-91
[11] Isah, A., Phang, C., New Operational Matrix of Derivative for Solving Non-Linear Fractional Differential Equations Via Genocchi Polynomials, *Journal of King Saud University-Science*, *31* (2017), 1, pp. 1-7

[12] Isah, A., Phang, C., Genocchi Wavelet-Like Operational Matrix and Its Application for Solving Non-Linear Fractional Differential Equations, *Open Phys*., *14* (2016), 1, pp. 463-472
[13] Isah, A., Phang, C., Legendre Wavelet Operational Matrix of Fractional Derivative through Wavelet-Polynomial Transformation and Its Applications in Solving Fractional Order Differential Equations, *International Journal of Pure and Applied Mathematics, 105* (2015), 1, pp. 97-114
[14] Doha, E. H., *et al*., A New Jacobi Operational Matrix: An Application for Solving Fractional Differential Equations, *Applied Mathematical Modelling, 36* (2012), 10, pp. 4931-4943
[15] Şenol, M., Dolapci, I. T., On the Perturbation-Iteration Algorithm for Fractional Differential Equations, *Journal of King Saud University-Science, 28* (2016), 1, pp. 69-74
[16] Secer, A., *et al*., Sinc-Galerkin Method for Approximate Solutions of Fractional Order Boundary Value Problems, *Boundary Value Problems, 2013* (2013), 281
[17] Khader, M. M., *et al*., A Computational Matrix Method for Solving Systems of High Order Fractional Differential Equations, *Applied Mathematical Modelling, 37* (2013), 6, pp. 4035-4050
[18] Kurulay, M., *et al*., A New Approximate Analytical Solution of Kuramoto-Sivashinsky Equation Using Homotopy Analysis Method, *Appl. Math. Inf. Sci*., *7*, (2013), 1, pp. 267-271
[19] Akinlar, M., *et al*., Numerical Solution of Fractional Benney Equation, *Appl. Math. Inf. Sci.*, *8*, (2014), 4, pp. 1633-1637
[20] Song, L., Wang, W., A New Improved Adomian Decomposition Method and Its Application to Fractional Differential Equations, *Applied Mathematical Modelling, 37* (2013), 3, pp. 1590-1598
[21] Mohammadi, F., Hosseini, M. M., A New Legendre Wavelet Operational Matrix of Derivative and its Applications in Solving the Singular Ordinary Differential Equations, *Journal of The Franklin Institute, 348* (2011), 8, pp. 1787-1796
[22] Venkatesh, S. G., *et al*., The Legendre Wavelet Method for Solving Initial Value Problems of Bratu-Type, *Computers and Mathematics with Applications, 63* (2012), 8, pp. 1287-1295
[23] Mohammadi, F., *et al*., Legendre Wavelet Galerkin Method for Solving Ordinary Differential Equations with Non-Analytic Solution, *International Journal of Systems Science 42* (2011), 4, pp. 579-585
[24] Mishra, V., Sabina, Wavelet Galerkin Solutions of Ordinary Differential Equations, *International Journal of Math. Analysis*, *5* (2011), 9, pp. 407-424
[25] Khellat, F., Yousefi, S. A., The Linear Legendre Mother Wavelets Operational Matrix of Integration and its Application, *Journal of The Franklin Institute, 343* (2006), 2, pp. 181-190
[26] Mohammadi, F., Hosseini, M. M., A Comparative Study of Numerical Methods for Solving Quadratic Riccati Differential Equations, *Journal of The Franklin Institute, 348* (2011), 2, pp. 156-164
[27] Kumar, P., *et al*., A Mathematical Model for Hyperbolic Space-Fractional Bioheat Transfer during Thermal Therapy, *Procedia Engineering, 127* (2015), Dec., pp. 56-62
[28] Kumar, P., *et al*., Numerical Study on Non-Fourier Bioheat Transfer during Thermal Ablation, *Procedia Engineering, 127* (2015), Dec., pp. 1300-1307
[29] Secer, A., Altun, S., A New Operational Matrix of Fractional Derivatives to Solve Systems of Fractional Differential Equations Via Legendre Wavelets, *Mathematics*, *6*, (2018), 11, 238

Paper submitted: September 20, 2018
Paper revised: October 24, 2018
Paper accepted: January 10, 2019