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On the 3-Parameter Spatial Motions in Lorentzian 3-Space

Handan Yıldırım[a], **Nuri Kuruoğlu**[b]

[a] *Istanbul University, Faculty of Science, Department of Mathematics, Vezneciler-Fatih, 34134, Istanbul, TURKEY*
[b] *Istanbul Gelişim University, Faculty of Engineering and Architecture, Department of Civil Engineering, Avcılar, 34310, Istanbul, TURKEY*

Abstract. In this paper, we obtain the formulas of the volume element and the volume of the region which is determined in the fixed space by any fixed point of the moving space under the 3-parameter spatial motions in Lorentzian 3-space L^3. Moreover, taking into account these formulas, we give Holditch-Type Theorems and some corollaries in Lorentzian sense.

1. Introduction

Lorentzian 3-*space* L^3 is the vector space $\mathbb{R}^3$ endowed with Lorentzian inner product

$$< \mathbf{x}, \mathbf{y} >= x_1 y_1 + x_2 y_2 - x_3 y_3$$

for $\mathbf{x} = (x_1, x_2, x_3)\,, \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

$$\text{A vector } \mathbf{x} \in L^3 \text{ is called} \begin{cases} \text{spacelike if } < \mathbf{x}, \mathbf{x} >> 0 \text{ or } \mathbf{x} = \mathbf{0}, \\ \text{lightlike if } < \mathbf{x}, \mathbf{x} >= 0, \\ \text{timelike if } < \mathbf{x}, \mathbf{x} >< 0. \end{cases}$$

Moreover, the norm of $\mathbf{x}$ is defined by $\|\mathbf{x}\| := \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, (cf. [13]).

Let R (respectively, R') be the moving (respectively, fixed) Lorentzian space L^3 and $\{O; \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ (respectively, $\{O'; \mathbf{e'_1}, \mathbf{e'_2}, \mathbf{e'_3}\}$) be the right-handed orthonormal frame of R (respectively, R'). If $\mathbf{e_j} = \mathbf{e_j}(t_1, t_2, t_3)$ and the vector $\overrightarrow{OO'} = \mathbf{u} = \mathbf{u}(t_1, t_2, t_3)$ are continuously differentiable functions of real parameters t_1, t_2 and t_3, then a 3-parameter spatial motion of R with respect to R' is defined. In what follows, such a motion will be denoted by $\mathfrak{B}_3$. A motion $\mathfrak{B}_3$ is given analytically by

$$\mathbf{x}' = A\mathbf{x} + C,$$

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Email addresses: handanyildirim@istanbul.edu.tr (Handan Yıldırım), nkuruoglu@gelisim.edu.tr (Nuri Kuruoğlu)

where $\mathbf{x}$ (respectively, $\mathbf{x}'$) corresponds to the position vector represented by the column matrix of any point $X \in R$ according to the moving (respectively, fixed) orthonormal frame; C is the translation vector represented by the column matrix and $A \in SO_1(3)$, that is,

$$A^{-1} = \mathcal{E}A^T\mathcal{E}. \tag{1}$$

Here, $\mathcal{E}$ is a sign matrix defined by

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

(cf. [13]). Furthermore, the elements of A and C are continuously differentiable functions of real parameters t_1, t_2 and t_3.

In this paper, taking into account the Holditch-Type Theorems in [10] (respectively, in [12] and in ([9] and [11])) and its corollaries in [5] (respectively, [6] and section 2.4 of [14]) for the 1-parameter closed (respectively, 2-parameter closed and 3-parameter) spatial motions in Euclidean 3-space E^3, we give Holditch-Type Theorems and some corollaries for the 3-parameter spatial motions $\mathfrak{B}_3$ in Lorentzian 3-space L^3, by means of [14]. Thus, we present Lorentzian versions of some results given in [9], [11] and section 2.4 of [14] in Euclidean sense. For our purpose, we first get the formulas of the volume element and the volume of the region which is determined in R' by any fixed point of R under the motions $\mathfrak{B}_3$ in L^3, taking into account [14]. We emphasize that such formulas and related Holditch-Type Theorems were obtained in [4] (respectively, in [1]) for 3-parameter spatial homothetic motions in E^3 (respectively, 3-parameter Galilean motions in Galilean space G^3). We refer [15] (respectively, [14]) about the Holditch-Type Theorems for the 1-parameter (respectively, 2-parameter) closed spatial motions in L^3.

2. Formulas of the volume element and the volume of the regions determined during the 3-parameter spatial motions in L^3

Under the 3-parameter spatial motions $\mathfrak{B}_3$ in L^3, set the following matrices E and E':

$$E = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \text{ and } E' = \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix}. \tag{2}$$

Then, we can write that

$$E = AE', \tag{3}$$

where $A \in SO_1(3)$. Since A is regular, it is obvious that

$$AA^{-1} = I_3.$$

The differentiation of this equation yields the following equation

$$dAA^{-1} + AdA^{-1} = 0.$$

By means of the last equation and equation (1), we deduce the following equation

$$\Omega^T = -\mathcal{E}\Omega\mathcal{E},$$

where

$$\Omega = dAA^{-1}. \tag{4}$$

If we denote the elements of the matrix Ω by ω_{ij} $(1 \leq i, j \leq 3)$ and take $\omega = (\omega_1, \omega_2, \omega_3)$ such that $\Omega\mathbf{x} = \omega \times \mathbf{x}$, where $\mathbf{x}$ on the left (respectively, right) side of this equality corresponds to the position vector represented by the column matrix (respectively, $\mathbf{x} = (x_1, x_2, x_3)$) of any point $X \in R$ (cf. [2]), then we can write that

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{5}$$

Here, ω_i $(1 \leq i \leq 3)$ are differential forms of real parameters t_1, t_2 and t_3. By using equations (3) and (4), it follows that

$$dE = \Omega E.$$

Thus, in terms of (2) and (5), we derive from the last equation that

$$\mathbf{de_i} = -(-1)^k (\omega_j \, \mathbf{e_k} - \omega_k \, \mathbf{e_j}) \tag{6}$$

for $i, j, k = 1, 2, 3$ (cyclic). Since $\mathbf{d}\,(\mathbf{de_i}) = \mathbf{0}$, we obtain the following conditions of integration:

$$d\omega_i = -(-1)^k \, \omega_j \wedge \omega_k, \tag{7}$$

where "$\wedge$" is the wedge product of the differential forms. If we denote $-\mathbf{du}$ by σ', from equation (6), we have

$$\sigma' = \sigma_1 \, \mathbf{e_1} + \sigma_2 \, \mathbf{e_2} + \sigma_3 \, \mathbf{e_3}, \tag{8}$$

where $\sigma_i = -du_i + (-1)^i \, u_j \, \omega_k - (-1)^j \, u_k \, \omega_j$. Moreover, since $\mathbf{d}\,(\sigma') = \mathbf{0}$, we get the following conditions of integration:

$$d\sigma_i = -(-1)^i \, \sigma_j \wedge \omega_k + (-1)^j \, \sigma_k \wedge \omega_j. \tag{9}$$

We note that, during the motions $\mathfrak{B}_3$, σ_i $(1 \leq i \leq 3)$ are differential forms of the real parameters t_1, t_2 and t_3. Furthermore, since ω_i $(1 \leq i \leq 3)$ are linearly independent, $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$.

Under the motions $\mathfrak{B}_3$, it is clear that

$$\mathbf{x}' = -\mathbf{u} + \mathbf{x},$$

where $\mathbf{x}$ (respectively, $\mathbf{x}'$) corresponds to the position vector of any point $X \in R$ according to the orthonormal frame of R (respectively, R'). This equation yields

$$\mathbf{dx}' = \sigma' + \mathbf{dx}.$$

If we take $\mathbf{dx}' = \sum_{i=1}^{3} \tau_i \, \mathbf{e_i}$ for any fixed point $X \in R$, we find

$$\tau_i = \sigma_i - (-1)^i \, x_j \, \omega_k + (-1)^j \, x_k \, \omega_j \tag{10}$$

by means of equations (6) and (8). During the motions $\mathfrak{B}_3$, the volume element and the volume of the region which is determined in R' by this fixed point $X \in R$ are given respectively by

$$dJ_X = \tau_1 \wedge \tau_2 \wedge \tau_3 \tag{11}$$

and

$$J_X = \int_G dJ_X. \tag{12}$$

Here, G is the domain of the parameter space (space of t_i). In terms of equations (10) and (11), we obtain the following volume element formula:

$$\begin{aligned} dJ_X &= \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + \sum_{i=1}^{3} \sigma_i \wedge \omega_j \wedge \omega_k \, x_i^2 + \sum_{i=1}^{3} (-1)^j \left(-\sigma_i \wedge \omega_i \wedge \omega_k + \sigma_j \wedge \omega_j \wedge \omega_k\right) x_i \, x_j \\ &\quad - \sum_{i=1}^{3} (-1)^k \left(\sigma_i \wedge \sigma_j \wedge \omega_j + \sigma_i \wedge \sigma_k \wedge \omega_k\right) x_i \end{aligned}$$

for $i, j, k = 1, 2, 3$ (cyclic). From this formula and (12), we get a quadratic polynomial for J_X. If we use Stokes formula and choose the moving coordinate system such that the coefficients of the mixture quadratic terms and the coefficients of x_i will vanish, we have the following volume formula of the region which is determined in R' by this fixed point $X \in R$:

$$J_X = J_O + \sum_{i=1}^{3} A_i x_i^2, \tag{13}$$

where $J_O = \int_G \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ is the volume of the region which is determined in R' by the point $O \in R$ and

$$A_i = \int_G \sigma_i \wedge \omega_j \wedge \omega_k = -\frac{1}{2}(-1)^k \left(\int_{R(G)} \sigma_j \wedge \omega_j + \int_{R(G)} \sigma_k \wedge \omega_k \right)$$

is obtained in terms of equations (7) and (9). We point out that the boundary of G denoted by $R(G)$ is a closed and orientable surface having the structure of connectedness of a sphere. We remark that the motion $\mathfrak{B}_3$ is related with a 2-parameter closed spatial motion which is denoted by $\mathfrak{B}_2$ and corresponds to $R(G)$ such that a fixed point $X \in R$ draws a closed trajectory surface in R' under the motion $\mathfrak{B}_2$ and this surface is the boundary of the region which is determined in R' by this fixed point during the motion $\mathfrak{B}_3$ (see [3], [9], [11] and [12] for the details in Euclidean sense).

By means of [14], we note that the above volume formula is the Lorentzian version of the volume formula given in [3], [9] and [11] in Euclidean sense.

So, taking into account [14], we have the following theorem which is the Lorentzian version of the theorem given in [3], [9] and [11] in Euclidean sense:

Theorem 2.1. *Under the motions $\mathfrak{B}_3$, all the fixed points of R which determine the regions having equal volume J_X in R' generally lie on the same quadric Φ_X.*

3. Holditch-Type Theorems during the 3-parameter spatial motions in L^3

In this section, we give Holditch-Type Theorems and some results under the 3-parameter spatial motions $\mathfrak{B}_3$ in L^3 taking into account [9] and [11] in Euclidean sense.

Let $X = (x_i)$ and $Y = (y_i)$ be two different fixed points in R and $Z = (z_i)$ be another fixed point on the line segment XY. In this case, we can write that

$$z_i = \lambda \, x_i + \mu \, y_i; \ \lambda + \mu = 1. \tag{14}$$

By means of equations (13) and (14), we obtain

$$J_Z = \lambda^2 J_X + 2 \lambda \, \mu \, J_{XY} + \mu^2 J_Y, \tag{15}$$

where the expression

$$J_{XY} = J_O + \sum_{i=1}^{3} A_i \, x_i \, y_i \tag{16}$$

is said to be the *mixture volume*. It is obvious that $J_{XY} = J_{YX}$ and $J_{XX} = J_X$. Since

$$J_X - 2J_{XY} + J_Y = \sum_{i=1}^{3} A_i \left(x_i - y_i\right)^2 , \tag{17}$$

taking into account $\lambda + \mu = 1$, we can rewrite equation (15) as follows:

$$J_Z = \lambda \, J_X - \lambda \, \mu \sum_{i=1}^{3} A_i \left(x_i - y_i\right)^2 + \mu \, J_Y. \tag{18}$$

Thus, we have a relation among the volumes of the regions which are determined in R' by the collinear three fixed points X, Y and Z of R during the motions $\mathfrak{B}_3$.

If we define the square of the distance $D\left(X, Y\right)$ between the points X and Y of R with respect to the motions $\mathfrak{B}_3$ and $\mathfrak{B}_2$ by

$$D^2\left(X, Y\right) = \varepsilon \sum_{i=1}^{3} A_i \left(x_i - y_i\right)^2 ; \; \varepsilon = \mp 1, \tag{19}$$

we can express equation (18) as follows:

$$J_Z = \lambda \, J_X + \mu \, J_Y - \varepsilon \, \lambda \, \mu \, D^2\left(X, Y\right). \tag{20}$$

In terms of the orientation of the line segment XY, it is clear that $D\left(X, Y\right) = -D\left(Y, X\right)$. And also since X, Y and Z are collinear, it is obvious that

$$D\left(X, Z\right) + D\left(Z, Y\right) = D\left(X, Y\right).$$

Moreover, from equation (14), we can write that

$$\lambda \, D\left(X, Y\right) = D\left(Z, Y\right) \text{ and } \mu \, D\left(X, Y\right) = D\left(X, Z\right).$$

As a result, if we substitute these equations into equation (20), we deduce

$$J_Z = \lambda \, J_X + \mu \, J_Y - \varepsilon \, D\left(X, Z\right) D\left(Z, Y\right). \tag{21}$$

Now, suppose that the fixed points X and Y of R determine the same region in R'. So, their volumes are equal, that is, $J_X = J_Y$. Moreover, assume that the fixed point Z determines another region in R'. In this regard, by means of $\lambda + \mu = 1$ and equation (21), we get

$$J_X - J_Z = \varepsilon \, D\left(X, Z\right) D\left(Z, Y\right). \tag{22}$$

Consequently, taking into account [14], we have the following Holditch-Type Theorem similar to the theorem given by Müller in [9] and [11] in Euclidean sense:

Theorem 3.1. *Under the motions $\mathfrak{B}_3$, consider a line segment XY with constant length in R. Suppose that the endpoints of XY determine the same region in R'. Then, another fixed point Z on this line segment determines another region in R'. The difference between the volumes of these regions depends only on distances $D\left(X, Z\right)$ and $D\left(Z, Y\right)$ which are measured in a special way with respect to the motions $\mathfrak{B}_3$ and $\mathfrak{B}_2$.*

We remark that this theorem is an extension of the classical Holditch Theorem in [8] to the 3-parameter spatial motions in L^3.

Now, let X and Y be two different fixed points of a quadric Φ_X. Then, $\Phi_X = \Phi_Y$ and so $J_X = J_Y$. If a point P is the harmonic conjugate of the point Z given in (14) with respect to X and Y, then we obtain the following equation

$$\sum_{i=1}^{3} A_i\, p_i\, z_i = J_X - J_O,$$

where $p_i = \lambda'\, x_i + \mu'\, y_i;\ \lambda' + \mu' = 1$ and $\mu\lambda' + \mu'\lambda = 0$. On the other hand, from equation (16), it is clear that

$$\sum_{i=1}^{3} A_i\, p_i\, z_i = J_{PZ} - J_O.$$

Therefore, we can easily see that

$$J_{PZ} = J_X.$$

So, taking into account [14], we can give the following theorem which is similar to the theorem expressed in [11] in Euclidean sense:

Theorem 3.2. *During the motions $\mathfrak{B}_3$, for all pairs of conjugate points P and Z with respect to the quadric Φ_X, the mixture volume J_{PZ} is equal to J_X which is the volume of each region determined in R' by each point of Φ_X.*

Now, choose the points X and Y on the same generator of the quadric Φ_X. In this respect, the point Z determined by equation (14) also lies on the quadric Φ_X. Hence, $J_X = J_Y = J_Z$. For this reason, by using $\lambda + \mu = 1$ and equation (20), we get $D(X, Y) = 0$.

Thus, taking into account [14], we can give the following theorem which is similar to the theorem expressed in [11] in Euclidean sense:

Theorem 3.3. *Let X and Y be two different fixed points of R. If X and Y lie on the same generator of the quadric Φ_X, then the distance $D(X, Y)$ which is measured in a special way with respect to the motions $\mathfrak{B}_3$ and $\mathfrak{B}_2$ vanishes.*

Now, under the motions $\mathfrak{B}_3$, we give a relation among the volumes of the regions determined in R' by non-collinear three fixed points $X_1 = (x_{1i})$, $X_2 = (x_{2i})$ and $X_3 = (x_{3i})$ of R and another fixed point $Q = (q_i)$ on the plane which is described by these three fixed points. In this regard, we can write that

$$q_i = \lambda_1\, x_{1i} + \lambda_2\, x_{2i} + \lambda_3\, x_{3i};\ \lambda_1 + \lambda_2 + \lambda_3 = 1, \tag{23}$$

where $1 \le i \le 3$. By means of equations (13), (16) and (23), we deduce

$$J_Q = \lambda_1^2\, J_{X_1} + \lambda_2^2\, J_{X_2} + \lambda_3^2\, J_{X_3} + 2\,\lambda_1\,\lambda_2\, J_{X_1X_2} + 2\,\lambda_2\,\lambda_3\, J_{X_2X_3} + 2\,\lambda_3\,\lambda_1\, J_{X_3X_1}.$$

Taking into account equations (17), (19) and (23) in the last equation, we obtain

$$\begin{aligned} J_Q &= \lambda_1\, J_{X_1} + \lambda_2\, J_{X_2} + \lambda_3\, J_{X_3} \\ &\quad - \left\{\varepsilon_{12}\,\lambda_1\,\lambda_2\, D^2(X_1, X_2) + \varepsilon_{23}\,\lambda_2\,\lambda_3\, D^2(X_2, X_3) + \varepsilon_{31}\,\lambda_3\,\lambda_1\, D^2(X_3, X_1)\right\} \end{aligned} \tag{24}$$

as a generalization of equation (20).

Let Q_i be the intersection points of lines X_iQ and X_jX_k. If any of the distances related with the moving triangle do not vanish, then we can take

$$\lambda_i = \frac{D(Q, Q_i)}{D(X_i, Q_i)} = \frac{D\left(X_j, Q\right) D\left(X_k, Q_j\right)}{D\left(X_j, Q_j\right) D(X_k, X_i)} = \frac{D(X_k, Q)\, D\left(Q_k, X_j\right)}{D(X_k, Q_k)\, D\left(X_i, X_j\right)} \tag{25}$$

for $i, j, k = 1, 2, 3$ (cyclic). In this respect, we get

$$J_Q = \sum_{i=1}^{3} \frac{D(Q, Q_i)}{D(X_i, Q_i)} J_{X_i} - \sum_{i=1}^{3} \varepsilon_{ij} \left(\frac{D(X_k, Q)}{D(X_k, Q_k)} \right)^2 D\left(Q_k, X_j\right) D(X_i, Q_k). \tag{26}$$

Assume that the fixed points X_1, X_2 and X_3 of R determine the same region in R' during the motions $\mathfrak{B}_3$. In this case, it is obvious that $J_{X_1} = J_{X_2} = J_{X_3}$. Denote this volume by J. Also suppose that the fixed point Q determines another region in R'. In this regard, by means of equation $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and equations (25) and (26), we deduce

$$J - J_Q = \sum_{i=1}^{3} \varepsilon_{ij} \left(\frac{D(X_k, Q)}{D(X_k, Q_k)} \right)^2 D\left(Q_k, X_j\right) D(X_i, Q_k).$$

So, taking into account [14], we have the following Holditch-Type Theorem which is similar to the theorem expressed in [9] in Euclidean sense:

Theorem 3.4. *Under the motions $\mathfrak{B}_3$, consider a triangle with the vertices X_1, X_2 and X_3 in R. Assume that the vertices of the triangle determine the same region in R'. Then, another fixed point Q on the plane described by the points X_1, X_2 and X_3 determines another region in R'. The difference between the volumes of these regions depends only on the distances of the moving triangle which are measured in a special way with respect to the motions $\mathfrak{B}_3$ and $\mathfrak{B}_2$.*

Now, consider a point $Q \notin \Phi_{X_1}$ which is on the tangent plane of Φ_{X_1} at the point X_1. There are two generators intersecting at X_1 in the tangent plane. Choose the fixed point X_2 on one of these generators and another fixed point X_3 on the other generator such that the projections of Q across to these generators give respectively the points Q_3 and Q_2 introduced as before. In terms of Theorem 3.3, we have $D(X_1, X_2) = D(X_1, X_3) = 0$. As a result, by using $J_{X_1} = J_{X_2} = J_{X_3} = J$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$ in (24), we obtain

$$J - J_Q = \varepsilon_{23} \frac{D(Q, Q_2)}{D(X_2, Q_2)} \frac{D(Q, Q_3)}{D(X_3, Q_3)} D^2(X_2, X_3),$$

where the first equality of (25) is valid for $i = 2$ and $i = 3$. This can also be thought as an extension of the classical Holditch Theorem in [8] to the 3-parameter spatial motions in L^3.

4. Corollaries of Holditch-Type Theorems during the 3-parameter spatial motions in L^3

In this section, we give some corollaries of Holditch-Type Theorems expressed in the previous section under the 3-parameter spatial motions $\mathfrak{B}_3$ in L^3, taking into account [5], [6], [7] and section 2.4 of [14] in Euclidean sense.

Let M, N, X and Y be four different fixed points in R. Moreover, let X and Y be on the line segment MN. During the motions $\mathfrak{B}_3$, assume that while M and N lie on the same quadric Φ_M, X (respectively, Y) lies on the quadric Φ_X (respectively, Φ_Y) which is different from Φ_M. Denote the difference between the volumes J_M and J_X by J and the difference between the volumes J_M and J_Y by J'. In this respect, if we respectively evaluate the collinear triple points M, X, N and M, Y, N in equation (22), we get

$$J = J_M - J_X = \varepsilon D(M, X) D(X, N)$$

and

$$J' = J_M - J_Y = \varepsilon D(M, Y) D(Y, N),$$

respectively. The last two equations yield

$$\frac{J}{J'} = \frac{D(M, X) D(X, N)}{D(M, Y) D(Y, N)}$$

or

$$\frac{J}{J'} = \left(\frac{D(M,X)}{D(M,Y)}\right)^2 \frac{D(M,Y)\,D(X,N)}{D(M,X)\,D(Y,N)}. \tag{27}$$

Here, we remark that the ratio J/J' depends only on the choices of the points X and Y on the line segment MN. Since $X \neq Y$, it follows that

$$\frac{D(M,X)}{D(M,Y)} \neq 1.$$

The following ratio in equation (27)

$$\frac{D(M,Y)\,D(X,N)}{D(M,X)\,D(Y,N)}$$

is the cross ratio denoted by (MN, YX) of the points M, N, X and Y.

Thus, taking into account [14], we have the following theorem:

Theorem 4.1. *Under the motions $\mathfrak{B}_3$, the ratio J/J' defined as above depends only on the relative positions of the points M, N, X and Y.*

And also, taking into account [14], we have the following corollary as a special case of the above theorem:

Corollary 4.2. *Let M, N, X and Y be four different fixed points in R. Moreover, let X and Y be on the line segment MN. During the motions $\mathfrak{B}_3$, suppose that while M and N determine the same region with volume J_M in R', X (respectively, Y) determines a region whose volume is J_X (respectively, J_Y) and different from J_M in R'. Furthermore, if we denote the difference between the volumes J_M and J_X by J and the difference between the volumes J_M and J_Y by J', then equation (27) holds.*

Now, let M, N, A and B be four different fixed points in R and another fixed point X be the intersection of line segments MN and AB. Moreover, under the motions $\mathfrak{B}_3$, assume that while M and N lie on the same quadric Φ_M, A and B lie on the same quadric Φ_A. In this regard, we get the following results and theorems:

Under the above conditions, if we respectively use the collinear triple points M, X, N and A, X, B in equation (22), we obtain

$$J_M - J_X = \varepsilon_1 D(M,X)\,D(X,N)\,;\, \varepsilon_1 = \mp 1 \tag{28}$$

and

$$J_A - J_X = \varepsilon_2 D(A,X)\,D(X,B)\,;\, \varepsilon_2 = \mp 1, \tag{29}$$

respectively. In this respect, we have the following two cases:

i) Let $\varepsilon_1 = \varepsilon_2 = \varepsilon$. In this case, by using equations (28) and (29), we get

$$J_M - J_A = \varepsilon[D(M,X)\,D(X,N) - D(A,X)\,D(X,B)]. \tag{30}$$

If $\Phi_M = \Phi_A$, then $J_M = J_A$. Therefore, from equation (30), it follows that

$$D(M,X)\,D(X,N) - D(A,X)\,D(X,B) = 0.$$

Conversely, if the last equation is valid, equation (30) yields

$$J_M = J_A.$$

This means that M, N, A and B lie on the same quadric ($\Phi_M = \Phi_A$) of R during the motions $\mathfrak{B}_3$.

ii) Let $\varepsilon_1 = -\varepsilon_2 = \varepsilon$. In this case, by using equations (28) and (29), we obtain

$$J_M - J_A = \varepsilon[D(M,X)D(X,N) + D(A,X)D(X,B)]. \tag{31}$$

If $\Phi_M = \Phi_A$, then $J_M = J_A$. So, we deduce the following equation from equation (31)

$$D(M,X)D(X,N) + D(A,X)D(X,B) = 0.$$

Conversely, if the last equation is valid, equation (31) gives

$$J_M = J_A.$$

This means that M, N, A and B lie on the same quadric ($\Phi_M = \Phi_A$) of R under the motions $\mathfrak{B}_3$.

Hence, taking into account [14], we have the following theorem:

Theorem 4.3. *Let M, N, A and B be four different fixed points in R and another fixed point X be the intersection of the line segments MN and AB. Moreover, during the motions $\mathfrak{B}_3$, suppose that while M and N lie on the same quadric Φ_M, A and B lie on the same quadric Φ_A. Then,*

i) Let $\varepsilon_1 = \varepsilon_2 = \varepsilon$, where ε_1 and ε_2 are indicated in equations (28) and (29). In this case, M, N, A and B lie on the same quadric ($\Phi_M = \Phi_A$) of R if and only if $D(M,X)D(X,N) - D(A,X)D(X,B) = 0$.

ii) Let $\varepsilon_1 = -\varepsilon_2 = \varepsilon$, where ε_1 and ε_2 are indicated in equations (28) and (29). In this case, M, N, A and B lie on the same quadric ($\Phi_M = \Phi_A$) of R if and only if $D(M,X)D(X,N) + D(A,X)D(X,B) = 0$.

Remark 4.4. *Let M, N, A and B be four different fixed points in R and another fixed point X be the intersection of the line segments MN and AB. Moreover, under the motions $\mathfrak{B}_3$, assume that while M and N lie on the same quadric Φ_M, A and B lie on the same quadric Φ_A. Furthermore, suppose that X lie on the quadric Φ_X. Let the quadrics Φ_M, Φ_A and Φ_X be the same. In this regard, $J_M = J_A = J_X$. Thus, if we respectively use the collinear triple points M, X, N and A, X, B in equation (22), then we find*

$$D(M,X)D(X,N) = 0 \tag{32}$$

and

$$D(A,X)D(X,B) = 0, \tag{33}$$

respectively. Conversely, if equations (32) and (33) are valid, by using the collinear triple points M, X, N (respectively, A, X, B) in equation (22), we obtain

$$J_M = J_X$$

and

$$J_A = J_X,$$

respectively. This means that M, N, A, B and X lie on the same quadric ($\Phi_M = \Phi_A = \Phi_X$) of R during the motions $\mathfrak{B}_3$.

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