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Extension of the Darboux frame into Euclidean 4-space and its invariants

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Abstract: In this paper, by considering a Frenet curve lying on an oriented hypersurface, we extend the Darboux frame field into Euclidean 4-space $\mathbb{E}^4$. Depending on the linear independency of the curvature vector with the hypersurface's normal, we obtain two cases for this extension. For each case, we obtain some geometrical meanings of new invariants along the curve on the hypersurface. We also give the relationships between the Frenet frame curvatures and Darboux frame curvatures in $\mathbb{E}^4$. Finally, we compute the expressions of the new invariants of a Frenet curve lying on an implicit hypersurface.

Key words: Curves on hypersurface, Darboux frame field, curvatures

1. Introduction

In differential geometry, frame fields constitute an important tool while studying curves and surfaces. The most familiar frame fields are the Frenet–Serret frame along a space curve, and the Darboux frame along a surface curve. In Euclidean 3-space, the Darboux frame is constructed by the velocity of the curve and the normal vector of the surface whereas the Frenet–Serret frame is constructed from the velocity and the acceleration of the curve. Expressing the derivatives of these frames' vector fields in terms of the vector fields themselves includes some real valued functions. These functions are called the curvature and the torsion for the Frenet–Serret frame, and the normal curvature, the geodesic curvature, and the geodesic torsion for the Darboux frame [2,6–8,10]. The generalizations of the Frenet–Serret frame into higher dimensional spaces are well known. However, the generalization of the Darboux frame even into 4-space is not available (in the literature, we do not come across any work that extends the above three curvatures of a surface curve in $\mathbb{E}^3$ into the hypersurface curve in $\mathbb{E}^4$).

In this paper, we construct a frame field (in which the first three vectors span the tangent space of the hypersurface along the curve) along a Frenet curve lying on an oriented hypersurface, and call this new frame field an "extended Darboux frame field" (we think that this extension will be a useful tool for studying curves on hypersurfaces in $\mathbb{E}^4$). Later, we obtain the derivative equations of this new frame field and give the geometrical meanings of the new curvatures of the curve with respect to the hypersurface. Finally, the expressions of our new curvatures are obtained for a curve lying on hypersurfaces defined by implicit equations. By using the obtained expressions for the new invariants, an example is also presented.

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2. Preliminaries

2.1. Darboux frame in $\mathbb{E}^3$

Let $S \subset \mathbb{E}^3$ be an oriented surface and $\gamma : I \subset \mathbb{R} \to S$ be a unit speed curve. Let $\mathbf{T}$ denote the unit tangent vector field of γ and $\mathbf{U}$ denote the unit normal vector field of S restricted to the curve γ. Then the Darboux frame field along γ is given by $\{\mathbf{T}, \mathbf{V}, \mathbf{U}\}$, where $\mathbf{V} = \mathbf{U} \times \mathbf{T}$. Thus, we can express the derivatives according to the arc-length of each vector field along the curve γ as [6]

$$\begin{cases} \mathbf{T}' &= \kappa_g \mathbf{V} + \kappa_n \mathbf{U}, \\ \mathbf{V}' &= -\kappa_g \mathbf{T} + \tau_g \mathbf{U}, \\ \mathbf{U}' &= -\kappa_n \mathbf{T} - \tau_g \mathbf{V}, \end{cases}$$

where κ_g, κ_n, and τ_g denote the geodesic curvature, the normal curvature, and the geodesic torsion of the curve γ, respectively.

2.2. Curves on a hypersurface in $\mathbb{E}^4$

Definition 1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$. The ternary product (or vector product) of the vectors $\mathbf{x} = \sum_{i=1}^{4} x_i \mathbf{e_i}$, $\mathbf{y} = \sum_{i=1}^{4} y_i \mathbf{e_i}$, and $\mathbf{z} = \sum_{i=1}^{4} z_i \mathbf{e_i}$ is defined by [5, 9]*

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

The ternary product has the following properties [9]:

1) $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = -\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{z} = \mathbf{y} \otimes \mathbf{z} \otimes \mathbf{x}$
2) $\langle \mathbf{x}, \mathbf{y} \otimes \mathbf{z} \otimes \mathbf{w} \rangle = \det\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$
3) $(\mathbf{x} + \mathbf{y}) \otimes \mathbf{z} \otimes \mathbf{w} = \mathbf{x} \otimes \mathbf{z} \otimes \mathbf{w} + \mathbf{y} \otimes \mathbf{z} \otimes \mathbf{w}$

Let $M \subset \mathbb{E}^4$ denote a regular hypersurface and $\beta : I \subset \mathbb{R} \to M$ be a unit speed curve. If $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ is the moving Frenet frame along β, then the Frenet formulas are given by [4]

$$\begin{cases} \mathbf{t}' &= k_1 \mathbf{n}, \\ \mathbf{n}' &= -k_1 \mathbf{t} + k_2 \mathbf{b}_1, \\ \mathbf{b}_1' &= -k_2 \mathbf{n} + k_3 \mathbf{b}_2, \\ \mathbf{b}_2' &= -k_3 \mathbf{b}_1, \end{cases} \tag{1}$$

where $\mathbf{t}, \mathbf{n}, \mathbf{b}_1$, and $\mathbf{b}_2$ denote the unit tangent, the principal normal, the first binormal, and the second binormal vector fields; k_1, k_2, k_3 are the curvature functions of the curve β.

Theorem 1 *Let $\alpha : I \to \mathbb{E}^4$ be an arbitrary-speed regular curve. Then the Frenet vectors of the curve are given by [1]*

$$\mathbf{t} = \frac{\dot{\alpha}}{||\dot{\alpha}||}, \quad \mathbf{b}_2 = \frac{\dot{\alpha} \otimes \ddot{\alpha} \otimes \dddot{\alpha}}{||\dot{\alpha} \otimes \ddot{\alpha} \otimes \dddot{\alpha}||}, \quad \mathbf{b}_1 = \frac{\mathbf{b}_2 \otimes \dot{\alpha} \otimes \ddot{\alpha}}{||\mathbf{b}_2 \otimes \dot{\alpha} \otimes \ddot{\alpha}||}, \quad \mathbf{n} = \frac{\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \dot{\alpha}}{||\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \dot{\alpha}||} \tag{2}$$

and the curvatures of the curve are given by

$$k_1 = \frac{\langle \mathbf{n}, \ddot{\alpha} \rangle}{||\dot{\alpha}||^2}, \quad k_2 = \frac{\langle \mathbf{b}_1, \dddot{\alpha} \rangle}{||\dot{\alpha}||^3 k_1}, \quad k_3 = \frac{\langle \mathbf{b}_2, \ddddot{\alpha} \rangle}{||\dot{\alpha}||^4 k_1 k_2}, \tag{3}$$

where "$\langle , \rangle$" denotes the scalar product.

Definition 2 *A unit speed curve $\beta : I \to \mathbb{E}^n$ of class C^n is called a Frenet curve if the vectors $\beta'(s), \beta''(s), ..., \beta^{(n-1)}(s)$ are linearly independent at each point along the curve.*

3. The extended Darboux frame field

3.1. The construction of the extended Darboux frame field

Let $\mathcal{M}$ be an orientable hypersurface oriented by the unit normal vector field N in $\mathbb{E}^4$ and β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on $\mathcal{M}$. We denote the unit tangent vector field of the curve by T, and denote the hypersurface unit normal vector field restricted to the curve by N, i.e.

$$\mathsf{T}(s) = \beta'(s) \quad \text{and} \quad \mathsf{N}(s) = N(\beta(s)).$$

We can construct the extended Darboux frame field along the Frenet curve β as follows:

Case 1. If the set $\{\mathsf{N}, \mathsf{T}, \beta''\}$ is linearly independent, then using the Gram–Schmidt orthonormalization method gives the orthonormal set $\{\mathsf{N}, \mathsf{T}, \mathsf{E}\}$, where

$$\mathsf{E} = \frac{\beta'' - \langle \beta'', \mathsf{N} \rangle \mathsf{N}}{||\beta'' - \langle \beta'', \mathsf{N} \rangle \mathsf{N}||}. \tag{4}$$

Case 2. If the set $\{\mathsf{N}, \mathsf{T}, \beta''\}$ is linearly dependent, i.e. if β'' is in the direction of the normal vector N, applying the Gram–Schmidt orthonormalization method to $\{\mathsf{N}, \mathsf{T}, \beta'''\}$ yields the orthonormal set $\{\mathsf{N}, \mathsf{T}, \mathsf{E}\}$, where

$$\mathsf{E} = \frac{\beta''' - \langle \beta''', \mathsf{N} \rangle \mathsf{N} - \langle \beta''', \mathsf{T} \rangle \mathsf{T}}{||\beta''' - \langle \beta''', \mathsf{N} \rangle \mathsf{N} - \langle \beta''', \mathsf{T} \rangle \mathsf{T}||}. \tag{5}$$

In each case, if we define $\mathsf{D} = \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{E}$, we have four unit vector fields $\mathsf{T}, \mathsf{E}, \mathsf{D}$, and N, which are mutually orthogonal at each point of β. Thus, we have a new orthonormal frame field $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ along the curve β instead of its Frenet frame field. It is obvious that $\mathsf{E}(s)$ and $\mathsf{D}(s)$ are also tangent to the hypersurface $\mathcal{M}$ for all s. Thus, the set $\{\mathsf{T}(s), \mathsf{E}(s), \mathsf{D}(s)\}$ spans the tangent hyperplane of the hypersurface at the point $\beta(s)$. We call these new frame fields

"*extended Darboux frame field of first kind*" or in short "ED-*frame field of first kind*"
in case 1,
and
"*extended Darboux frame field of second kind*" or in short "ED-*frame field of second kind*"
in case 2, respectively.

Remark 1 *The Darboux frame field $\{\mathbf{T}, \mathbf{V}, \mathbf{U}\}$ along the Frenet curve γ in 3-space can also be constructed by the method explained in Case 1 and Case 2 depending on the linear independency of $\{\mathbf{U}, \mathbf{T}, \gamma''\}$.*

Remark 2 *If a Frenet curve β parametrized by arc-length s lies in a hyperplane with the unit normal vector* N, *we may write* $\langle \beta(s)-\beta(0), \mathsf{N}\rangle = 0$. *Thus, we have* $\langle \beta'(s), \mathsf{N}\rangle = 0$, $\langle \beta''(s), \mathsf{N}\rangle = 0$, $\langle \beta'''(s), \mathsf{N}\rangle = 0$, *i.e. Case 1 is valid. If we substitute* $\langle \beta''(s), \mathsf{N}\rangle = 0$ *into (4), we obtain* $\mathsf{E}(s) = \mathbf{n}(s)$. *Moreover, since* $\beta', \beta'', \beta'''$ *are perpendicular to* N, *using (2) we get* N *and* $\mathbf{b}_2$ *are parallel. Hence, if we take* $\mathsf{N} = \mathbf{b}_2$, *we obtain* $\mathsf{D}(s) = \mathbf{b}_1(s)$, *i.e.* ED*-frame field of first kind coincides with the Frenet frame.*

Remark 3 *If a Frenet curve β parametrized by arc-length s is a geodesic on a hypersurface, by the proper orientiation of the hypersurface with* $\mathsf{N}(s) = \mathbf{n}(s)$, *Case 2 is valid. In this case, since* $\beta'' = k_1\mathbf{n}$, *substituting* $\beta''' = -k_1^2\mathbf{t} + k_1'\mathbf{n} + k_1k_2\mathbf{b}_1$ *into (5) yields* $\mathsf{E} \parallel \mathbf{b}_1$. *If we take* $\mathsf{E}(s) = \mathbf{b}_1(s)$, *we obtain* $\mathsf{D}(s) = \mathbf{b}_2(s)$, *i.e. the frame* $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ *coincides with the frame* $\{\mathbf{T}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{n}\}$.

3.2. The derivative equations

Let us now express the derivatives of these vector fields in terms of themselves in each case. Since $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ is orthonormal we have

$$\begin{array}{rcl} \mathsf{T}' &=& \langle \mathsf{T}', \mathsf{E}\rangle\, \mathsf{E} + \langle \mathsf{T}', \mathsf{D}\rangle\, \mathsf{D} + \langle \mathsf{T}', \mathsf{N}\rangle\, \mathsf{N}, \\ \mathsf{E}' &=& \langle \mathsf{E}', \mathsf{T}\rangle\, \mathsf{T} + \langle \mathsf{E}', \mathsf{D}\rangle\, \mathsf{D} + \langle \mathsf{E}', \mathsf{N}\rangle\, \mathsf{N}, \\ \mathsf{D}' &=& \langle \mathsf{D}', \mathsf{T}\rangle\, \mathsf{T} + \langle \mathsf{D}', \mathsf{E}\rangle\, \mathsf{E} + \langle \mathsf{D}', \mathsf{N}\rangle\, \mathsf{N}, \\ \mathsf{N}' &=& \langle \mathsf{N}', \mathsf{T}\rangle\, \mathsf{T} + \langle \mathsf{N}', \mathsf{E}\rangle\, \mathsf{E} + \langle \mathsf{N}', \mathsf{D}\rangle\, \mathsf{D}. \end{array} \tag{6}$$

Case 1. In this case, ED-frame field is first kind. Since we have

$$\mathsf{E} = \frac{\beta'' - \langle \beta'', \mathsf{N}\rangle \mathsf{N}}{||\beta'' - \langle \beta'', \mathsf{N}\rangle \mathsf{N}||} = \frac{\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||},$$

we get

$$\mathsf{T}' = ||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||\mathsf{E} + \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}$$

i.e. $\langle \mathsf{T}', \mathsf{D}\rangle = 0$.

Case 2. In this case, ED-frame field is second kind. Thus $\{\mathsf{N}, \mathsf{T}, \beta''\}$ is linearly dependent and

$$\mathsf{E} = \frac{\beta''' - \langle \beta''', \mathsf{N}\rangle \mathsf{N} - \langle \beta''', \mathsf{T}\rangle \mathsf{T}}{||\beta''' - \langle \beta''', \mathsf{N}\rangle \mathsf{N} - \langle \beta''', \mathsf{T}\rangle \mathsf{T}||}. \tag{7}$$

The linear dependency of $\{\mathsf{N}, \mathsf{T}, \beta''\}$ gives $\beta'' = \lambda \mathsf{N}$, that is, $\langle \mathsf{T}', \mathsf{E}\rangle = \langle \mathsf{T}', \mathsf{D}\rangle = 0$. Moreover, if we substitute $\beta''' = \lambda' \mathsf{N} + \lambda \mathsf{N}'$ into (7), we obtain $\langle \mathsf{N}', \mathsf{D}\rangle = 0$.

We denote

$$\langle \mathsf{E}', \mathsf{N}\rangle = \tau_g^1, \quad \langle \mathsf{D}', \mathsf{N}\rangle = \tau_g^2 \tag{8}$$

and call τ_g^i the geodesic torsion of order i. Similarly, we put

$$\langle \mathsf{T}', \mathsf{E}\rangle = \kappa_g^1, \quad \langle \mathsf{E}', \mathsf{D}\rangle = \kappa_g^2 \tag{9}$$

and define κ_g^i as the geodesic curvature of order i.

Lastly, if we use $\langle \mathsf{T}', \mathsf{N} \rangle = \kappa_n$, we obtain the differential equations of $\mathsf{ED}-$frame fields in matrix notation as

$$\textbf{Case 1:} \quad \begin{bmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix}, \tag{10}$$

$$\textbf{Case 2:} \quad \begin{bmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix}. \tag{11}$$

3.3. Geometrical interpretations

Now let us investigate the geometrical interpretations of the real valued functions κ_n, κ_g^1, κ_g^2, τ_g^1, τ_g^2.

3.3.1. κ_n, κ_g^1 and their geometrical interpretations

It is obvious from its definition that $\kappa_n = \langle \mathsf{T}', \mathsf{N} \rangle$ is the normal curvature of the hypersurface in the direction of the tangent vector T in each case. Hence, β is an asymptotic curve if and only if $\kappa_n = 0$ along β.

The following result can be easily seen according to the corollary 3.1 given by [3]:

Theorem 2 *Let $\beta(s)$ be a unit-speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space, and $\mathcal{M}_1$, $\mathcal{M}_2$ be the hyperplanes at $\beta(s_0) \in \mathcal{M}$ determined by $\{\mathsf{T}(s_0), \mathsf{E}(s_0), \mathsf{N}(s_0)\}$ and $\{\mathsf{T}(s_0), \mathsf{D}(s_0), \mathsf{N}(s_0)\}$, respectively. Then the first curvature, at the point $\beta(s_0)$, of the intersection curve of the hypersurfaces $\mathcal{M}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ is $|\kappa_n(s_0)|$, where κ_n is the normal curvature of the hypersurface $\mathcal{M}$ in the direction of the tangent vector T.*

Theorem 3 *Let $\beta(s)$ be a unit-speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space. If α denotes the orthogonal projection of the curve β onto the tangent hyperplane at the point $\beta(s_0)$, then the first curvature of the projection curve α is given by $k_1^\alpha(s_0) = |\kappa_g^1(s_0)|$.*

Proof Since α denotes the orthogonal projection curve of β onto the tangent hyperplane at $\beta(s_0)$, we may write

$$\alpha(s) = \beta(s) - \langle \beta(s) - \beta(s_0), \mathsf{N}(s_0) \rangle \, \mathsf{N}(s_0).$$

Differentiating both sides of the last equation with respect to s yield

$$\alpha'(s_0) = \mathsf{T}(s_0),$$

$$\alpha''(s_0) = \kappa_g^1(s_0)\mathsf{E}(s_0),$$

$$\begin{aligned} \alpha'''(s_0) \quad = \quad & \left\{-(\kappa_g^1)^2(s_0) - (\kappa_n)^2(s_0)\right\} \mathsf{T}(s_0) \\ & + \left\{(\kappa_g^1)'(s_0) - \kappa_n(s_0)\tau_g^1(s_0)\right\} \mathsf{E}(s_0) \\ & + \left\{\kappa_g^2(s_0)\kappa_g^1(s_0) - \kappa_n(s_0)\tau_g^2(s_0)\right\} \mathsf{D}(s_0) \end{aligned}$$

at the point $\alpha(s_0) = \beta(s_0)$. Hence, using Theorem 1 gives the result as we desired. □

Theorem 4 *Let $\beta(s)$ be a unit-speed asymptotic curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space. If γ denotes the orthogonal projection of the curve β onto the hyperplane determined by $\{\mathsf{T}(s_0), \mathsf{E}(s_0), \mathsf{N}(s_0)\}$ at the point $\beta(s_0)$, then the first curvature of γ is given by $k_1^{\gamma}(s_0) = |\kappa_g^1(s_0)|$.*

Proof The proof can be given similar to the proof of Theorem 3. □

Now let us consider the moving Frenet frame $\{\mathsf{T}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ along β. Since $\mathbf{n}$, $\mathbf{b}_1$, $\mathbf{b}_2$, E, D, N are perpendicular to T, we may write

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi_1 & \cos\phi_2 & \cos\phi_3 \\ \cos\psi_1 & \cos\psi_2 & \cos\psi_3 \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix}.$$

Using the orthogonality of above 3×3 coefficient matrix, we get

$$\begin{bmatrix} \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix} = \begin{bmatrix} \cos\phi_1 & \cos\psi_1 & \cos\theta_1 \\ \cos\phi_2 & \cos\psi_2 & \cos\theta_2 \\ \cos\phi_3 & \cos\psi_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} \mathbf{n} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}. \tag{12}$$

Hence, using Frenet formula $\mathsf{T}' = k_1\mathbf{n}$ we obtain

$$\kappa_g^1 = \langle \mathsf{T}', \mathsf{E}\rangle = k_1 \cos\phi_1, \qquad \kappa_n = \langle \mathsf{T}', \mathsf{N}\rangle = k_1\cos\phi_3. \tag{13}$$

3.3.2. τ_g^1, τ_g^2, κ_g^2 and their geometrical interpretations

It is clear that the curve β lying on $\mathcal{M}$ is a line of curvature if and only if

$$\tau_g^1(s) = \tau_g^2(s) = 0, \quad \text{in Case 1.}$$

On the other hand, since we have $\tau_g^1 = \langle \mathsf{E}', \mathsf{N}\rangle$, (12) gives us

$$\tau_g^1 = \left\langle \frac{d}{ds}\big\{(\cos\phi_1)\mathbf{n} + (\cos\psi_1)\mathbf{b}_1 + (\cos\theta_1)\mathbf{b}_2\big\}, (\cos\phi_3)\mathbf{n} + (\cos\psi_3)\mathbf{b}_1 + (\cos\theta_3)\mathbf{b}_2 \right\rangle.$$

Thus, by using the Frenet formulas for β, we get the geodesic torsion of order 1 as

$$\begin{aligned} \tau_g^1 &= -\phi_1'\sin\phi_1\cos\phi_3 - \psi_1'\sin\psi_1\cos\psi_3 - \theta_1'\sin\theta_1\cos\theta_3 \\ &\quad + k_2(\cos\phi_1\cos\psi_3 - \cos\psi_1\cos\phi_3) + k_3(\cos\psi_1\cos\theta_3 - \cos\theta_1\cos\psi_3). \end{aligned} \tag{14}$$

Similarly, for the geodesic torsion of order 2 and the geodesic curvature of order 2, we obtain

$$\begin{aligned} \tau_g^2 &= -\phi_2'\sin\phi_2\cos\phi_3 - \psi_2'\sin\psi_2\cos\psi_3 - \theta_2'\sin\theta_2\cos\theta_3 \\ &\quad + k_2(\cos\phi_2\cos\psi_3 - \cos\psi_2\cos\phi_3) + k_3(\cos\psi_2\cos\theta_3 - \cos\theta_2\cos\psi_3) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \kappa_g^2 &= -\phi_1'\sin\phi_1\cos\phi_2 - \psi_1'\sin\psi_1\cos\psi_2 - \theta_1'\sin\theta_1\cos\theta_2 \\ &\quad + k_2(\cos\phi_1\cos\psi_2 - \cos\psi_1\cos\phi_2) + k_3(\cos\psi_1\cos\theta_2 - \cos\theta_1\cos\psi_2), \end{aligned} \tag{16}$$

respectively. Therefore, τ_g^1, and κ_g^2 have the following geometrical interpretations for a geodesic curve.

Theorem 5 *Let β be a unit-speed geodesic curve parametrized by arc-length s on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space. Let $\{\mathsf{T}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denote the Frenet frame field and* ED*-frame field of β, respectively. Then we have*

$$\kappa_g^2 = k_3, \quad \tau_g^1 = -k_2, \quad \kappa_n = k_1,$$

where $k_i\,(i = 1, 2, 3)$ denotes the i-th curvature functions of β.

Proof Since β is a geodesic curve on $\mathcal{M}$, the curvature vector is perpendicular to the tangent hyperplane, i.e. $\mathbf{n}$ and N are linearly dependent (Case 2 is valid). Thus, if we use Remark 3 we have

$$\phi_1(s) = \phi_2(s) = \psi_2(s) = \psi_3(s) = \theta_1(s) = \theta_3(s) = \frac{\pi}{2}, \quad \phi_3(s) = \psi_1(s) = \theta_2(s) = 0$$

along β. Substituting these equations into (13), (14), and (16) yields the desired results. □

Theorem 6 *Let β be a unit-speed asymptotic curve parametrized by arc-length s on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space. Let $\{\mathsf{T}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denote the Frenet frame field and* ED*-frame field of β, respectively. Then we have*

$$\kappa_g^1 = k_1, \quad \kappa_g^2 = k_2 \cos\varphi, \quad \tau_g^1 = -k_2 \sin\varphi, \quad \tau_g^2 = k_3 + \frac{d\varphi}{ds},$$

where φ denotes the angle between D and $\mathbf{b}_1$, and $k_i\,(i = 1, 2, 3)$ denotes the i-th curvature functions of β.

Proof Since β is an asymptotic curve on $\mathcal{M}$, we have $\kappa_n = 0$, i.e. $\mathbf{t}' = k_1\mathbf{n} = \kappa_g^1\mathsf{E}$. In this case $\mathbf{n}$ and E are linearly dependent (Case 1 is valid). Thus, we obtain

$$\phi_1(s) = 0, \quad \phi_2(s) = \phi_3(s) = \psi_1(s) = \theta_1(s) = \frac{\pi}{2}$$

along β. Furthermore, since in this particular case $\mathsf{D}, \mathsf{N}, \mathbf{b}_1, \mathbf{b}_2$ lie in a plane, we also have

$$\psi_2(s) = \theta_3(s) = \varphi(s), \quad \theta_2(s) = \frac{\pi}{2} - \varphi(s), \quad \psi_3(s) = \frac{\pi}{2} + \varphi(s).$$

Substituting these equations into (13)–(16) yields the desired results. □

As a consequence of the above theorem, we may give the following corollaries:

Corollary 1 *Let β be an asymptotic curve on $\mathcal{M}$. If $\varphi(s) = constant$, then the geodesic torsion of order 2 of β is equal to its third curvature, i.e. $\tau_g^2 = k_3$.*

Corollary 2 *Let β be an asymptotic curve on $\mathcal{M}$. If $\varphi(s) = 0$, then we obtain $\kappa_g^1 = k_1, \kappa_g^2 = k_2, \tau_g^1 = 0, \tau_g^2 = k_3$, i.e. the* ED*-frame field of first kind along β coincides with the Frenet frame.*

Corollary 3 *Let β be an asymptotic curve on $\mathcal{M}$. If $\varphi(s) = -\pi/2$, then the geodesic torsion of order 1,2 of β is equal to its second and third curvatures, respectively, i.e. $\tau_g^1 = k_2, \tau_g^2 = k_3$.*

Corollary 4 *Let β be an asymptotic curve on $\mathcal{M}$. In this case, we have $\left(\tau_g^1\right)^2 + \left(\kappa_g^2\right)^2 = (k_2)^2$.*

4. Computations of new invariants on implicit hypersurfaces

Let us consider a hypersurface $\mathcal{M}$ given by its implicit equation $f(x,y,z,w)=0$, and let $\beta(s)=(x(s),y(s),z(s),w(s))$ be a Frenet curve of class $C^n\,(n\geq 4)$ on $\mathcal{M}$. Then the unit normal vector field along β is given by $\mathsf{N}(s)=\frac{\nabla f}{||\nabla f||}(s)$. Moreover, we have [1]

$$\langle \nabla f, \beta' \rangle = 0, \tag{17}$$

$$\langle \nabla f, \beta'' \rangle = -\beta' H_f \left(\beta'\right)^t, \tag{18}$$

$$\langle \nabla f, \beta''' \rangle = -3\beta' H_f \left(\beta''\right)^t - \beta' \frac{d(H_f)}{ds} \left(\beta'\right)^t, \tag{19}$$

where $\beta'=[x'\ y'\ z'\ w']$, $\beta''=[x''\ y''\ z''\ w'']$, $\beta'''=[x'''\ y'''\ z'''\ w''']$, $\nabla f=[f_x\ f_y\ f_z\ f_w]$, and

$$H_f=\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} & f_{xw} \\ f_{yx} & f_{yy} & f_{yz} & f_{yw} \\ f_{zx} & f_{zy} & f_{zz} & f_{zw} \\ f_{wx} & f_{wy} & f_{wz} & f_{ww} \end{bmatrix},$$

$$\frac{d\left(H_f\right)}{ds}=\begin{bmatrix} \frac{\partial H_f}{\partial x}\left(\beta'\right)^t & \cdots & \frac{\partial H_f}{\partial w}\left(\beta'\right)^t \end{bmatrix},$$

$$\frac{\partial H_f}{\partial x}=\begin{bmatrix} f_{xxx} & f_{xyx} & f_{xzx} & f_{xwx} \\ f_{yxx} & f_{yyx} & f_{yzx} & f_{ywx} \\ f_{zxx} & f_{zyx} & f_{zzx} & f_{zwx} \\ f_{wxx} & f_{wyx} & f_{wzx} & f_{wwx} \end{bmatrix},\ldots,\ \frac{\partial H_f}{\partial w}=\begin{bmatrix} f_{xxw} & f_{xyw} & f_{xzw} & f_{xww} \\ f_{yxw} & f_{yyw} & f_{yzw} & f_{yww} \\ f_{zxw} & f_{zyw} & f_{zzw} & f_{zww} \\ f_{wxw} & f_{wyw} & f_{wzw} & f_{www} \end{bmatrix}.$$

We may also write

$$(\nabla f)' = \beta' H_f, \tag{20}$$

$$(\nabla f)'' = \beta'' H_f + \beta' \frac{d(H_f)}{ds}. \tag{21}$$

Let us now compute the expressions of the new invariants of β with respect to the hypersurface in each case.

4.1. Extended Darboux frame field of first kind (Case 1)

4.1.1. The expressions for κ_g^1 and κ_n

Since $\kappa_g^1=\langle \mathsf{T}',\mathsf{E}\rangle$, $\kappa_n=\langle \mathsf{T}',\mathsf{N}\rangle$ and $\mathsf{E}=\frac{\mathsf{T}'-\langle \mathsf{T}',\mathsf{N}\rangle\mathsf{N}}{||\mathsf{T}'-\langle \mathsf{T}',\mathsf{N}\rangle\mathsf{N}||}$, we obtain

$$\kappa_g^1=\left\langle \mathsf{T}', \frac{\mathsf{T}'-\langle \mathsf{T}',\mathsf{N}\rangle\mathsf{N}}{||\mathsf{T}'-\langle \mathsf{T}',\mathsf{N}\rangle\mathsf{N}||}\right\rangle=\sqrt{\langle \mathsf{T}',\mathsf{T}'\rangle-\langle \mathsf{T}',\mathsf{N}\rangle^2}$$

or

$$\kappa_g^1=\left\{\beta''(\beta'')^t-\frac{1}{||\nabla f||^2}\left(\beta' H_f(\beta')^t\right)^2\right\}^{\frac{1}{2}} \tag{22}$$

and

$$\kappa_n=\langle \mathsf{T}',\mathsf{N}\rangle=\frac{1}{||\nabla f||}\langle \beta'',\nabla f\rangle=\frac{-1}{||\nabla f||}\beta' H_f(\beta')^t. \tag{23}$$

4.1.2. The expression for τ_g^1

If we differentiate $\mathsf{E} = \frac{\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}$ with respect to s, we get

$$\mathsf{E}' = \frac{1}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}\Big(\mathsf{T}'' - <\mathsf{T}'', \mathsf{N}> \mathsf{N} - <\mathsf{T}', \mathsf{N}'> \mathsf{N} - <\mathsf{T}', \mathsf{N}> \mathsf{N}'\Big) \tag{24}$$

$$+\frac{d}{ds}\left(\frac{1}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}\right)\Big(\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}\Big).$$

Thus, we deduce

$$\tau_g^1 = \langle \mathsf{E}', \mathsf{N}\rangle = \frac{-\langle \mathsf{T}', \mathsf{N}'\rangle}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}$$

or

$$\tau_g^1 = \frac{-1}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}\left\langle \mathsf{T}', \frac{(\nabla f)'}{||\nabla f||} - \frac{1}{||\nabla f||^3}\langle \nabla f, (\nabla f)'\rangle \nabla f\right\rangle.$$

Hence, in matrix notation we have

$$\tau_g^1 = -\frac{1}{\xi}\left\{\frac{1}{||\nabla f||}\beta' H_f(\beta'')^t + \frac{1}{||\nabla f||^3}\Big(\beta' H_f(\nabla f)^t\Big)\Big(\beta' H_f(\beta')^t\Big)\right\}, \tag{25}$$

where

$$\xi = \left\{\beta''(\beta'')^t - \frac{1}{||\nabla f||^2}\Big(\beta' H_f(\beta')^t\Big)^2\right\}^{\frac{1}{2}}.$$

4.1.3. The expression for κ_g^2

If we use (24), and $\langle \mathsf{T}', \mathsf{D}\rangle = 0$, we obtain

$$\kappa_g^2 = \langle \mathsf{E}', \mathsf{D}\rangle = \frac{1}{\xi}\Big(\langle \mathsf{T}'', \mathsf{D}\rangle - \langle \mathsf{T}', \mathsf{N}\rangle\langle \mathsf{N}', \mathsf{D}\rangle\Big). \tag{26}$$

On the other hand, substituting $\mathsf{E} = \frac{\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}$ into $\mathsf{D} = \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{E}$ yields $\mathsf{D} = \xi^{-1}\mathsf{N} \otimes \mathsf{T} \otimes \mathsf{T}'$. Then from (26) we have the expression for the geodesic curvature of order 2 as

$$\kappa_g^2 = \frac{1}{\xi^2||\nabla f||}\left\{\begin{vmatrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ x''' & y''' & z''' & w''' \\ f_x & f_y & f_z & f_w \end{vmatrix} + \frac{1}{||\nabla f||^2}\beta' H_f(\beta')^t \begin{vmatrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ a & b & c & d \\ f_x & f_y & f_z & f_w \end{vmatrix}\right\}, \tag{27}$$

where $(\nabla f)' = \alpha' H_f = [a\ b\ c\ d]$.

4.1.4. The expression for τ_g^2

Since $\tau_g^2 = \langle \mathsf{D}', \mathsf{N}\rangle = -\langle \mathsf{N}', \mathsf{D}\rangle = -\left\langle \frac{(\nabla f)'}{||\nabla f||} - \frac{1}{||\nabla f||^3}\langle \nabla f, (\nabla f)'\rangle \nabla f, \mathsf{D}\right\rangle$, we obtain

$$\tau_g^2 = \frac{-1}{||\nabla f||}\Big\langle (\nabla f)', \mathsf{D}\Big\rangle = \frac{-1}{\xi||\nabla f||^2}\begin{vmatrix} x' & y' & z' & w' \\ x'' & y'' & z'' & w'' \\ a & b & c & d \\ f_x & f_y & f_z & f_w \end{vmatrix}. \tag{28}$$

4.2. Extended Darboux frame field of second kind (Case 2)

4.2.1. The expression for κ_n and κ_g^2

The normal curvature $\kappa_n = \langle \mathsf{T}', \mathsf{N}\rangle$ is obtained by (23).

On the other hand, since $\beta'' = \lambda \mathsf{N}$ in Case 2, from (5) we obtain

$$\mathsf{E} = \frac{\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}}{||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}||}$$

and

$$\mathsf{E}' = \frac{1}{||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}||}\Big(\mathsf{N}'' - < \mathsf{N}'', \mathsf{T} > \mathsf{T} - < \mathsf{N}', \mathsf{T}' > \mathsf{T} - < \mathsf{N}', \mathsf{T} > \mathsf{T}'\Big)$$

$$+\frac{d}{ds}\left(\frac{1}{||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}||}\right)\Big(\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}\Big).$$

Thus, since $\langle \mathsf{T}', \mathsf{D}\rangle = 0,\ \langle \mathsf{N}', \mathsf{D}\rangle = 0$ in Case 2 we find

$$\kappa_g^2 = \langle \mathsf{E}', \mathsf{D}\rangle = \frac{\langle \mathsf{N}'', \mathsf{D}\rangle}{||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}||} = \frac{-\langle \mathsf{N}', \mathsf{D}'\rangle}{||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}||} = \frac{-1}{\mu}\Big\langle \mathsf{N}', \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{N}''\Big\rangle,$$

where

$$\mu = ||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T}\rangle \mathsf{T}||^2 = \Big\langle \mathsf{N}', \mathsf{N}'\Big\rangle - \Big\langle \mathsf{N}', \mathsf{T}\Big\rangle^2.$$

If we substitute $\mathsf{N}' = \frac{(\nabla f)'}{||\nabla f||} - \frac{1}{||\nabla f||^3}\Big\langle \nabla f, (\nabla f)'\Big\rangle \nabla f$ into the last equations, we obtain

$$\kappa_g^2 = \frac{-1}{\mu||\nabla f||^3}\Big\langle (\nabla f)', \nabla f \otimes \mathsf{T} \otimes (\nabla f)''\Big\rangle$$

or

$$\kappa_g^2 = \frac{1}{\mu||\nabla f||^3}\begin{vmatrix} x' & y' & z' & w' \\ f_x & f_y & f_z & f_w \\ a & b & c & d \\ p & q & r & s \end{vmatrix}, \tag{29}$$

and

$$\mu = \frac{1}{||\nabla f||^2}\Big\{\Big\langle (\nabla f)', (\nabla f)'\Big\rangle - \frac{1}{||\nabla f||^2}\Big\langle \nabla f, (\nabla f)'\Big\rangle^2 - \Big\langle (\nabla f)', \mathsf{T}\Big\rangle^2\Big\}$$

$$= \frac{1}{||\nabla f||^2}\Big\{\beta' H_f (\beta' H_f)^t - \frac{1}{||\nabla f||^2}\big(\beta' H_f (\nabla f)^t\big)^2 - \big(\beta' H_f (\beta')^t\big)^2\Big\}, \tag{30}$$

where $(\nabla f)' = [a\ b\ c\ d]$, $(\nabla f)'' = \beta'' H_f + \beta' \frac{d(H_f)}{ds} = [p\ q\ r\ s]$.

4.2.2. The expression for τ_g^1

Similarly, for the geodesic torsion of order 1, we may write

$$\tau_g^1 = \langle \mathsf{E}', \mathsf{N} \rangle = \frac{1}{||\mathsf{N}' - \langle \mathsf{N}', \mathsf{T} \rangle \mathsf{T}||} \left(\left\langle \mathsf{N}'', \mathsf{N} \right\rangle + \left\langle \mathsf{T}', \mathsf{N} \right\rangle^2 \right)$$

$$= -\left\{ \left\langle \mathsf{N}', \mathsf{N}' \right\rangle - \left\langle \mathsf{N}', \mathsf{T} \right\rangle^2 \right\}^{\frac{1}{2}}$$

or we have

$$\tau_g^1 = -\frac{1}{||\nabla f||} \left\{ \beta' H_f (\beta' H_f)^t - \frac{1}{||\nabla f||^2} \left(\beta' H_f (\nabla f)^t \right)^2 - \left(\beta' H_f (\beta')^t \right)^2 \right\}^{\frac{1}{2}}. \tag{31}$$

5. Example

Example 1 *Let us consider the unit-speed curve*

$$\beta(s) = \left(\cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \cos\left(\frac{2s}{\sqrt{5}}\right), \sin\left(\frac{2s}{\sqrt{5}}\right) \right)$$

lying on the hypersphere $\mathcal{M}...x^2+y^2+z^2+w^2 = 2$. *The unit normal vector field of* $\mathcal{M}$ *along* β *is* $\mathsf{N}(s) = \frac{1}{\sqrt{2}}\beta(s)$ *and the unit tangent vector field of* β *is*

$$\mathsf{T}(s) = \left(-\frac{1}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right), \frac{1}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right), -\frac{2}{\sqrt{5}} \sin\left(\frac{2s}{\sqrt{5}}\right), \frac{2}{\sqrt{5}} \cos\left(\frac{2s}{\sqrt{5}}\right) \right).$$

Since the curvature vector field

$$\mathsf{T}'(s) = \beta''(s) = \left(-\frac{1}{5} \cos\left(\frac{s}{\sqrt{5}}\right), -\frac{1}{5} \sin\left(\frac{s}{\sqrt{5}}\right), -\frac{4}{5} \cos\left(\frac{2s}{\sqrt{5}}\right), -\frac{4}{5} \sin\left(\frac{2s}{\sqrt{5}}\right) \right)$$

is linear independent with $\mathsf{N}(s)$, *Case 1 is valid. Thus, if we apply the method given in Case 1, we obtain*

$$\mathsf{E}(s) = \left(\frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{5}}\right), \frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{5}}\right), -\frac{1}{\sqrt{2}} \cos\left(\frac{2s}{\sqrt{5}}\right), -\frac{1}{\sqrt{2}} \sin\left(\frac{2s}{\sqrt{5}}\right) \right),$$

$$\mathsf{D}(s) = \left(\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right), -\frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right), -\frac{1}{\sqrt{5}} \sin\left(\frac{2s}{\sqrt{5}}\right), \frac{1}{\sqrt{5}} \cos\left(\frac{2s}{\sqrt{5}}\right) \right).$$

On the other hand, if we use the formulas (22), (23), (25), (27), and (28), the geodesic curvatures of order 1, 2 are obtained as $\kappa_g^1(s) = \frac{3}{5\sqrt{2}}$, $\kappa_g^2(s) = \frac{-4}{5\sqrt{2}}$, *the geodesic torsions of order 1, 2 are* $\tau_g^1(s) = \tau_g^2(s) = 0$, *and the normal curvature of* β *is* $\kappa_n(s) = \frac{-1}{\sqrt{2}}$. *As expected,* β *is a line of curvature on* $\mathcal{M}$.

6. Conclusion

The Darboux frame field in Euclidean 3-space $\mathbb{E}^3$ is extended into $\mathbb{E}^4$. By using Gram–Schmidt orthonormalization, we construct the extended Darboux frame field along a Frenet curve lying on an oriented hypersurface. We obtain some geometrical meanings of new invariants of the new frame field. The relationships between the new invariants according to the hypersurface and the curvatures according to $\mathbb{E}^4$ are given. Finally, the expressions of the new invariants of a Frenet curve lying on an implicit hypersurface are obtained. These expressions are given in matrix notation to shorten the formulas. Computing the expressions of these new invariants for a curve lying on a parametric hypersurface is a future work.

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