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Exploration of Soliton Solutions for the Kaup–Newell Model Using Two Integration Schemes in Mathematical Physics

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ABSTRACT

This research deals with the Kaup–Newell model, a class of nonlinear Schrödinger equations with important applications in plasma physics and nonlinear optics. Soliton solutions are essential for analyzing nonlinear wave behaviors in different physical systems, and the Kaup–Newell model is also significant in this context. The model's ability to represent subpicosecond pulses makes it a significant tool for the research of nonlinear optics and plasma physics. Overall, the Kaup–Newell model is an important research domain in these areas, with ongoing efforts focused on understanding its various solutions and potential applications. A new version of the generalized exponential rational function method and $\left(\frac{G'}{G^2}\right)$-expansion function method are utilized to discover diverse soliton solutions. The generalized exponential rational function method facilitates the generation of multiple solution types, including singular, shock, singular periodic, exponential, combo trigonometric, and hyperbolic solutions in mixed forms. Thanks to $\left(\frac{G'}{G^2}\right)$-expansion function method, we obtain trigonometric, hyperbolic, and rational solutions. The modulation instability of the proposed model is examined, with numerical simulations complementing the analytical results to provide a better understanding of the solutions' dynamic behavior. These results offer a foundation for future research, making the solutions effective, manageable, and reliable for tackling complex nonlinear problems. The methodologies used in this study are robust, influential, and practicable for diverse nonlinear partial differential equations; to our knowledge, for this equation, these methods of investigation have not been explored before. The accuracy of each solution has been verified using the Maple software program.

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1 | Introduction

The advancement of many scientific and engineering fields has been greatly supported by nonlinear partial differential equations (NLPDEs), which provide valuable insights into the complexities of physical and biological processes. The historical development of NLPDEs dates back to the 18th and 19th centuries, when mathematicians and physicists first began to formulate mathematical models to describe natural phenomena. Initially emerging in the study of fluid dynamics, heat transfer, and wave propagation, these equations have since found applications in a wide array of disciplines, including electrical engineering, magnetism, quantum mechanics, and biological systems. Over time, the study and solution of NLPDEs have evolved, driven by both theoretical advancements and the increasing power of computational methods, leading to a deeper understanding and more accurate modeling of intricate systems [1–11].

In the 19th century, Scottish engineer John Scott Russell first observed solitons, which are extraordinary nonlinear wave phenomena that sustain their shape while propagating at a uniform speed. He noticed a solitary wave that traveled long distances without dissipating, a phenomenon that intrigued scientists for decades. The mathematical foundation of solitons was later developed in the 1960s with the advent of the Korteweg–de Vries (KdV) equation, which describes shallow water waves. Solitons have since found applications across various fields, including fiber optic communications, where they enable stable, long-distance data transmission; plasma physics, where they help explain phenomena in magnetic fields; and even in biological systems, such as in the propagation of nerve impulses. The study of solitons has not only deepened our understanding of nonlinear wave dynamics but also spurred technological innovations in multiple domains [12–16].

Analytical solutions to NLPDEs provide profound insights into the behavior of complex systems governed by these equations. Unlike their linear counterparts, NLPDEs often exhibit intricate and diverse phenomena such as shock waves, solitons, and turbulence, making their solutions highly valuable but challenging to obtain. Analytical methods aim to find the exact solutions or simplifications that reveal the underlying structure and dynamics of the problem. Techniques such as the separation of variables, similarity transformations, the method of characteristics, and the inverse scattering transform have been developed to tackle specific types of NLPDEs. These methods not only help in understanding the fundamental properties of the solutions but also serve as benchmarks for validating numerical and approximate solutions. Despite their complexity, analytical solutions remain a cornerstone in the study of NLPDEs, offering clarity and depth to the exploration of nonlinear systems in physics, engineering, and beyond. In the present day, there are numerous trustworthy and well-developed techniques for finding both analytic and numerical solutions to NLPDEs, including the well-regarded modified auxiliary equation method [17], extended sinh-Gordon expansion technique [18], improved tanh approach [19], extended $\left(\frac{G'}{G^2}\right)$-expansion method [20], improved F-expansion method [21], polynomial expansion technique [22], Sumudu transform homotopy perturbation method (STHPM) [23], Painlevé analysis [24], Lie group analysis, group invariant solutions, and conservation laws [25], Hirota bilinear method [26], $\left(\frac{G'}{G}\right)$-expansion method [27], multiwaves solutions [28], semi-inverse variational principle (SIVP) schemes [29], Bernoulli $\left(\frac{G'}{G}\right)$-expansion method [30], complex method [31] and so on [32–42].

In this work, we will examine the Kaup–Newell model (KNM), which is described by [43] the following:

$$
\begin{aligned}
i\frac{\partial\Gamma(x,t)}{\partial t}+\theta\frac{\partial^2\Gamma(x,t)}{\partial x^2}+i\alpha\frac{\partial}{\partial x}\left(|\Gamma(x,t)|^2\Gamma(x,t)\right)& \\
-i\beta\frac{\partial}{\partial x}\left(|\Gamma(x,t)|^2\right)\Gamma(x,t)=0,&
\end{aligned}
\tag{1}
$$

in which $i^2=-1,\ \Gamma=\Gamma(x,t)$ represents a complex-valued wave function. The specific meanings of the nonzero real coefficients θ, α, and β represent coefficient of group velocity dispersion, coefficient of self-steepening term, and coefficient of nonlinear dispersion, respectively. The renowned KNM, a complex-valued NLPDE, was initially introduced by David Kaup and Alan Newell while researching the situation of subpicosecond pulse propagation. This equation is useful to describe patterns in plasma physics and optical fiber, especially in Alfven waves. Equation (1) is a powerful mathematical model for understanding and predicting the behavior of ultrashort optical pulses optical fiber systems and other nonlinear dispersive media. Its solutions yield important insights into the propagation of optical solitons and other nonlinear wave phenomena.

The nonlinear KNM is a crucial mathematical model in the field of optics, utilized to characterize the propagation of subpicosecond pulses through single mode optical fibers. In [44], Qarni discovered some dispersive solitons of the nonlinear Kaup–Newell system using the Riccati equation approach. In [45], Biswas et al. explored the solutions of Equation (1) via the trial equation scheme and modifed simple equation method. Jawad and colleagues, in Jawad et al. [46], determined some bright and singular soliton solutions of Equation (1) using the csch-function approach.

This work highlights the need for novel approaches to problem-solving to identify the underlying mathematical structures that underpin physical systems in the actual world. We introduce effective analytical methods utilizing finite series expansion to produce novel and precise wave solutions for the KNM, unveiling previously unpublished findings. These methods are effective resources for scientists working on difficult mathematical problems in a variety of scientific domains. The presented methods overcome the limitations of existing methods by providing effective solutions, especially for complex and high-dimensional systems. In this way, it provides practical and feasible solutions to solution challenges frequently encountered in the literature. They also add depth to existing theories and enable researchers to develop new hypotheses.

The principal aim of this study is to investigate different and novel optical soliton solutions for Equation (1). In this sense, the new version of the generalized exponential rational function method (nGERFM) and the $\left(\frac{G'}{G^2}\right)$-expansion function method will be used. It can be easily concluded that these analytical solution methods are very effective and successful in finding exact solutions to NLPDEs.

The paper is structured as follows: Section 2 describes the techniques, specifically the nGERFM and $\left(\frac{G'}{G^2}\right)$ -expansion function method. Analytical solutions for the model are discussed in Section 3. Modulation instability (MI) analysis is presented in Section 4. Section 5 offers comparisons, Section 6 presents graphical explanations, and Section 7 explains the conclusions.

2 | Methodology

Consider the NLPDE as follows:

$$
Y_1\left[\Gamma(x,t),\ \frac{\partial}{\partial t}\Gamma(x,t),\ \frac{\partial}{\partial x}\Gamma(x,t),\ \frac{\partial^2}{\partial t^2}\Gamma(x,t),\ \frac{\partial^2}{\partial x^2}\Gamma(x,t),\ \dots\right]=0. \tag{2}
$$

By implementing the complex wave transformation shown below

$$\Gamma(x,t) = \Phi(\xi)\exp(i\tau(x,t)), \quad \xi = x - \omega t, \quad \tau(x,t) = -sx + ct + c_0, \tag{3}$$

then, Equation (2) is transferred to

$$Y_2[\Phi, \Phi', \Phi'', \Phi''', \dots] = 0. \tag{4}$$

In Equation (3), ω, s, and c are the nonzero real constants.

2.1 | The Details of the nGERFM

Step 1. Let us set up the solution of Equation (4) as [47] follows:

$$\Phi(\xi) = k_0 + \sum_{n=1}^{n_0} k_n \left(\frac{\daleth'(\xi)}{\daleth(\xi)}\right)^n + \sum_{n=1}^{n_0} l_n \left(\frac{\daleth'(\xi)}{\daleth(\xi)}\right)^{-n}, \tag{5}$$

in which

$$\daleth(\xi) = \frac{\varsigma_1 \exp(J_1\xi) + \varsigma_2 \exp(J_2\xi)}{\varsigma_3 \exp(J_3\xi) + \varsigma_4 \exp(J_4\xi)}. \tag{6}$$

In the pre-assumed structures Equations (6) and (5), ς_i, J_i $(i \in [1,4])$ and k_0, k_n, l_n $(n \in [1, n_0])$ are unknown coefficients. In addition, to determine the value of the positive integer n_0, we can make use of various balancing rules documented in the literature.

Step 2. Substituting the expression from Equation (5) into Equation (4) produces a polynomial equation $\Pi(\varrho_1, \varrho_2, \varrho_3, \varrho_4) = 0$ in terms of $\varrho_n = e^{(J_\upsilon \xi)}$ for $\upsilon = 1,2,3,4$.

Step 3. At last, we obtain analytical solutions for Equation (2) by substituting the results derived from solving this system into the general expression of Equation (5).

2.2 | The Summary of $\left(\frac{G'}{G^2}\right)$ - Expansion Function Method

Step 1. Consider the solution of Equation (4) is characterized by the expression in $\left(\frac{G'}{G^2}\right)$-expansion method as follows [48]:

$$\Phi(\xi) = \rho_0 + \sum_{n=1}^{n_0} \left(\rho_n \left(\frac{G'}{G^2}\right)^n + \sigma_n \left(\frac{G'}{G^2}\right)^{-n}\right), \tag{7}$$

in which $G = G(\xi)$ holds

$$\left(\frac{G'}{G^2}\right)' = \mu + \lambda \left(\frac{G'}{G^2}\right)^2, \tag{8}$$

where $\lambda \neq 0$, $\mu \neq 1$ are integers. To complete the process, it is necessary to find the constants ρ_0, ρ_n, σ_n $(n = 1,2,3, \dots, n_0)$.

Step 2. As a result, $\left(\frac{G'}{G^2}\right)$ provides the following three types of solutions:

Scenario I, Trigonometric family: If $\mu\lambda > 0$, then

$$\left(\frac{G'}{G^2}\right) = \sqrt{\frac{\mu}{\lambda}} \left(\frac{\Theta_1 \cos\left(\sqrt{\mu\lambda}\xi\right) + \Theta_2 \sin\left(\sqrt{\mu\lambda}\xi\right)}{\Theta_2 \cos\left(\sqrt{\mu\lambda}\xi\right) - \Theta_1 \sin\left(\sqrt{\mu\lambda}\xi\right)}\right). \tag{9}$$

Scenario II, Hyperbolic family: When $\mu\lambda < 0$, then

$$\left(\frac{G'}{G^2}\right) = -\frac{\sqrt{|\mu\lambda|}}{\lambda} \left(\frac{\Theta_1 \sinh\left(2\sqrt{|\mu\lambda|}\xi\right) + \Theta_1 \cosh\left(2\sqrt{|\mu\lambda|}\xi\right) + \Theta_2}{\Theta_1 \sinh\left(2\sqrt{|\mu\lambda|}\xi\right) + \Theta_1 \cosh\left(2\sqrt{|\mu\lambda|}\xi\right) - \Theta_2}\right). \tag{10}$$

Scenario III, Rational family: When $\mu = 0$, $\lambda \neq 0$, then

$$\left(\frac{G'}{G^2}\right) = \left(-\frac{\Theta_1}{\lambda(\Theta_1\xi + \Theta_2)}\right), \tag{11}$$

in which Θ_1 and Θ_2 are constants.

Step 3. To find the n_0, the homogeneous balance rule is applied.

Step 4. By inserting from Equations (7) and (8) into Equation (4), we get a set of algebraic equations.

Step 5. By investigating the specific terms, we found the necessary set of parameters.

3 | Extraction of the Soliton Solutions

The real and imaginary parts of Equation (1) can be obtained by adopting Equation (3) as follows:

$$\theta\Phi'' - (c + \theta s^2)\Phi + \alpha s\Phi^3 = 0, \tag{12}$$

and

$$(-\omega - 2\theta s)\Phi' + (3\alpha - 2\beta)\Phi^2\Phi' = 0. \tag{13}$$

From Equation (13), we can establish the following constraint conditions:

$$\alpha = \frac{2}{3}\beta, \tag{14}$$

with the speed of soliton:

$$\omega = -2\theta s. \tag{15}$$

Utilizing Equation (14) into Equation (12) yields

$$\theta\Phi'' - (c + \theta s^2)\Phi + \frac{2}{3}s\beta\Phi^3 = 0. \tag{16}$$

$3n_0 = n_0 + 2$ is obtained by using certain standard balancing principles in Equation (16) between Φ^3 and Φ''. Hence, one should take $n_0 = 1$.

3.1 | Main Results of Solving Model Equation (1) Using Technique I

Equation (5) can be expressed as

$$\Phi(\xi) = k_0 + k_1 \left(\frac{\daleth'(\xi)}{\daleth(\xi)}\right) + l_1 \left(\frac{\daleth'(\xi)}{\daleth(\xi)}\right)^{-1}, \tag{17}$$

$\daleth(\xi)$ is specified by Equation (6).

Class 1:

Taking $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, 1, 1, 0]$ and $[J_1, J_2, J_3, J_4] = [0, 1, 2, 0]$ in Equation (6) yields

$$ℸ(\xi) = \frac{1 + \exp(\xi)}{\exp(2\xi)}. \tag{18}$$

To obtain the parameter values, we solve the algebraic equations with Maple (or Mathematica). The collection of responses generated might be given as follows:

Type 1.1:

$$c = -\frac{\theta}{2}\left(2s^2 + 1\right),$$
$$k_0 = \mp\frac{5}{2}\sqrt{-\frac{3\theta}{\beta s}}, \quad k_1 = \pm\sqrt{-\frac{3\theta}{\beta s}}, \quad l_1 = 0.$$

By plugging the aforementioned values of k_0, k_1, l_1 into Equation (17), we get

$$\Phi^{\pm}(\xi) = \frac{\sqrt{3}}{2}\sqrt{-\frac{\theta}{\beta s}} \times \left(\frac{\exp(\xi) - 1}{\exp(\xi) + 1}\right). \tag{19}$$

We find the following exponential function solution for Equation (1) using Equation (19)

$$\Gamma^{\pm}_{1,1}(x,t) = \frac{\sqrt{3}}{2}\sqrt{-\frac{\theta}{\beta s}} \times \left(\frac{\exp(x - \omega t) - 1}{\exp(x - \omega t) + 1}\right) \times \exp\left(i\left\{-sx + ct + c_0\right\}\right). \tag{20}$$

Type 1.2:

$$c = -\frac{\theta}{2}\left(2s^2 + 1\right),$$
$$k_0 = \mp\frac{5}{2}\sqrt{-\frac{3\theta}{\beta s}}, \quad k_1 = 0, \quad l_1 = \pm 6\sqrt{-\frac{3\theta}{\beta s}}.$$

By inserting the aforementioned values of k_0, k_1, l_1 into Equation (17), we get

$$\Phi^{\mp}(\xi) = -\frac{\sqrt{3}}{2}\sqrt{-\frac{\theta}{\beta s}} \times \left(\frac{3\exp(\xi) - 2}{3\exp(\xi) + 2}\right). \tag{21}$$

The exponential function solution for Equation (1) is found using Equation (21) in the following manner:

$$\Gamma^{\mp}_{1,2}(x,t) = -\frac{\sqrt{3}}{2}\sqrt{-\frac{\theta}{\beta s}} \times \left(\frac{3\exp(x - \omega t) - 2}{3\exp(x - \omega t) + 2}\right) \times \exp\left(i\left\{-sx + ct + c_0\right\}\right). \tag{22}$$

Class 2: When we pick $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 2i, 0]$ and $[J_1, J_2, J_3, J_4] = [i, -i, 1, 0]$ in Equation (6), we reach

$$ℸ(\xi) = \frac{\sin(\xi)}{\exp(\xi)}. \tag{23}$$

To acquire parameter values, we use package programs to solve algebraic equations; the collection of solutions that result can be stated as follows:

Type 2.1:

$$c = -\theta(s^2 - 2),$$
$$k_0 = \pm\sqrt{-\frac{3\theta}{\beta s}}, \quad k_1 = 0, \; l_1 = \pm 2\sqrt{-\frac{3\theta}{\beta s}}.$$

These values are entered into Equation (17), yielding

$$\Phi^{\pm}(\xi) = \sqrt{3}\sqrt{-\frac{\theta}{\beta s}} \times \left(\frac{\sin(\xi) + \cos(\xi)}{\cos(\xi) - \sin(\xi)}\right). \tag{24}$$

By employing Equation (24), we derive the following combo trigonometric soliton solution for Equation (1):

$$\Gamma^{\pm}_{2,1}(x,t) = \sqrt{-\frac{3\theta}{\beta s}} \times \left(\frac{\cos(x - \omega t) + \sin(x - \omega t)}{\cos(x - \omega t) - \sin(x - \omega t)}\right) \times \exp\left(i\left\{-sx + ct + c_0\right\}\right). \tag{25}$$

Type 2.2:

$$c = -\theta(s^2 - 2),$$
$$k_0 = \mp\frac{3\theta}{\beta s\sqrt{-\frac{3\theta}{\beta s}}}, \quad k_1 = \sqrt{-\frac{3\theta}{\beta s}}, \quad l_1 = 0.$$

These values are entered into Equation (17), yielding

$$\Phi^{\pm}(\xi) = \sqrt{3}\sqrt{-\frac{\theta}{\beta s}} \times \cot(\xi). \tag{26}$$

By substituting Equation (26), we derive the singular periodic solution for Equation (1) as follows:

$$\Gamma^{\pm}_{2,2}(x,t) = \sqrt{3}\sqrt{-\frac{\theta}{\beta s}} \times \cot(x - \omega t) \times \exp\left(i\left\{-sx + ct + c_0\right\}\right). \tag{27}$$

Class 3: For $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 2, 0]$ and $[J_1, J_2, J_3, J_4] = [1, -1, -1, 0]$ in Equation (6) provides

$$ℸ(\xi) = \frac{\sinh(\xi)}{\exp(-\xi)}. \tag{28}$$

By implementing **the nGERFM** procedure, we obtain

$$c = -\theta\left(s^2 + 2\right),$$
$$k_0 = \mp\frac{3\theta}{\beta s\sqrt{-\frac{3\theta}{\beta s}}}, \quad k_1 = \pm\sqrt{-\frac{3\theta}{\beta s}}, \quad l_1 = 0.$$

These values are entered into Equation (17), yielding

$$\Phi^{\pm}(\xi) = \frac{\theta\sqrt{3}}{\beta s\sqrt{-\frac{\theta}{\beta s}}} \times \tanh(\xi). \tag{29}$$

By using Equation (29), we reach the next shock solution for Equation (1)

$$\Gamma^{\pm}_{3}(x,t) = \frac{\theta\sqrt{3}}{\beta s\sqrt{-\frac{\theta}{\beta s}}} \times \tanh(x - \omega t) \times \exp\left(i\left\{-sx + ct + c_0\right\}\right). \tag{30}$$

Class 4: In Equation (6), picking $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4]=[2,0,1,1]$ and $[J_1,J_2,J_3,J_4]=[1,0,i,-i]$ produces

$$\daleth(\xi)=\frac{\exp(\xi)}{\cos(\xi)}. \tag{31}$$

We also achieve

Type 4.1:

$$c=-\theta\left(s^2-2\right),$$
$$k_0=\pm\sqrt{-\frac{3\theta}{\beta s}},\quad k_1=0,\ l_1=\mp 2\sqrt{-\frac{3\theta}{\beta s}}.$$

These findings combined with Equation (17) produce

$$\Phi^{\pm}(\xi)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\left(\frac{2\cos(\xi)\sin(\xi)-1}{2\cos(\xi)^2-1}\right). \tag{32}$$

As a result of our findings, the following combo trigonometric soliton solution is achieved:

$$\Gamma^{\pm}_{4,1}(x,t)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\left(\frac{2\cos(x-\omega t)\sin(x-\omega t)-1}{2\cos(x-\omega t)^2-1}\right)\times\exp\left(i\left\{-sx+ct+c_0\right\}\right). \tag{33}$$

Type 4.2:

$$c=-\theta\left(s^2-2\right),$$
$$k_0=\pm\frac{3\theta}{\beta s\sqrt{-\frac{3\theta}{\beta s}}},\ k_1=\pm\sqrt{-\frac{3\theta}{\beta s}},\quad l_1=0.$$

These findings combined with Equation (17) yield

$$\Phi^{\mp}(\xi)=\frac{\theta\sqrt{3}}{\beta s\sqrt{-\frac{\theta}{\beta s}}}\times\tan(\xi). \tag{34}$$

Therefore, one way to describe the singular periodic solution is as

$$\Gamma^{\pm}_{4,1}(x,t)=\frac{\theta\sqrt{3}}{\beta s\sqrt{-\frac{\theta}{\beta s}}}\times\tan(x-\omega t)\times\exp\left(i\left\{-sx+ct+c_0\right\}\right). \tag{35}$$

Class 5: In Equation (6), considering $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4]=[2,0,1,1]$ and $[J_1,J_2,J_3,J_4]=[0,0,1,-1]$ generates

$$\daleth(\xi)=\frac{1}{\cosh(\xi)}. \tag{36}$$

We accomplish

Type 5.1:

$$c=-\theta\left(s^2+2\right),$$
$$k_0=0,\quad k_1=0,\quad l_1=\pm\sqrt{-\frac{3\theta}{\beta s}}.$$

By plugging the values of $k_0,\ k_1,\ l_1$ into Equation (17), we gain

$$\Phi^{\mp}(\xi)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\coth(\xi). \tag{37}$$

As a result, we can derive singular soliton solutions given by the following:

$$\Gamma^{\mp}_{5,1}(x,t)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\coth(x-\omega t)\times\exp\left(i\left\{-sx+ct+c_0\right\}\right). \tag{38}$$

Type 5.2:

$$c=-\theta\left(s^2+8\right),$$
$$k_0=0,\qquad k_1=l_1=\pm\sqrt{-\frac{3\theta}{\beta s}}.$$

For these reasons, along with Equation (17), the following result is most likely to be achieved:

$$\Phi^{\mp}(\xi)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\left(\frac{2\cosh(\xi)^2-1}{\cosh(\xi)\sinh(\xi)}\right). \tag{39}$$

The mixed form of the hyperbolic solution is expressed as

$$\Gamma^{\mp}_{5,2}(x,t)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\left(\frac{-1+2\cosh(x-\omega t)^2}{\sinh(x-\omega t)\cosh(x-\omega t)}\right)\times\exp\left(i\left\{-sx+ct+c_0\right\}\right). \tag{40}$$

Class 6: In Equation (6), considering $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4]=[2,0,1,1]$ and $[J_1,J_2,J_3,J_4]=[-2,0,1,-1]$ generates

$$\daleth(\xi)=\frac{\exp(-2\xi)}{\cosh(\xi)}. \tag{41}$$

We reach

$$c=-\theta\left(s^2+2\right),$$
$$k_0=\pm 2\sqrt{-\frac{3\theta}{\beta s}},\qquad k_1=0,\qquad l_1=\pm 3\sqrt{-\frac{3\theta}{\beta s}}.$$

By entering the values of $k_0,\ k_1,\ l_1$ into Equation (17), one gets

$$\Phi^{\pm}(\xi)=\sqrt{-\frac{1}{\rho}}\times\left(\frac{\cosh(\xi)+2\sinh(\xi)}{\sinh(\xi)+2\cosh(\xi)}\right). \tag{42}$$

We arrive at the combo soliton solution by using the Equation (42)

$$\Gamma^{\pm}_{6}(x,t)=\sqrt{3}\sqrt{-\frac{\theta}{\beta s}}\times\left(\frac{2\sinh(x-\omega t)+\cosh(x-\omega t)}{2\cosh(x-\omega t)+\sinh(x-\omega t)}\right)\times\exp\left(i\left\{-sx+ct+c_0\right\}\right). \tag{43}$$

Class 7:

If we take $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, 1, 2, 0]$ and $[J_1, J_2, J_3, J_4] = [i, -i, 0, 0]$ in Equation (6), it provides

$$\daleth(\xi) = \cos(\xi). \tag{44}$$

We have

$$c = -\theta\left(s^2 + 4\right),$$

$$k_0 = 0, \qquad k_1 = l_1 = \pm\sqrt{-\frac{3\theta}{\beta s}}.$$

When these results are analyzed in conjunction with Equation (17), we get the following outcome:

$$\Phi^{\mp}(\xi) = \sqrt{3}\sqrt{-\frac{\theta}{\beta s}} \times \frac{1}{\cos(\xi)\sin(\xi)}. \tag{45}$$

As a result, we have the periodic solution given below:

$$\Gamma_7^{\mp}(x,t) = \sqrt{3}\sqrt{-\frac{\theta}{\beta s}} \times \frac{1}{\cos(x-\omega t)\sin(x-\omega t)} \times \exp\left(i\left\{-sx + ct + c_0\right\}\right). \tag{46}$$

3.2 | Main Results of Solving Model Equation (1) Using Technique II

Equation (7) can be expressed as

$$\Phi(\xi) = \rho_0 + \rho_1\left(\frac{G'}{G^2}\right) + \sigma_1\left(\frac{G'}{G^2}\right)^{-1}, \tag{47}$$

in which ρ_0, ρ_1, and σ_1 are real parameters and are determined later. The use of Equations (16) and (47) together with Equation (8) gives the cluster of algebraic expression. By accumulating and equating to zero, the power coefficient is $\left(\frac{G'}{G^2}\right)$. The values of ρ_0, ρ_1, and σ_1 are derived by handling the resulting algebraic system.

Cluster 1: $c = -4\lambda\mu\theta - s^2\theta$, $\quad \rho_0 = 0$, $\quad \rho_1 = \pm\lambda\sqrt{-\frac{3\theta}{\beta s}}$, $\quad \sigma_1 = \pm\mu\sqrt{-\frac{3\theta}{\beta s}}$.

Cluster 2: $c = 2\lambda\mu\theta - s^2\theta$, $\rho_0 = 0$, $\rho_1 = \pm\lambda\sqrt{-\frac{3\theta}{\beta s}}$, $\sigma_1 = 0$.

Cluster 3: $c = 2\lambda\mu\theta - s^2\theta$, $\rho_0 = 0$, $\rho_1 = 0$, $\sigma_1 = \pm\mu\sqrt{-\frac{3\theta}{\beta s}}$.

For **Cluster 1**, the following solutions are built.

Case-1: If $\mu\lambda > 0$, then by using Equation (3), family of trigonometric solutions can be presented as follows:

$$\Gamma_{1,1}^{\pm}(x,t) = \left\{\frac{\lambda\sqrt{-\frac{3\theta}{\beta s}}\sqrt{\frac{\mu}{\lambda}}\left(\Theta_1\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) + \Theta_2\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}{\left(\Theta_2\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) - \Theta_1\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)} + \frac{\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_2\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) - \Theta_1\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}{\sqrt{\frac{\mu}{\lambda}}\left(\Theta_1\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) + \Theta_2\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}\right\} \times \exp(i \times \left\{-sx + ct + c_0\right\}). \tag{48}$$

Case-2: If $\mu\lambda < 0$, then by using Equation (3), family of hyperbolic solutions is written as follows:

$$\Gamma_{1,2}^{\pm}(x,t) = \left\{\frac{\lambda\sqrt{-\frac{3\theta}{\beta s}}\sqrt{-\frac{\mu}{\lambda}}\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_2\right)}{\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) - \Theta_2\right)} - \frac{\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) - \Theta_2\right)}{\sqrt{-\frac{\mu}{\lambda}}\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_2\right)}\right\} \times \exp(i \times \left\{-sx + ct + c_0\right\}). \tag{49}$$

Case-3: If $\mu = 0,\ \lambda \neq 0$, then by using Equation (3), a family of rational solutions is written as follows:

$$\Gamma_{1,3}^{\pm}(x,t) = \left\{\frac{\sqrt{-\frac{3\theta}{\beta s}}\Theta_1}{\left(\Theta_1(x-\omega t) + \Theta_2\right)} - \frac{\lambda\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_1(x-\omega t) + \Theta_2\right)}{\Theta_1}\right\} \times \exp(i \times \left\{-sx + ct + c_0\right\}). \tag{50}$$

For **Cluster 2,** the following solutions are built.

Case-1: If $\mu\lambda > 0$, then by using Equation (3), family of trigonometric solutions can be shown as follows:

$$\Gamma_{2,1}^{\pm}(x,t) = \left\{\frac{\lambda\sqrt{-\frac{3\theta}{\beta s}}\sqrt{\frac{\mu}{\lambda}}\left(\Theta_1\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) + \Theta_2\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}{\left(\Theta_2\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) - \Theta_1\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)} + \frac{\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_2\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) - \Theta_1\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}{\sqrt{\frac{\mu}{\lambda}}\left(\Theta_1\cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) + \Theta_2\sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}\right\} \times \exp(i \times \left\{-sx + ct + c_0\right\}). \tag{51}$$

Case-2: If $\mu\lambda < 0$, then by using Equation (3), family of hyperbolic solutions is written as follows:

$$\Gamma_{2,2}^{\pm}(x,t) = \left\{\frac{\lambda\sqrt{-\frac{3\theta}{\beta s}}\sqrt{-\frac{\mu}{\lambda}}\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_2\right)}{\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) - \Theta_2\right)} - \frac{\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) - \Theta_2\right)}{\sqrt{-\frac{\mu}{\lambda}}\left(\Theta_1\cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1\sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_2\right)}\right\} \times \exp(i \times \left\{-sx + ct + c_0\right\}). \tag{52}$$

Case-3: If $\mu = 0,\ \lambda \neq 0$, then by using Equation (3), a family of rational solutions is written as follows:

$$\Gamma_{2,3}^{\pm}(x,t) = \left\{\frac{\sqrt{-\frac{3\theta}{\beta s}}\Theta_1}{\left(\Theta_1(x-\omega t) + \Theta_2\right)} - \frac{\lambda\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_1(x-\omega t) + \Theta_2\right)}{\Theta_1}\right\} \times \exp(i \times \left\{-sx + ct + c_0\right\}). \tag{53}$$

For **Cluster 3,** the following solutions are built.

Case-1: If $\mu\lambda > 0$, then by using Equation (3), family of trigonometric solutions can be illustrated as follows:

$$\Gamma^{\pm}_{3,1}(x,t) = \left\{ \frac{\lambda\sqrt{-\frac{3\theta}{\beta s}}\sqrt{\frac{\mu}{\lambda}}\left(\Theta_1 \cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) + \Theta_2 \sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}{\left(\Theta_2 \cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) - \Theta_1 \sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)} + \frac{\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_2 \cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) - \Theta_1 \sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)}{\sqrt{\frac{\mu}{\lambda}}\left(\Theta_1 \cos\left(\sqrt{\mu\lambda}(x-\omega t)\right) + \Theta_2 \sin\left(\sqrt{\mu\lambda}(x-\omega t)\right)\right)} \right\} \times \exp(i \times \{-sx + ct + c_0\}). \tag{54}$$

Case-2: If $\mu\lambda < 0$, then by using (3), family of hyperbolic solutions is written as follows:

$$\Gamma^{\pm}_{3,2}(x,t) = \left\{ \frac{\lambda\sqrt{-\frac{3\theta}{\beta s}}\sqrt{-\frac{\mu}{\lambda}}\left(\Theta_1 \cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1 \sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_2\right)}{\left(\Theta_1 \cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1 \sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) - \Theta_2\right)} - \frac{\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_1 \cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1 \sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) - \Theta_2\right)}{\sqrt{-\frac{\mu}{\lambda}}\left(\Theta_1 \cosh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_1 \sinh\left(2\sqrt{-\mu\lambda}(x-\omega t)\right) + \Theta_2\right)} \right\} \times \exp(i \times \{-sx + ct + c_0\}). \tag{55}$$

Case-3: If $\mu = 0,\ \lambda \neq 0$, then by using Equation (3), a family of rational solutions is written as follows:

$$\Gamma^{\pm}_{3,3}(x,t) = \left\{ \frac{\sqrt{-\frac{3\theta}{\beta s}}\Theta_1}{\left(\Theta_1(x-\omega t) + \Theta_2\right)} - \frac{\lambda\mu\sqrt{-\frac{3\theta}{\beta s}}\left(\Theta_1(x-\omega t) + \Theta_2\right)}{\Theta_1} \right\} \times \exp(i \times \{-sx + ct + c_0\}). \tag{56}$$

4 | Modulation Instability (MI) Analysis

In many nonlinear systems, the interplay between dispersive and nonlinear effects leads to modulation instability in steady states. Specifically, in the context of optical fibers, MI is a well-researched phenomenon that is significant for designing fiber optic communication systems. Here, the optical signal propagates through the fiber and is represented by the wave field, with the Kerr effect causing nonlinearity by affecting the fiber's refractive index based on light intensity. In nonlinear wave equations, modulation instability describes how small perturbations can grow exponentially to form new structures or patterns. This section focuses on investigating the MI of the equation using linear stability techniques [49].

Equation (1) provides a definition for the MI of traveling waves. Assume that the perturbed solution is represented by

$$\Gamma(x,t) = \left(\sqrt{P} + u(x,t)\right)e^{i\lambda x}, \tag{57}$$

in which λ is used to denote the normalized optical power. When we use Equation (57) in Equation (1), we obtain

$$iu_t - \theta\lambda^2 u + 2i\theta\lambda u_x + \theta u_{xx} = 0. \tag{58}$$

Let us consider a solution to Equation (58) expressed as

$$u(x,t) = l_1 e^{i(kx-wt)} + l_2 e^{-i(kx-wt)}. \tag{59}$$

Here, k is the normalized wave number and w denotes the frequency. By integrating Equation (59) into Equation (58) and isolating the coefficients of $e^{i(kx-wt)}$ and $e^{-i(kx-wt)}$ to solve the determinant of the coefficient matrix, we obtain the following dispersion relation:

$$\left(-(k-\lambda)^2\theta + w\right) - (k+\lambda)^2\theta - w = 0. \tag{60}$$

By analyzing dispersion relation Equation (60) for k, we can derive

$$k = \pm\frac{\sqrt{-\theta\left(\theta\lambda^2 + w\right)}}{\theta}.$$

The dispersion relation obtained provides insights into the stability of the steady state. A real component for k indicates that the steady state is stable against small perturbations, whereas an imaginary k signifies instability and results in an exponential growth of perturbations.

5 | Comparisons

In this section, we compare the results of our investigation with those from previous studies that employed different analytical methods and are documented in the literature. Highlighting the unique qualities and innovative contributions of our present research is the aim of this comparative analysis. In [50], the

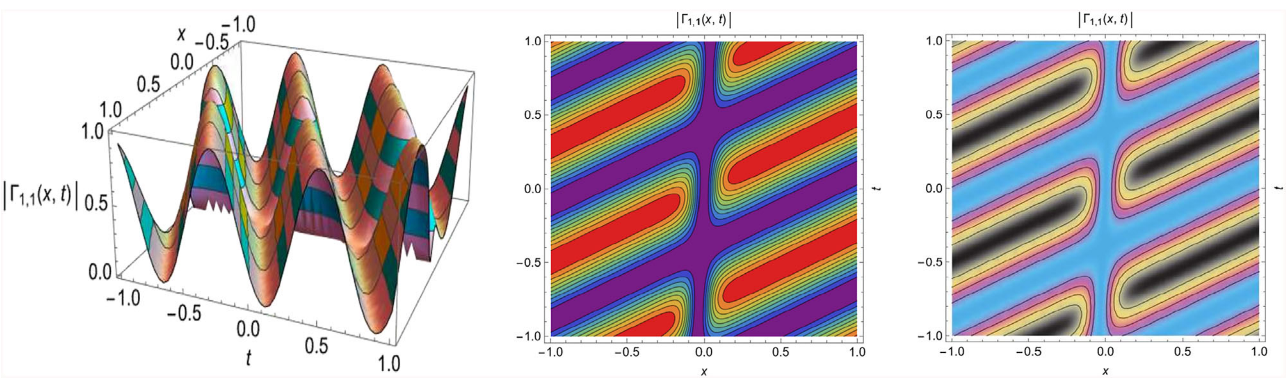

FIGURE 1 | The 3D, contour, and density plots for the solution $|\Gamma_{1,1}(x,t)|$ in Equation (20) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4, \omega = 24,\ c = 2, c_0 = 0$, and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

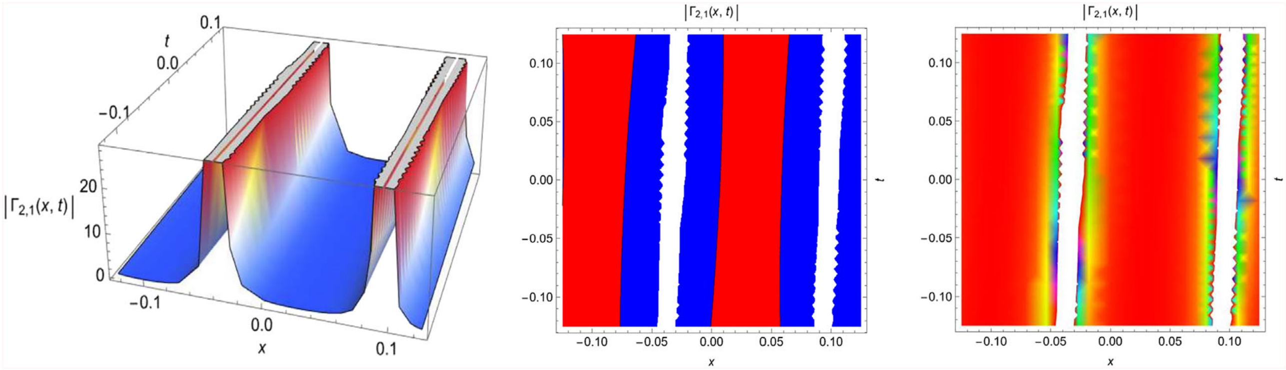

FIGURE 2 | The 3D, contour, and density plots for the solution $|\Gamma_{2,1}(x,t)|$ in Equation (25) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4, \omega = 24,\ c = 2, c_0 = 0,$ and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

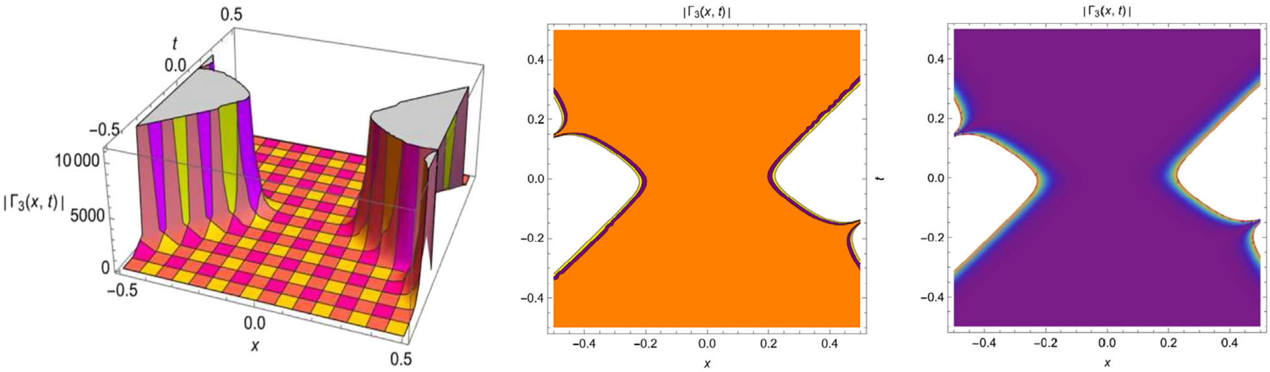

FIGURE 3 | The 3D, contour, and density plots for the solution $|\Gamma_3(x,t)|$ in Equation (30) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4, \omega = 24,\ c = 2, c_0 = 0,$ and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

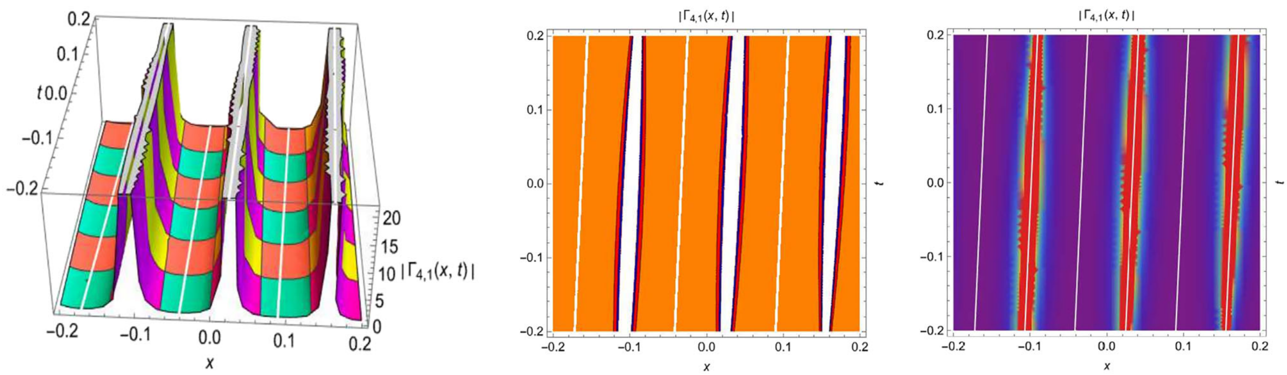

FIGURE 4 | The 3D, contour, and density plots for the solution $|\Gamma_{4,1}(x,t)|$ in Equation (33) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4, \omega = 24,\ c = 2, c_0 = 0,$ and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

KNM was studied by using the Jacobian elliptic function to find new type solutions. In [43], the authors found various kinds of exact solutions, such as bright soliton, singular soliton, dark soliton, and other traveling wave solutions, expressed as rational functions, Exp functions, sin-cos functions, sinh-cosh functions, hyperbolic functions (sech function and csch function), and trigonometric functions (sec function and csc function). In [51], the author uncovered analytical solutions, including bright, dark, and singular solitons, and other types of solutions by using the extended simplest equation method. To get exact-wave solutions to the KNM, the authors in [52] employed the generalized projective Riccati equations method (GPREM) together with the new Kudryashov approach. Our work synthesizes this research to apply the more effective technique, namely, the nGERFM and $\left(\frac{G'}{G^2}\right)$-expansion function method. Thus, by implementing these two methods, we give novel soliton solutions, such as exponential function, singular periodic, shock, singular, combo trigonometric, hyperbolic solutions in mixed form, trigonometric, hyperbolic, and rational solutions.

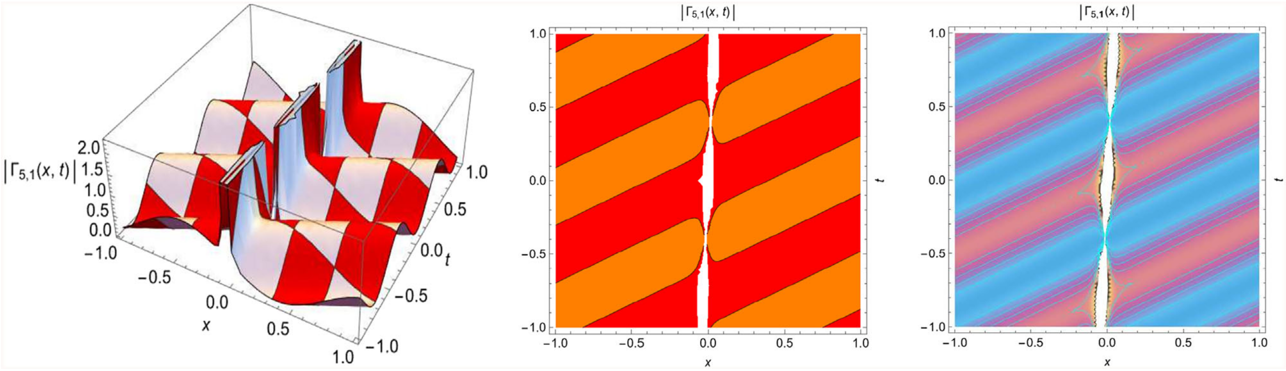

FIGURE 5 | The 3D, contour, and density plots for the solution $|\Gamma_{5,1}(x,t)|$ in Equation (38) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4, \omega = 24,\ c = 2, c_0 = 0$, and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

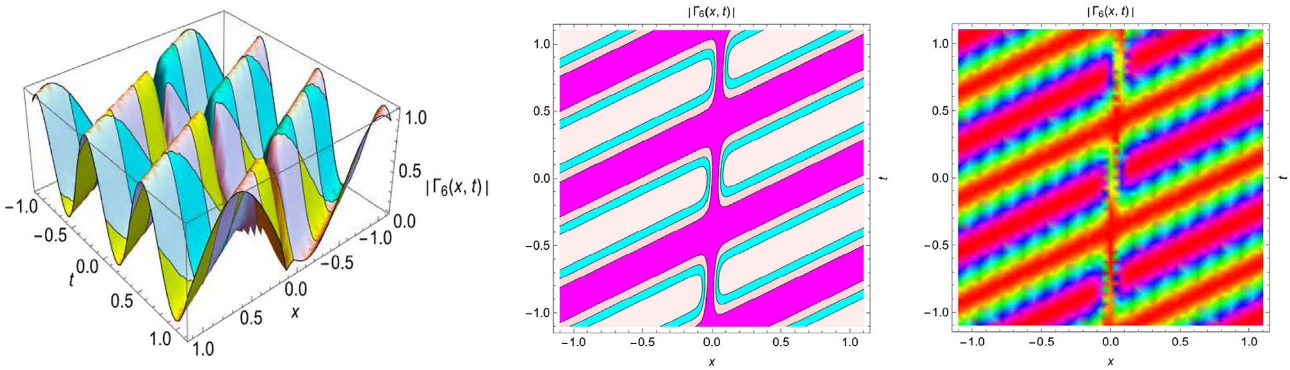

FIGURE 6 | The 3D, contour, and density plots for the solution $|\Gamma_6(x,t)|$ in Equation (43) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4, \omega = 24,\ c = 2, c_0 = 0$, and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

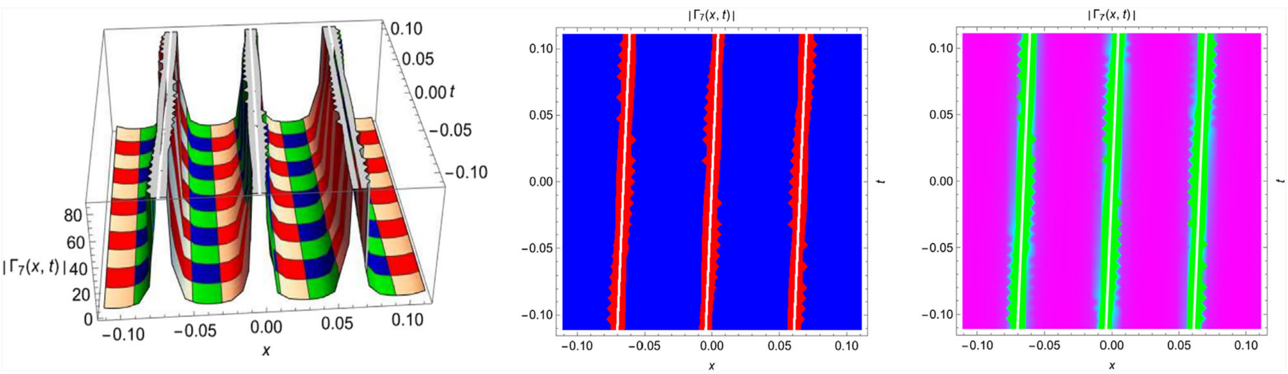

FIGURE 7 | The 3D, contour, and density plots for the solution $|\Gamma_6(x,t)|$ in Equation (46) when $\beta = 3,\ \alpha = 2,\ \theta = -3,\ s = 4,\ \omega = 24,\ c = 2,\ c_0 = 0$, and $\xi = x - \omega t$. [Colour figure can be viewed at wileyonlinelibrary.com]

As a result of carefully selecting the study's parameters, it becomes evident that many of the study's conclusions are in line with previous research findings. Recall that these outcomes were originally expressed differently from ours, even though there may have been some parallels. Despite an extensive search of relevant publications, no solution identical to the one revealed in this work could be found. The fact that the remaining results we have obtained do not seem to have been published before highlights their novelty and innovation. The outcomes displayed in this investigation were obtained by utilizing the nGERFM and $\left(\frac{G'}{G^2}\right)$-expansion function method.

6 | Graphical Explanation

In this section, we will describe the parameters that were used to generate the plots. Using these parameters, we will illustrate 3D, contour, and density plots for some of the solutions that we have derived (Figures 1–7).

7 | Conclusion

In this study, the the Kaup–Newell models the glamorous technology of subpicosecond pulses that spread via single-mode optical fibers is investigated using two innovative methods, namely the nGERFM and $\left(\frac{G'}{G^2}\right)$-expansion function method. To begin with, we regarded the nGERFM. The use of nGERFM enabled us to obtain several classes of soliton solutions, such as shock, singular, singular periodic, exponential, combo trigonometric, and hyperbolic solutions in mixed forms. Secondly, we considered the $\left(\frac{G'}{G^2}\right)$ -expansion function method. By using this method, we get trigonometric, hyperbolic, and rational soliton solutions. A key benefit of these techniques is their capability to predict and organize solution structures from the start. By using Mathematica to create 3D, contour, and density plots, we were able to graphically represent the obtained soliton solutions based on selected parameter values. The logical and lucid methodology makes it possible for interested parties to solve their NLPDEs right away. The outcomes of this research show that the offered methodology is promising for uncovering a variety of soliton solutions in nonlinear optical equations. The MI analysis performed on the model also contributes new insights.

The soliton solutions we have achieved are original and unique to the KNM. Furthermore, in comparison to many previous studies, our discoveries offer a more exhaustive range of functions, including hyperbolic, periodic, rational, trigonometric, and exponential solutions. These validated solutions are applicable for analyzing NLPDEs across fields such as applied sciences, plasma physics, mathematical physics, nonlinear dynamics, and engineering. The KNM is relatively straightforward, making it easier to understand and implement in various applications. It often allows for analytical solutions, which can be beneficial for theoretical studies and understanding wave behavior. Also, the model provides valuable insights into wave propagation and interactions, which are critical in meteorological contexts. Future research could focus on examining the long-term behavior of these solitary wave solutions and exploring how different parameters affect system dynamics to discover new phenomena.

Author Contributions

Bahadır Kopçasız: conceptualization; methodology; software; investigation; validation; writing – original draft; writing – review and editing; visualization. **Fatma Nur Kaya Sağlam:** conceptualization; investigation; writing – original draft; methodology; validation; visualization; writing – review and editing; software.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

References

1. E. Yaşar, Y. Yıldırım, and A. R. Adem, "Perturbed Optical Solitons With Spatio-Temporal Dispersion in (2+ 1)-Dimensions by Extended Kudryashov Method," *Optik* 158 (2018): 1–14.

2. B. Kopçasız, "Qualitative Analysis and Optical Soliton Solutions Galore: Scrutinizing the (2+1)-Dimensional Complex Modified Korteweg–de Vries System," *Nonlinear Dynamics* 112 (2024): 21321–21341, https://doi.org/10.1007/s11071-024-10036-9.

3. B. Kopçasız and E. Yaşar, "Inquisition of Optical Soliton Structure and Qualitative Analysis for the Complex-Coupled Kuralay System," *Modern Physics Letters B* (2024): 2450512, https://doi.org/10.1142/S0217984924505122.

4. B. Kopçasız and E. Yaşar, "Solitonic Structures and Chaotic Behavior in the Geophysical Korteweg–de Vries Equation: A μ-Symmetry and g'-Expansion Approach," *Modern Physics Letters B* 2024 (2024): 2450419., https://doi.org/10.1142/S0217984924504190.

5. B. Kopçasız and E. Yaşar, "Novel Exact Solutions and Bifurcation Analysis to Dual-Mode Nonlinear Schrödinger Equation," *Journal of Ocean Engineering and Science* (2022), https://doi.org/10.1016/j.joes.2022.06.007.

6. Z. P. Izgi, F. N. Saglam, S. Sahoo, H. Rezazadeh, and L. Akinyemi, "A Partial Offloading Algorithm Based on Intelligent Sensing," *International Journal of Modern Physics B* 36, no. 17 (2022): 2250097.

7. F. N. K. Sağlam and S. Ahmad, "Stability Analysis and Retrieval of New Solitary Waves of (2+ 1)-and (3+ 1)-Dimensional Potential Kadomtsev–Petviashvili and B-Type Kadomtsev–Petviashvili Equations Using Auxiliary Equation Technique," *Modern Physics Letters B* 2024 (2024): 2450413.

8. F. N. K. Sağlam and S. Malik, "Various Traveling Wave Solutions for (2+ 1)-Dimensional Extended Kadomtsev–Petviashvili Equation Using a Newly Created Methodology," *Chaos, Solitons & Fractals* 186 (2024): 115318.

9. G. Akram, M. Sadaf, S. Arshed, and M. S. Riaz, "Exact Solutions of Paraxial Equation via Extended Hyperbolic Function Method," *Optical and Quantum Electronics* 56, no. 10 (2024): 1621.

10. K. U. Tariq, J. G. Liu, and S. Nisar, "Study of Explicit Travelling Wave Solutions of Nonlinear (2+ 1)-Dimensional Zoomeron Model in Mathematical Physics," *Journal of Nonlinear, Complex and Data Science* 25, no. 1 (2024): 109–124.

11. L. Tang, A. Biswas, Y. Yıldırım, and A. S. Alshomrani, "Bifurcations and Optical Soliton Perturbation for the Lakshmanan–Porsezian–Daniel System With Kerr Law of Nonlinear Refractive Index," *Journal of Optics* 2024 (2024): 1–6.

12. G. H. Tipu, W. A. Faridi, D. Rizk, Z. Myrzakulova, R. Myrzakulov, and L. Akinyemi, "The Optical Exact Soliton Solutions of Shynaray-IIA Equation With Φ^6-Model Expansion Approach," *Optical and Quantum Electronics* 56, no. 2 (2024): 226.

13. M. N. Hossain, F. Alsharif, M. M. Miah, and M. Kanan, "Abundant New Optical Soliton Solutions to the Biswas–Milovic Equation With Sensitivity Analysis for Optimization," *Mathematics* 12, no. 10 (2024): 1585.

14. M. Ur Rahman, M. Sun, S. Boulaaras, and D. Baleanu, "Bifurcations, Chaotic Behavior, Sensitivity Analysis, and Various Soliton Solutions for the Extended Nonlinear Schrödinger Equation," *Boundary Value Problems* 2024, no. 1 (2024): 15.

15. U. Younas, F. Yao, N. Nasreen, A. Khan, and T. Abdeljawad, "On the Dynamics of Soliton Solutions for the Nonlinear Fractional Dynamical System: Application in Ultrasound Imaging," *Results in Physics* 57 (2024): 107349.

16. M. M. Khater, "Modeling Wave Propagation With Gravity and Surface Tension: Soliton Solutions for the Generalized Hietarinta-Type Equation," *Qualitative Theory of Dynamical Systems* 23, no. 2 (2024): 86.

17. S. Arshed, G. Akram, M. Sadaf, A. U. Nabi, and A. S. Alzaidi, "Optical Soliton Solutions of Perturbed Nonlinear Schrödinger Equation With Parabolic Law Nonlinearity," *Optical and Quantum Electronics* 56, no. 1 (2024): 50.

18. N. Raza, A. Jaradat, G. A. Basendwah, A. Batool, and M. M. M. Jaradat, "Dynamic Analysis and Derivation of New Optical Soliton Solutions for the Modified Complex Ginzburg-Landau Model in Communication Systems," *Alexandria Engineering Journal* 90 (2024): 197–207.

19. M. T. Islam, T. R. Sarkar, F. A. Abdullah, and J. F. Gómez-Aguilar, "Distinct Optical Soliton Solutions to the Fractional Hirota Maccari System Through Two Separate Strategies," *Optik* 300 (2024): 171656.

20. M. A. U. Khan, G. Akram, and M. Sadaf, "Dynamics of Novel Exact Soliton Solutions of Concatenation Model Using Effective Techniques," *Optical and Quantum Electronics* 56, no. 3 (2024): 385.

21. J. Ahmad, K. Noor, S. Anwar, and S. Akram, "Stability Analysis and Soliton Solutions of Truncated M-Fractional Heisenberg Ferromagnetic Spin Chain Model Via Two Analytical Methods," *Optical and Quantum Electronics* 56, no. 1 (2024): 95.

22. M. Adel, K. U. Tariq, H. Ahmad, and S. R. Kazmi, "Soliton Solutions, Stability, and Modulation Instability of the (2+ 1)-Dimensional Nonlinear Hyperbolic Schrödinger Model," *Optical and Quantum Electronics* 56, no. 2 (2024): 182.

23. M. Kapoor, "A Robust Regime Via Sumudu HPM for Schrö dinger Equation in Different Dimensions," *Results in Optics* 14 (2024): 100619.

24. A. M. Wazwaz, "Extended (3+ 1)-Dimensional Kairat-II and Kairat-X Equations: Painlevé Integrability, Multiple Soliton Solutions, Lump Solutions, and Breather Wave Solutions," *International Journal of Numerical Methods for Heat & Fluid Flow* 34, no. 5 (2024): 2177–2194.

25. B. Kopçasız and E. Yaşar, "Dual-Mode Nonlinear Schrödinger Equation (DMNLSE): Lie Group Analysis, Group Invariant Solutions, and Conservation Laws," *International Journal of Modern Physics B* 38, no. 02 (2024): 2450020.

26. Y. Gu, L. Peng, Z. Huang, and Y. Lai, "Soliton, Breather, Lump, Interaction Solutions and Chaotic Behavior for the (2+ 1)-Dimensional KPSKR Equation," *Chaos, Solitons & Fractals* 187 (2024): 115351.

27. Y. Gu, X. Zhang, Z. Huang, L. Peng, Y. Lai, and N. Aminakbari, "Soliton and Lump and Travelling Wave Solutions of the (3+ 1) Dimensional KPB Like Equation With Analysis of Chaotic Behaviors," *Scientific Reports* 14, no. 1 (2024): 20966.

28. Y. Gu, J. Manafian, S. Malmir, B. Eslami, and O. A. Ilhan, "Lump, Lump-Trigonometric, Breather Waves, Periodic Wave and Multi-Waves Solutions for a Konopelchenko–Dubrovsky Equation Arising in Fluid Dynamics," *International Journal of Modern Physics B* 37, no. 15 (2023): 2350141.

29. Y. Gu, S. M. Zia, M. Isam, J. Manafian, A. Hajar, and M. Abotaleb, "Bilinear Method and Semi-Inverse Variational Principle Approach to the Generalized (2+ 1)-Dimensional Shallow Water Wave Equation," *Results in Physics* 45 (2023): 106213.

30. Y. Gu and N. Aminakbari, "New Optical Soliton Solutions for the Variable Coefficients Nonlinear Schrödinger Equation," *Optical and Quantum Electronics* 54, no. 4 (2022): 255.

31. Y. Gu, W. Yuan, N. Aminakbari, and J. Lin, "Meromorphic Solutions of Some Algebraic Differential Equations Related Painlevé Equation IV and its Applications," *Mathematical Methods in the Applied Sciences* 41, no. 10 (2018): 3832–3840.

32. S. Chávez-Vázquez, J. E. Lavín-Delgado, J. F. Gómez-Aguilar, J. R. Razo-Hernández, S. Etemad, and S. Rezapour, "Trajectory Tracking of Stanford Robot Manipulator by Fractional-Order Sliding Mode Control," *Applied Mathematical Modelling* 120 (2023): 436–462.

33. H. Mohammadi, S. Kumar, S. Rezapour, and S. Etemad, "A Theoretical Study of the Caputo–Fabrizio Fractional Modeling for Hearing Loss Due to Mumps Virus With Optimal Control," *Chaos, Solitons & Fractals* 144 (2021): 110668.

34. D. Baleanu, S. M. Aydogn, H. Mohammadi, and S. Rezapour, "On Modelling of Epidemic Childhood Diseases With the Caputo-Fabrizio Derivative by Using the Laplace Adomian Decomposition Method," *Alexandria Engineering Journal* 59, no. 5 (2020): 3029–3039.

35. H. Khan, K. Alam, H. Gulzar, S. Etemad, and S. Rezapour, "A case Study of Fractal-Fractional Tuberculosis Model in China: Existence and Stability Theories Along With Numerical Simulations," *Mathematics and Computers in Simulation* 198 (2022): 455–473.

36. D. Baleanu, S. Etemad, H. Mohammadi, and S. Rezapour, "A Novel Modeling of Boundary Value Problems on the Glucose Graph," *Communications in Nonlinear Science and Numerical Simulation* 100 (2021): 105844.

37. D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, "A New Study on the Mathematical Modelling of Human Liver With Caputo–Fabrizio Fractional Derivative," *Chaos, Solitons & Fractals* 134 (2020): 109705.

38. N. H. Tuan, H. Mohammadi, and S. Rezapour, "A Mathematical Model for COVID-19 Transmission by Using the Caputo Fractional Derivative," *Chaos, Solitons & Fractals* 140 (2020): 110107.

39. M. Ahmad, A. Zada, M. Ghaderi, R. George, and S. Rezapour, "On the Existence and Stability of a Neutral Stochastic Fractional Differential System," *Fractal and Fractional* 6, no. 4 (2022): 203.

40. S. Hussain, E. N. Madi, H. Khan, et al., "Investigation of the Stochastic Modeling of COVID-19 With Environmental Noise From the Analytical and Numerical Point of View," *Mathematics* 9, no. 23 (2021): 3122.

41. K. Dehingia, A. A. Mohsen, S. A. Alharbi, R. D. Alsemiry, and S. Rezapour, "Dynamical Behavior of a Fractional Order Model for Within-Host SARS-CoV-2," *Mathematics* 10, no. 13 (2022): 2344.

42. H. Khan, J. Alzabut, A. Shah, et al., "On Fractal-Fractional waterborne Disease Model: A Study on Theoretical and Numerical Aspects of Solutions Via Simulations," *Fractals* 31, no. 04 (2023): 2340055.

43. K. J. Wang, G. D. Wang, and F. Shi, "Sub-Picosecond Pulses in Single-Mode Optical Fibres With the Kaup–Newell Model Via Two Innovative Methods," *Pramana* 98, no. 1 (2024): 26.

44. A. A. Qarni, "A Riccati Equation Approach to Construct New Dispersive Solitons to the Kaup-Newell Equation," *Memoirs of the Scientific Sections of the Romanian Academy* 43 (2020): 113.

45. A. Biswas, Y. Yıdırım, E. Yaşar, Q. Zhou, S. P. Moshokoa, and M. Belic, "Sub Pico-Second Pulses in Mono-Mode Optical Fibers With Kaup–Newell Equation by a Couple of Integration Schemes," *Optik* 167 (2018): 121–128.

46. A. J. A. M. Jawad, F. J. I. Al Azzawi, A. Biswas, et al., "Bright and Singular Optical Solitons for Kaup–Newell Equation With Two Fundamental Integration Norms," *Optik* 182 (2019): 594–597.

47. B. Ghanbari and D. Baleanu, "Applications of Two Novel Techniques in Finding Optical Soliton Solutions of Modified Nonlinear Schrödinger Equations," *Results in Physics* 44 (2023): 106171.

48. M. Bilal, A. R. Seadawy, M. Younis, S. T. R. Rizvi, and H. Zahed, "Dispersive of Propagation Wave Solutions to Unidirectional Shallow Water Wave Dullin–Gottwald–Holm System and Modulation Instability Analysis," *Mathematical Methods in the Applied Sciences* 44, no. 5 (2021): 4094–4104.

49. A. Ali, J. Ahmad, and S. Javed, "Exact Soliton Solutions and Stability Analysis to (3+ 1)-Dimensional Nonlinear Schrö dinger Model," *Alexandria Engineering Journal* 76 (2023): 747–756.

50. A. H. Salas, S. A. El-Tantawy, and A. A. A. R. Youssef, "New Solutions for Chirped Optical Solitons Related to Kaup-Newell Equation: Application to Plasma Physics," *Optik* 218 (2020): 165203.

51. H. M. Ahmed, W. B. Rabie, A. H. Arnous, and A. M. Wazwaz, "Optical Solitons in Birefringent Fibers of Kaup-Newell's Equation With Extended Simplest Equation Method," *Physica Scripta* 95, no. 11 (2020): 115214.

52. H. Esen, A. Secer, M. Ozisik, and M. Bayram, "Dark, Bright and Singular Optical Solutions of the Kaup–Newell Model With Two Analytical Integration Schemes," *Optik* 261 (2022): 169110.