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On the Solution Structure of Sequential φ-Hilfer Fractional Differential Equations With p-Laplacian Operator

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ABSTRACT

This work researches in a class of φ-Hilfer $\mathbb{FDE}$s with p-Laplacian operator by evolving an appropriate analytical framework. We demonstrate the existence and uniqueness of solutions utilizing Banach's fixed-point theorem. Subsequently, an alternative theorem is applied to verify the existence of at least a single solution. In addition to the theoretical analysis, we provide demonstrative examples to confirm the development of the solution.

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1 | Introduction

The concept of fractional calculus originates from to the 17th century, when mathematicians such as Leibniz, Euler, and L'Hôpital commenced to question the significance of a derivative of noninteger order. Over time, multiple definitions were proposed, among them the Caputo and Riemann–Liouville (RL) derivatives which authorized the modeling of systems with memory.

In the 1990s, R. Hilfer presented a new type of fractional derivative, now established as the Hilfer derivative, which integrates and extends the RL and Caputo expressions through an interpolation parameter [1], for more applications in the articles [2–4]. This novelty led to the progress of sequential Hilfer fractional differential (H$\mathbb{FD}$), where multiple Hilfer-type mappings are utilized successively. Such equations offer a powerful and flexible structure for clarifying complex dynamical processes comprising multiple time scales. In current decades, they have motivated by increasing notice in both theoretical and adopted research, stressing on the existence, uniqueness, and stability of solutions, as well as on the organization of efficient numerical techniques for computation. The authors in [5] discuss the existence and uniqueness of solutions for a newly presented generalization of the Langevin equation. This generalization integrates Caputo-type fractional derivatives of diverse orders and is formulated as

$$\mathrm{C}\mathcal{D}_{c^+}^{\tau,\phi}\left(\mathrm{C}\mathcal{D}_{c^+}^{\alpha,\phi}\left(\mathrm{C}\mathcal{D}_{c^+}^{\beta,\phi}(\psi(z)+\eta)\right)\right)=f\left(z,\psi(z),\mathrm{C}\mathcal{D}_{a^+}^{\gamma,\phi}\psi(z)\right),\ \ t\in(c,d),$$

subject to the limiting conditions

$$\psi(c)=\mathrm{C}\mathcal{D}_{c^+}^{\gamma,\phi}\psi(c),\quad c>0,\quad u(d)=\lambda\sum_{l=1}^{m}\psi\left(\xi_l\right),$$

where $\eta,\lambda>0$, $c<\xi_i<b$, and $\tau,\alpha,\beta,\gamma\in(0,1)$ with $\gamma<\beta$, the function $\phi:(c,d)\longrightarrow\mathbb{R}$ is strictly increasing, $\phi'(t)$ is not equal to zero for all $t\in(c,d)$, and the function $f:(c,d)\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}^+$ is given and is assumed to verify certain conditions. Vatsala and Sambandham examined the historical progression and analytical approach of sequential Caputo fractional derivatives, stressing how these operators have been implemented to formulate and determine linear initial value problems concerning derivatives of iterated fractional order, and in particular, they investigate the sequential Caputo fractional differential equation ($\mathbb{FDE}$) with cumulative order

$$A_m\mathrm{C}\mathcal{D}_{0^+}^{ml}\psi(z)+A_{m-1}\mathrm{C}\mathcal{D}_{0^+}^{(m-1)l}\psi(z)+\cdots+A_1\mathrm{C}\mathcal{D}_{0^+}^{l}\psi(z)+A_0\psi(z)=h(z),$$

which initial conditions

$$\mathrm{C}\mathcal{D}_{0^+}^{pl}\psi(0)=C_p,\quad\left(C_p\right)_{p=\overline{0,m-1}}\in\mathbb{R}$$

where h is continuous function [6]. In addition, the solution method parallels standard integer-order case, with the main difference developing in the inversion of the Laplace transform and one necessity relies on modify expressions specifically formed for $\mathbb{FD}$ operators. Bekri et al. examined the sequential fractional boundary value problem ($\mathbb{FBVP}$) of the form

$$\begin{cases}\mathrm{C}\mathcal{D}_{e^+}^{\eta_1}\left(\mathrm{C}\mathcal{D}_{e^+}^{\eta_2}\omega(z)\right)=f(z)\omega(z), & 1<\eta_1+\eta_2\leq 2,\\ \omega(e)=\omega(d)=0, & z\in[e,d],\end{cases}$$

where $f\in C[e,d]$ and ω is continuous [7]. In a recent examination [8], Kiataramkul et al. assessed the following coupled system including sequential RL and Hadamard–Caputo fractional derivatives which is enriched by nonlocal boundary conditions formulated via fractional integrals:

$$\begin{cases}\mathrm{RL}\mathcal{D}_{0^+}^{\eta_1}\left(\mathrm{H}\mathcal{D}_{0^+}^{\eta_2}\omega(z)\right)=h\left(z,\omega(z),\widetilde{\omega}(z)\right),\\ \mathrm{RL}\mathcal{D}_{0^+}^{\eta_3}\left(\mathrm{H}\mathcal{D}_{0^+}^{\eta_4}\widetilde{\omega}(z)\right)=k\left(z,\omega(z),\widetilde{\omega}(z)\right),\end{cases}$$

for $z\in\mathrm{J}_a:=[0,a]$, with conditions

$$\begin{cases}\mathrm{HC}\mathcal{D}_{0^+}^{\eta_2}\omega(0)=0, & \omega(a)=\displaystyle\sum_{l=1}^{p}\delta_l\mathrm{RL}\mathcal{I}^{\sigma_l}\widetilde{\omega}\left(\lambda_l\right), & \lambda_l,\delta_l\in\mathbb{R},\\ \mathrm{HC}\mathcal{D}_{0^+}^{\eta_4}\widetilde{\omega}(0)=0, & \widetilde{\omega}(a)=\displaystyle\sum_{\theta=1}^{r}\xi_\theta\mathrm{RL}\mathcal{I}^{\sigma_\theta}\omega\left(\widetilde{\lambda}_\theta\right), & \widetilde{\lambda}_\theta,\xi_\theta\in\mathbb{R},\end{cases}$$

where $0<\eta_1,\eta_2,\eta_3,\eta_4<1$, the nonlinear associations $h,k:\mathrm{J}_a\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are considered to be continuous, and the operator $\mathrm{RL}\mathcal{I}^{\sigma_l},\mathrm{RL}\mathcal{I}^{\sigma_\theta}$ represents the RL fractional integral of order $\sigma_l>0$ and $\sigma_\theta>0$, respectively [8]. In reference [7], Erkan et al. prolong their examination to the multiple values framework of the nonlinear type sequential (κ,φ)-Hilfer and (κ,φ)-Caputo fractional problem:

$$\kappa,\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_1,\tau_1,\varphi}\left(\kappa,\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2,\tau_2,\varphi}\phi(l)\right)=d(l,\phi(l)),\quad l\in\mathrm{J}_a,$$

with the boundary conditions

$$\begin{cases}C_1\phi(0)=C_2\phi(a),\\ C_3\kappa,\mathrm{C}\mathcal{D}_{0^+}^{\gamma_1,\varphi}\phi(0)=C_4\kappa,\mathrm{C}\mathcal{D}_{0^+}^{\gamma_2,\varphi}\phi(a), & \left(C_l\right)_{l=\overline{1,4}},\left(\gamma_l\right)_{l=\overline{1,2}}\in\mathbb{R},\end{cases}$$

where $d:\mathrm{J}_a\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ indicates a set-valued operator, the existence of solutions is established via the Leray–Schauder nonlinear alternative, although for nonconvex values, existence is extracted from the fixed-point theorem (FPT) of Covitz–Nadler for multiple values contractions. Gouari et al. examined the following sequential $\mathbb{FDE}$:

$$\mathrm{C}\mathcal{D}_{0^+}^{\alpha_1}\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_2}\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_3}w(t)\right)\right)=L\left(t,w(t),A_1\mathrm{C}\mathcal{D}_{0^+}^{\alpha_4}w(t),A_2\mathrm{C}\mathcal{D}_{0^+}^{\alpha_5}\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_6}w(t)\right)\right),$$

for $t\in\mathrm{J}_a$, $a=1$ with conditions

$$\begin{cases}w(0)+w(1)=A_3\displaystyle\int_0^\xi w(s)\mathrm{d}s, & 0<\xi<1,\\ \mathrm{C}\mathcal{D}_{0^+}^{\alpha_4}w(0)+\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_4}w(1)\right)=0,\\ \mathrm{C}\mathcal{D}_{0^+}^{\alpha_6}\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_6}w(0)\right)+\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_6}\left(\mathrm{C}\mathcal{D}_{0^+}^{\alpha_6}w(1)\right)\right)=0,\end{cases}$$

where $\left(\alpha_l\right)_{l=\overline{1,5}}$ such that $0<\alpha_l<1$ and $\left(A_l\right)_{l=\overline{1,3}}\in\mathbb{R}$, L be a continuous function defined on the interval J [9].

The sequential Hilfer $\mathbb{FDE}$s naturally developed from this framework. Those include the sequential application of many Hilfer-type fractional operators, resulting in models that can capture multiscale dynamic replies and operations with multiple recollection levels. Such systems are particularly helpful in highlighting complex dynamics that improve on different temporal distribution. Then, in this study, we consider the following $\mathbb{BVP}$ involving the φ-Hilfer derivative (H$\mathbb{FDE}$) and p-Laplacian operator:

$$\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_1,\mu_1;\varphi}\psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t)\right)\right)=h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t)),\ \ t\in\mathrm{J}_a,\tag{1}$$

with

$$\begin{cases} \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(0)=0, \quad \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(a)=0, \\ \mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(a)=\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(k_j), \quad k_j\in \mathrm{J}_a, \\ v(0)=0, \end{cases} \tag{2}$$

$$\begin{cases} \psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(0)\right)\right)=0, \\ \psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(a)\right)\right)=0, \end{cases} \tag{3}$$

where $1<(\eta_l)_{l=\overline{1,3}}<2$, three parameters $0\le(\mu_l)_{l=\overline{1,3}}\le 1$, and $\psi_p(r)=r|r|^{p-2}$, $(p>0)$ indicates the p-Laplacian operator such that there exists a real positive q with the property that

$$\frac{1}{p}+\frac{1}{q}=1, \quad (\psi_p)^{-1}=\psi_q. \tag{4}$$

Additionally, $\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}$ is the left-sided φ- RL fractional integral of order $\sigma>0$ with an increasing and positive monotone function φ on J_a having a continuous derivative φ' that does not zero on J_a and $h\in C(\mathrm{J}_a\times\mathbb{R})$.

The main contributions of this work, in comparison with existing studies, can be summarized as follows:

- Unlike [5, 6], which deal with Caputo-type operators in initial value problems, this work considers a boundary value problem involving sequential φ-Hilfer fractional derivatives, providing a more general framework.

- In contrast to [7, 8], where standard nonlinearity and classical fractional operators are used, the present model includes a nonlinear p-Laplacian operator, allowing the analysis of more complex nonlinear behaviors.

- While [8–10] focus on specific nonlocal or multipoint boundary conditions, this study introduces more general boundary conditions expressed via φ-Riemann–Liouville fractional integrals.

- Compared to [7–10], the assumptions imposed on the nonlinear term are less restrictive, which enlarges the class of admissible functions and enhances the applicability of the results.

It is worth noting that, when $\varphi(t)=t$, the sequential φ-Hilfer fractional derivative reduces to the classical Hilfer derivative. Moreover, appropriate choices of the type parameter yield the Caputo and Riemann–Liouville derivatives as limiting cases, while specific form of $\varphi(t)=\log(t)$ leads to the Hadamard-type derivatives, and for $\varphi(t)=t^\rho$, $\rho>0$, returns to the Katugampola-type derivatives. This highlights the general and unifying nature of the proposed framework.

In this research work, Section 2 provides the primary definitions concerning with fractional integrals and derivatives. Section 3 is assigned to establishing the corresponding integral representation of the problem and developing the assumptions necessary for our analysis. We then invoke Banach's FPT to verify the existence and uniqueness of a solution, whereas Schauder's FPT is applied to confirm the existence of at least one solution. As a final step, a concrete example is shown to highlight and illustrate the main findings of this investigation.

2 | Basic Definitions and Lemmas

In this unit, we initiate some definitions and lemmas affiliated to fractional integral and derivatives of Hilfer, see [1, 11–22].

Definition 2.1 [11, 22]**.** Let $\varphi:[e,f]\longrightarrow\mathbb{R},(-\infty\le e<f\le\infty)$ be a continuous function increasing positive on $[e,f]$ and $\varphi'(l)\neq 0$ for each $l\in[e,f]$. We describe it using the RL fractional integral of order $\omega>0$ for function p in relation to the function φ defined by

$$\mathrm{RL}\mathcal{I}_{e^+}^{\omega;\varphi}p(l)=\int_e^l\frac{\varphi'(s)}{\Gamma(\omega)}(\varphi_s(l))^{\omega-1}p(s)\mathrm{d}s, \quad \varphi_s(l):=\varphi(l)-\varphi(s).$$

Based on this interpretation we get the property

$$\mathrm{RL}\mathcal{I}_{e^+}^{\omega;\varphi}(\varphi_e(l))^{\delta-1}=\frac{\Gamma(\delta)}{\Gamma(\omega+\delta)}(\varphi_e(l))^{\omega+\delta-1}, \tag{5}$$

$$\left|\mathrm{RL}\mathcal{I}_{e^+}^{\omega;\varphi}p(l)\right|\le\frac{\|p\|_\infty}{\Gamma(\omega+1)}(\varphi_e(f))^{\omega}, \tag{6}$$

where

$$\|p\|_\infty=\sup_{l\in[e,f]}|p(l)|. \tag{7}$$

Lemma 2.2 [12, 13, 16]**.** We assume that the function p is continuous on the interval $[e,f]$. Then, the RL fractional integral $(RL\mathcal{I}_{e^+}^{\omega_r;\varphi})$ in relation to the function φ conforms for all $\omega_r>0$ such that $r=1,2$ and $l\in[e,f]$ the following characteristics of semigroups:

$$\mathrm{RL}\mathcal{I}_{e^+}^{\omega_1;\varphi}\left(\mathrm{RL}\mathcal{I}_{e^+}^{\omega_2;\varphi}p(l)\right)=\mathrm{RL}\mathcal{I}_{e^+}^{\omega_1+\omega_2;\varphi}p(l).$$

Lemma 2.3 [14, 17]**.** Assuming that φ is a continuous function strictly increasing belonging to $C^1[e,f]$ whose derivative is not identical to zero for all $l\in[e,f]$ and p is a continuous function on this segment. Then, for any $y_1,\widetilde{y}_1\in[e,f]$ with $y_1\le\widetilde{y}_1$, we have

$$\left|(\mathrm{RL}\mathcal{I}_{e^+}^{\omega;\varphi})p(\widetilde{y}_1)-(\mathrm{RL}\mathcal{I}_{e^+}^{\omega;\varphi})p(y_1)\right|\le 2\frac{(\varphi_{y_1}(\widetilde{y}_1))^{\omega}}{\Gamma(\omega+1)}\|p\|_\infty.$$

Definition 2.4 [1, 20]**.** Let g and φ be continuous functions belonging to $C^m[c,d]$, $m\in\mathbb{N}$ where $m=[\xi]+1$ such that $[\xi]$ represents the integer part real number of ξ. In addition, φ is an increasing function and $\varphi'(l)\neq 0$ for all $l\in[e,f]$. Then, the RL fractional derivative in relation to the function φ is given as

$$\mathrm{RL}\mathcal{D}_{e^+}^{\xi;\varphi}g(l) = \left(\frac{1}{\varphi'(l)}\frac{\mathrm{d}}{\mathrm{d}l}\right)^m \mathrm{RL}\mathcal{I}_{e^+}^{m-\xi;\varphi}g(l)$$
$$= \left(\frac{1}{\varphi'(l)}\frac{\mathrm{d}}{\mathrm{d}l}\right)^m \int_e^l \frac{\varphi'(s)}{\Gamma(m-\xi)}\left(\varphi_s(l)\right)^{m-\xi-1}g(s)\mathrm{d}s.$$

Definition 2.5 [18, 20]. Assuming that g and φ be continuous functions belonging to $C^m\left[e,f\right]$. The Hilfer fractional derivative of order $m-1<\xi<m$ with $m\in\mathbb{N}$ and type $0\leq\delta\leq 1$ in the relation to the function φ, where this is increasing function on $\left[e,f\right]$ and $\varphi'(l)\neq 0$ for all l in the interval, is defined by

$$\mathrm{Hr}\mathcal{D}_{e^+}^{\xi,\delta;\varphi}g(l) = \mathrm{RL}\mathcal{I}_{e^+}^{\delta(m-\xi);\varphi}\left(\frac{1}{\varphi'(l)}\frac{\mathrm{d}}{\mathrm{d}l}\right)^m \mathrm{RL}\mathcal{I}_{e^+}^{(1-\delta)(m-\xi);\varphi}g(l), \quad l\in e,f].$$

Let $\omega=\xi+\delta(m-\xi)$ where $m=[\xi]+1$. Then, we get

$$\begin{cases} m-\omega=(1-\delta)(m-\xi), \\ \mathrm{Hr}\mathcal{D}_{e^+}^{\xi,\delta;\varphi}g(l) = \mathrm{RL}\mathcal{I}_{e^+}^{\omega-\xi;\varphi}\left(\frac{1}{\varphi'(l)}\frac{\mathrm{d}}{\mathrm{d}l}\right)^m \mathrm{RL}\mathcal{I}_{e^+}^{m-\omega;\varphi}g(l), \end{cases}$$

and from the Definition 2.4, we obtain $\mathrm{Hr}\mathcal{D}_{e^+}^{\xi,\delta;\varphi}g(l) = \mathrm{RL}\mathcal{I}_{e^+}^{\omega-\xi;\varphi}\mathrm{RL}\mathcal{D}_{e^+}^{\omega;\varphi}g(l)$. For example, let $m-1<\xi<m$ and $\tau>1$. Then, for each $l\in\left[e,f\right]$ we get

$$\mathrm{Hr}\mathcal{D}_{e^+}^{\xi,\delta;\varphi}\varphi_{\tau-1}(l,e) = \frac{\Gamma(\tau)}{\Gamma(\tau-\xi)}\left(\varphi_e(l)\right)^{\tau-\xi-1}.$$

Lemma 2.6 [19]. Let g be a continuous function belonging to $C^m\left[e,f\right]$ and assuming that $m-1<\xi<m$, $m\in\mathbb{N}$. Furthermore, $0\leq\delta\leq 1$ and for all $l\in\left[e,f\right]$, we have

$$\mathrm{RL}\mathcal{I}_{e^+}^{\xi;\varphi}\mathrm{Hr}\mathcal{D}_{e^+}^{\xi,\delta;\varphi}g(l) = g(l) - \sum_{p=1}^{p=m}\frac{\left(\varphi_e(l)\right)^{\omega-p}}{\Gamma(\omega-p+1)}a_p,$$

where

$$a_p = \sqsubseteq_\varphi^{[n-p]}\mathrm{RL}\mathcal{I}_{e^+}^{(1-\delta)(m-\xi);\varphi}g(e) = \sqsubseteq_\varphi^{[n-p]}\mathrm{RL}\mathcal{I}_{e^+}^{m-\omega;\varphi}g(e),$$

such that

$$\begin{cases} \sqsubseteq_\varphi^{[m]}g(l) = \left(\frac{1}{\varphi'(l)}\frac{d}{dl}\right)^m g(l), \\ \omega=\xi+\delta(m-\xi). \end{cases}$$

Proposition 2.7 ([17], Page 4, Lemma 2.4). For all $y_1,\widetilde{y}_1\in\left[e,f\right]$. Then, the following estimates hold:

i. We suppose that $1<p\leq 2$, and there exists a positive real number C_1 such that

$$\begin{cases} |y_1|>C_1, |\widetilde{y}_1|>C_1, \\ y_1\cdot\widetilde{y}_1>0. \end{cases}$$

From this, we arrive at the estimate

$$\left|\psi_p(\widetilde{y}_1)-\psi_p(y_1)\right|\leq(p-1)C_1^{p-2}|\widetilde{y}_1-y_1|.$$

ii. Let $p>2$, and there exists a positive real number C_2 under the condition that

$$|y_1|<C_2, \quad |\widetilde{y}_1|<C_2.$$

We can thus establish the estimate as follows:

$$\left|\psi_p(\widetilde{y}_1)-\psi_p(y_1)\right|\leq(p-1)C_2^{p-2}|\widetilde{y}_1-y_1|.$$

Lemma 2.8 [15]. We work with real numbers $(y_p)_{p=1,l}$ each taking positive values and suppose that $\xi\geq 1$. We have the following inequality:

$$\left[\sum_{p=1}^{l}y_p\right]^\xi \leq l^{\xi-1}\sum_{p=1}^{l}y_p^\xi.$$

Theorem 2.9 (Banach FPT [23]). Let (X,d) be a complete metric space and let $T:X\longrightarrow X$ be a contraction, that is, there exists a constant $k\in(0,1)$ such that

$$d(Tx,Ty)\leq kd(x,y), \quad \forall x,y\in X.$$

Then, T admits a unique fixed point in X.

Theorem 2.10 (Schauder FPT [23]). Let X be a Banach space and let $C\subset X$ be a nonempty, closed, bounded, and convex subset. If $T:C\longrightarrow C$ is continuous and compact, then T has at least one fixed point in C.

Theorem 2.11 (Arzelà–Ascoli Theorem [23]). Let $K\subset\mathbb{R}^n$ be compact and let $\mathcal{F}\subset C(K)$. Then, $\mathcal{F}$ is relatively compact in $C(K)$ (with respect to the uniform norm) if and only if

1. $\mathcal{F}$ is uniformly bounded;
2. $\mathcal{F}$ is equicontinuous on K.

3 | Essential Results

We consider the subsequent Banach space $Z=C\left(\mathrm{J}_a\right)$, with the norm defined by Equation (7).

Theorem 3.1. Suppose that $L\in Z$. If

$$\Delta := \left(\varphi_0(\mathrm{a})\right)^{\sigma+\widetilde{\delta}-1} - \sum_{j=1}^{m}\left(\varphi_0\left(k_j\right)\right)^{\sigma+\widetilde{\delta}-1}\neq 0, \tag{8}$$

here $\widetilde{\delta}=\eta_3+\mu_3\left(2-\eta_3\right)$. Then, the solution integral of the following equation

$$\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_1,\mu_1;\varphi}\psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t)\right)\right)=L(t), \quad t\in \mathrm{J}_a,$$

with boundary conditions

$$\begin{cases} \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(0)=0, \ \ \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(a)=0, \\ \mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(a)=\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(k_j), \ \ k_j\in \mathrm{J}_a, \\ v(0)=0, \end{cases}$$
$$\begin{cases} \psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(0)\right)\right)=0, \\ \psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(a)\right)\right)=0, \end{cases}$$

is given by

$$\begin{aligned} v(t) &=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t)-Y_v(t)\right) \\ &+\frac{\Gamma\left(\sigma+\widetilde{\delta}\right)\left(\varphi_0(t)\right)^{\widetilde{\delta}-1}}{\Delta\Gamma\left(\widetilde{\delta}\right)}\left[\sum_{j=1}^{m}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q\left(\mathcal{I}_{0^+}^{\eta_1;\varphi}h(k_j)-Y_v(k_j)\right)\right] \end{aligned}$$

where

$$Y_v(t)=\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a), \quad \delta=\eta_1+\mu_1\left(2-\eta_1\right).$$

Proof 3.1. The equation introduced below is recognized as

$$\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_1,\mu_1;\varphi}\psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t)\right)\right)=L(t).$$

By employing the operator $\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}$ to this equation and exploiting the findings from Lemma 2.6, the equation can be illustrated in the following form:

$$\begin{aligned} &\psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t)\right)\right) \\ &\quad=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t)+\frac{\left(\varphi_0(t)\right)^{\delta-2}}{\Gamma(\delta-1)}z_0+\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\Gamma(\delta)}z_1, \end{aligned} \tag{9}$$

where z_0 and z_1 are two real numbers under examination. Additionally, we observe $\delta-1\geq 0$, and

$$2-\delta=\left(1-\mu_1\right)\left(2-\eta_1\right)\geq 0, \quad \delta-2\leq 0.$$

Using Condition (3) and the consequence estimates, we can then present

$$\begin{aligned} &\psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(0)\right)\right) \\ &=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(0)+\frac{\left[\left(\varphi_0(t)\right)^{\delta-2}\right]_{t\longrightarrow 0}}{\Gamma(\delta-1)}z_0+\frac{\left[\left(\varphi_0(t)\right)^{\delta-1}\right]_{t=0}}{\Gamma(\delta)}z_1=0, \end{aligned}$$

which implies that z_0 equals 0. Once again, by applying Condition (3) in Equation (9), we obtain

$$\psi_p\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(a)\right)\right)=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)+\frac{\left(\varphi_0(a)\right)^{\delta-1}}{\Gamma(\delta)}z_1=0.$$

By simplifying the prior terms, we derive

$$z_1=-\frac{\Gamma(\delta)}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a).$$

By replacing the value z_1 into Equation (9) and utilizing the property of the operator ψ defined in the same Formula (4), we achieve the following result:

$$\begin{aligned} &\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;\mu_2;\varphi}\left(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t)\right) \\ &\quad=\psi_p^{-1}\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t)-\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right). \end{aligned}$$

Upon functioning on this equation with the operator $\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2;\varphi}$, the resulting representation is deduced as follows:

$$\begin{aligned} \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t) &=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t)-\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &+\frac{\left(\varphi_0(t)\right)^{\varsigma-2}}{\Gamma(\varsigma-1)}z_2+\frac{\left(\varphi_0(t)\right)^{\varsigma-1}}{\Gamma(\varsigma)}z_3, \end{aligned} \tag{10}$$

where $\varsigma=\eta_2+\mu_2\left(2-\eta_2\right)$ and z_2, z_3 are two real numbers under examination and exploiting the findings from Lemma 2.6, such that the following inequality holds:

$$2-\varsigma=\left(1-\mu_2\right)\left(2-\eta_2\right)\geq 0. \tag{11}$$

To determine the value of z_2, we set $t=0$ in order to apply the boundary conditions. However, after this substitution, some denominators vanish. Therefore, the value is obtained by considering the corresponding limit. Thus, using Condition (2) and the consequence estimates, we can then present

$$\begin{aligned} \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(0) &=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(0)-\frac{\left[\varphi_0(t)\right]_{t=0}^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &+\left[\frac{\left(\varphi_0(t)\right)^{\varsigma-2}}{\Gamma(\varsigma-1)}\right]_{t\longrightarrow 0}z_2+\frac{\left[\varphi_0(t)\right]_{t=0}^{\varsigma-1}}{\Gamma(\varsigma)}z_3=0. \end{aligned}$$

Using Interrelation (11), we also acquire the following observations:

$$\begin{cases} \varsigma-1\geq 0, \\ \varsigma-2\leq 0, \end{cases} \Rightarrow \begin{cases} \left[\left(\varphi_0(t)\right)\right]_{t=0}^{\varsigma-1}=0, \\ \left(\varphi_0(t)\right)^{\varsigma-2}\longrightarrow\infty, \quad t\longrightarrow 0. \end{cases}$$

By executing these formulas, we can promptly evaluate that $z_2=0$. Using Condition (2) in Equation (10), we obtain the following result:

$$\begin{aligned} \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(a) &=\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)-\frac{\left[\left(\varphi_0(t)\right)\right]_{t=a}^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &+\frac{\left(\varphi_0(t)\right)^{\varsigma-1}}{\Gamma(\varsigma)}z_3=0. \end{aligned}$$

From this, it follows that

$$z_3 = -\frac{\Gamma(\varsigma)}{(\varphi_0(a))^{\varsigma-1}}\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a) - \mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right)\right) = 0.$$

By replacing the value z_3 into the equation, we get

$$\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;\mu_3;\varphi}v(t) = \mathrm{RL}\mathcal{I}_{0^+}^{\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t) - \frac{(\varphi_0(t))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right).$$

Employing the operator $\mathrm{RL}\mathcal{I}_{0^+}^{\eta_3;\varphi}$ in this equation and exploiting the findings of Lemma 2.6, we find the following:

$$\begin{aligned} v(t) &= \mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t) - \frac{(\varphi_0(t))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &\quad + \frac{(\varphi_0(t))^{\widetilde{\delta}-2}}{\Gamma(\widetilde{\delta}-1)}z_4 + \frac{(\varphi_0(t))^{\widetilde{\delta}-1}}{\Gamma(\widetilde{\delta})}z_5, \end{aligned} \tag{12}$$

where z_4 and z_5 are two real numbers under examination. Similar to the former case, the result becomes $\widetilde{\delta} - 1 \geq 0$ and $2 - \widetilde{\delta} = (1-\mu_3)(2-\eta_3) \geq 0$. When Equation (12) is evaluated at $t = 0$ and from the Condition (2), the statement becomes

$$\begin{aligned} v(0) &= \mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(0) - \frac{\left[(\varphi_0(t))\right]_{t=0}^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &\quad + \left(\frac{(\varphi_0(t))^{\widetilde{\delta}-2}}{\Gamma(\widetilde{\delta}-1)}\right)_{t\longrightarrow 0} z_4 + \left(\frac{(\varphi_0(t))^{\widetilde{\delta}-1}}{\Gamma(\widetilde{\delta})}\right)_{t=0} z_5 = 0, \end{aligned}$$

which yields $z_4 = 0$. To apply the condition $\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(a) = \sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(k_j)$, we substitute the value $t = a$ and z_4 into Equation (12), which yields the following result:

$$\begin{aligned} \mathcal{I}_{0^+}^{\sigma;\varphi}v(a) &= \mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a) - \frac{\left[(\varphi_0(t))\right]_{t=a}^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &\quad + \mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}\frac{(\varphi_0(a))^{\widetilde{\delta}-1}}{\Gamma(\delta_n)}z_5 = \frac{z_5}{\Gamma(\widetilde{\delta})}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}\left((\varphi(a)-\varphi(0))^{\widetilde{\delta}-1}\right). \end{aligned}$$

Invoking Property (5) allows us to express it as

$$\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(a) = \frac{(\varphi_0(a))^{\sigma+\widetilde{\delta}-1}}{\Gamma(\sigma+\widetilde{\delta})}z_5. \tag{13}$$

Now, we substitute the value $t = k_j$ into Equation (12), we obtain

$$\begin{aligned} &\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}v(k_j) \\ &= \sum_{j=1}^{m}\mathcal{I}_{0^+}^{\sigma+\eta_3+\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(k_j) - \frac{(\varphi_0(k_j))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &\quad + \sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma;\varphi}\frac{(\varphi_0(k_j))^{\delta_n-1}}{\Gamma(\widetilde{\delta})}z_5. \end{aligned}$$

Again, invoking Property (5) allows us to express it as

$$\begin{aligned} &\sum_{j=1}^{m}I_{0^+}^{\sigma;\varphi}v(k_j) \\ &= \sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_3+\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(k_j) - \frac{(\varphi_0(k_j))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &\quad + \frac{z_5}{\Gamma(\sigma+\widetilde{\delta})}\left(\sum_{j=1}^{m}(\varphi_0(k_j))^{\sigma+\widetilde{\delta}-1}\right). \end{aligned}$$

Setting Expressions (13) and (14) equal to each other yields

$$\begin{aligned} &\frac{1}{\Gamma(\sigma+\widetilde{\delta})}\left[(\varphi_0(a))^{\sigma+\widetilde{\delta}-1} - \sum_{j=1}^{m}(\varphi_0(k_j))^{\sigma+\widetilde{\delta}-1}\right]z_5 \\ &= \sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_3+\eta_2;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(k_j) - \frac{(\varphi_0(k_j))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right). \end{aligned}$$

And thanks to the Term (8), we can write

$$\begin{aligned} z_5 &= \frac{\Gamma(\sigma+\widetilde{\delta})}{\Delta}\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_3+\eta_2;\varphi}\psi_q\Bigg(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(k_j) \\ &\quad - \frac{(\varphi_0(k_j))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\Bigg). \end{aligned}$$

By replacing the value z_5 into the equation, we get

$$\begin{aligned} v(t) &= \mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(t) - \frac{(\varphi_0(t))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right) \\ &\quad + \frac{(\varphi_0(t))^{\widetilde{\delta}-1}}{\Gamma(\widetilde{\delta})}\Bigg[\frac{\Gamma(\sigma+\widetilde{\delta})}{\Delta}\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q \\ &\quad \left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(k_j) - \frac{(\varphi_0(k_j))^{\delta-1}}{(\varphi_0(a))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}L(a)\right)\Bigg]. \end{aligned}$$

The demonstration has been completed.

To proceed with the primary results, we assume the following assumptions:

$\mathcal{A}_1$) The function h displays continuity.

$\mathcal{A}_2$) Assuming for all $u, v \in \mathbb{R}$, there exists a positive real $\mathcal{M}$ such that

$$|h(t,u)-h(t,v)| \le \mathcal{M}|u-v|.$$

Proposition 3.2. Let $t \in J_a$ and we assign

$$\Omega_1 = \mathcal{M}\frac{\left(\varphi_0(a)\right)^{\sigma+\eta_1}}{\Gamma(\sigma+1)\Gamma\left(\eta_1+1\right)}. \tag{14}$$

Then, the following estimates are valid:

i. We suppose that $u, v \in Z$, then $\left|Y_v(t)-Y_u(t)\right| \le \Omega_1 \|v-u\|$.

ii. For all $q>1$, we get

$$\begin{aligned}&\left|\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-Y_v(t)\right)-\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}u(t))-Y_u(t)\right)\right|\\ &\le 2\Omega_1(q-1)C^{q-2}\|v-u\|,\end{aligned}$$

where C represents a constant positive.

Proof 3.2. i. We have

$$\left|Y_v(t)-Y_u(t)\right| \le \sup_{t\in J_a}\left|\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\right|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}|h(a,\mathrm{RL}\mathcal{I}^{\sigma}v(a))-h(a,\mathrm{RL}\mathcal{I}^{\sigma}u(a))|.$$

From the definition of RL Integral 2.1 and employing hypothesis $(\mathcal{A}_2)$, we get

$$\left|Y_v(t)-Y_u(t)\right| \le \mathcal{M}\mathrm{RL}\mathcal{I}^{\sigma}|v-u|\frac{\left(\varphi_0(a)\right)^{\eta_1}}{\Gamma\left(\eta_1+1\right)} \le \mathcal{M}\frac{\left(\varphi_0(a)\right)^{\sigma+\eta_1}}{\Gamma(\sigma+1)\Gamma\left(\eta_1+1\right)}|v-u|;$$

subsequently, the results are as follow:

ii. According to Proposition 2.7 and employing hypothesis ($\mathcal{A}_2$), there exists a positive real number C satisfying

$$\begin{aligned}&\left|\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-Y_v(t)\right)-\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}u(t))-Y_u(t)\right)\right|\\ &\le (q-1)C^{q-2}\left[\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}|h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-h(t,\mathrm{RL}\mathcal{I}^{\sigma}u(t))|+\left|Y_v(t)-Y_u(t)\right|\right]\\ &\le (q-1)C^{q-2}\left[\mathcal{M}\frac{\left(\varphi_0(a)\right)^{\sigma+\eta_1}}{\Gamma(\sigma+1)\Gamma\left(\eta_1+1\right)}\|v-u\|+\Omega_1\|v-u\|\right].\end{aligned}$$

Thereafter, the findings are proved.

Also, we propose the following additional hypothesis:

$\mathcal{A}_3$) For all $t \in \mathrm{J}_a$ and $u \in \mathbb{R}$, there exists a constant $\widetilde{\mathcal{M}}>0$ such that $|h(t,u)| < \widetilde{\mathcal{M}}$.then

$$\left|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-Y_v(t)\right|^{q-1} \le 2^{q-1}\widetilde{\Omega}_1.$$

Corollary 3.3. Let $t \in J_a$, $q>1$ and

$$\widetilde{\Omega}_1 = \left[\widetilde{\mathcal{M}}\frac{\left(\varphi_0(a)\right)^{\eta_1}}{\Gamma\left(\eta_1+1\right)}\right]^{q-1}, \tag{15}$$

Proof 3.3. Using the Lemma 2.8, we get

$$\begin{aligned}&\left|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-Y_v(t)\right|^{q-1}\\ &\le 2^{q-2}\sup_{t\in J_a}\left(\left|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))\right|^{q-1}+\left|Y_v(t)\right|^{q-1}\right)\\ &\le 2^{q-2}\left[\left|\widetilde{\mathcal{M}}\frac{\left(\varphi_0(a)\right)^{\eta_1}}{\Gamma\left(\eta_1+1\right)}\right|^{q-1}+\left|Y_v(t)\right|^{q-1}\right].\end{aligned}$$

From the given hypotheses $(\mathcal{A}_3)$, it follows that

$$\begin{aligned}\left|Y_v(t)\right| &\le \sup_{t\in J_a}\left(\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\right)\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}|h(a,\mathrm{RL}\mathcal{I}^{\sigma}v(a))|\\ &\le \left[\widetilde{\mathcal{M}}\frac{\left(\varphi_0(a)\right)^{\eta_1}}{\Gamma\left(\eta_1+1\right)}\right]^{q-1}.\end{aligned}$$

These calculations enable us to conclude that

$$\left|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-Y_v(t)\right|^{q-1} \le 2^{q-2}\left[2\left[\widetilde{\mathcal{M}}\frac{\left(\varphi_0(a)\right)^{\eta_1}}{\Gamma\left(\eta_1+1\right)}\right]^{q-1}\right].$$

The proof has been completed.

According to Theorem 3.1, we analyze the operator $\mathcal{B}: Z \longrightarrow Z$, which is determined as follows:

$$\begin{aligned}\mathcal{B}v(t) &= \mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}v(t))-Y_v(t)\right)\\ &+\frac{\Gamma\left(\sigma+\widetilde{\delta}\right)\left(\varphi_0(t)\right)^{\widetilde{\delta}-1}}{\Delta\Gamma\left(\widetilde{\delta}\right)}\left[\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q\right.\\ &\left.\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h\left(k_j,\mathrm{RL}\mathcal{I}^{\sigma}v\left(k_j\right)\right)-Y_v\left(k_j\right)\right)\right],\end{aligned} \tag{16}$$

where

$$Y_v(t)=\frac{\left(\varphi_0(t)\right)^{\delta-1}}{\left(\varphi_0(a)\right)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(a,\mathrm{RL}\mathcal{I}^{\sigma}v(a)). \tag{17}$$

Theorem 3.4. We assume that hypotheses $(\mathcal{A}_1)-(\mathcal{A}_3)$ are gratified and that the additional expressed by the relation $(q-1)\Omega_2 < 1/2$, also holds, where

$$\Omega_2 = C^{q-2}\left[\frac{\left(\varphi_0(a)\right)^{\eta_2+\eta_3}}{\Gamma\left(\eta_2+\eta_3+1\right)}+\frac{\Gamma\left(\sigma+\widetilde{\delta}\right)\left(\varphi_0(a)\right)^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)}\sum_{j=1}^{m}\frac{\left(\varphi_0\left(k_j\right)\right)^{\sigma+\eta_2+\eta_3}}{\Gamma\left(\sigma+\eta_2+\eta_3+1\right)}\right]\Omega_1. \tag{18}$$

Then, there exists exactly one solution to $\mathbb{BVP}$ (1)–(3) in the interval J_a, and this solution is unique.

Proof 3.4. We consider the ball

$$\begin{cases} \mathcal{A}_\theta = \{v \in Z : \|v\|_Z \le \theta\}, \\ \Omega_3 \le \dfrac{\theta}{2^{q-1}}, \end{cases}$$

where

$$\Omega_3 = \left[\frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} + \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(a))^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \left[\sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \right] \right] \widetilde{\Omega}_1. \tag{19}$$

Let $v \in \mathcal{A}_\theta$. We obtain

$$\begin{aligned} |\mathcal{B}v(t)| \le& \sup_{t\in J_a} \left| \mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi} \psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(t, \mathrm{RL}\mathcal{I}^\sigma v(t)) - Y_v(t)\right) \right| \\ &+ \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \sup_{t\in J_a} \left| (\varphi_0(t))^{\widetilde{\delta}-1} \right| \\ &\times \left| \sum_{j=1}^{m} \mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi} \psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(k_j, \mathrm{RL}\mathcal{I}^\sigma v(k_j)) - Y_v(k_j)\right) \right| \\ \le& \frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} \psi_q\left(\left| \mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(t, \mathrm{RL}\mathcal{I}^\sigma v(t)) - Y_v(t) \right| \right) \\ &+ \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(a))^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \times \left[\sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \right] \sup_{k_j\in J_a} \\ &\psi_q \left| \mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(k_j, \mathrm{RL}\mathcal{I}^\sigma v(k_j)) - Y_v(k_j) \right|. \end{aligned}$$

Based on the definition of the operator ψ_q, it can be written as

$$\begin{aligned} |\mathcal{B}v(t)| \le& \frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} \left| \mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(t, \mathrm{RL}\mathcal{I}^\sigma v(t)) - Y_v(t) \right|^{q-1} \\ &+ \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(a))^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \left[\sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \right] \\ &\times \left| \mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(k_j, \mathrm{RL}\mathcal{I}^\sigma v(k_j)) - Y_v(k_j) \right|^{q-1}. \end{aligned}$$

From Corollary 3.3, we obtain

$$\begin{aligned} |\mathcal{B}v(t)| \le& \left[\frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} + \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(a))^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \right. \\ &\left. \left[\sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \right] \right] 2^{q-1} \widetilde{\Omega}_1 \le 2^{q-1}\Omega_3 \le \theta. \end{aligned}$$

We show that the set $\mathcal{B}(\mathcal{A}_\theta) \subset (\mathcal{A}_\theta)$.

Let $u, v \in Z$. The following values have been determined:

$$\|\mathcal{B}v - \mathcal{B}u\|_Z = \sup_{t\in J_a} |\mathcal{B}v(t) - \mathcal{B}u(t)|.$$

From the defining format of the RL Integral 2.1 and the Association 2.2, we may represent it as

$$\begin{aligned} &|\mathcal{B}v(t) - \mathcal{B}u(t)| \\ &\le \frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} \left| \psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(t, \mathrm{RL}\mathcal{I}^\sigma v(t)) - Y_v(t)\right) \right. \\ &\left. - \psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(t, \mathrm{RL}\mathcal{I}^\sigma u(t)) - Y_u(t)\right) \right| + \sup_{t\in J_a} \left| \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(t))^{\widetilde{\delta}-1}}{\Delta\Gamma\left(\widetilde{\delta}\right)} \right| \\ &\times \left[\sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \left| \psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(k_j, \mathrm{RL}\mathcal{I}^\sigma v(k_j)) - Y_v(k_j)\right) \right. \right. \\ &\left. \left. - \psi_q\left(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi} h(k_j, \mathrm{RL}\mathcal{I}^\sigma u(k_j)) - Y_u(k_j)\right) \right| \right]. \end{aligned}$$

Using the Proposition 3.2, we obtain

$$\begin{aligned} |\mathcal{B}v(t) - \mathcal{B}u(t)| \le& 2\frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} \Omega_1(q-1)C^{q-2}\|v-u\| \\ &+ 2\frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(a))^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \Omega_1(q-1)C^{q-2} \\ &\left[\sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \right] \|v-u\| \\ \le& 2\left[\frac{(\varphi_0(a))^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)} + \frac{\Gamma\left(\sigma+\widetilde{\delta}\right)(\varphi_0(a))^{\widetilde{\delta}-1}}{|\Delta|\Gamma\left(\widetilde{\delta}\right)} \right. \\ &\left. \sum_{j=1}^{m} \frac{(\varphi_0(k_j))^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)} \right] \Omega_1(q-1)C^{q-2} \\ \le& 2\Omega_2(q-1)\|v-u\|. \end{aligned}$$

The proof has been completed.

To demonstrate the existence of at least one solution, we will employ the FPT of Schauder.

Theorem 3.5. We assume that hypotheses $(\mathcal{A}_1) - (\mathcal{A}_3)$ are valid. Then, the Problem (1)–(3) has at least one solution on J_a.

Proof 3.5. We consider a sequence $(p_n)_{n\in\mathbb{N}}$ that converges to p, which means

$$\lim_{n\longrightarrow+\infty} \|p_n - p\| = 0.$$

With the help of Formula (16), we find

$$
\begin{aligned}
&\left|\mathcal{B}(p_n)(t)-\mathcal{B}(p)(t)\right| \\
&\le \Big|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\big(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}p_n(t))-Y_{p_n}(t)\big) \\
&\quad -\mathrm{RL}\mathcal{I}_{0^+}^{\eta_2+\eta_3;\varphi}\psi_q\big(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(t,\mathrm{RL}\mathcal{I}^{\sigma}p(t))-Y_p(t)\big)\Big| \\
&\quad +\sup_{t\in \mathrm{J}_a}\left|\frac{\Gamma\big(\sigma+\widetilde{\delta}\big)\big(\varphi_0(t)\big)^{\widetilde{\delta}-1}}{\Delta\Gamma\big(\widetilde{\delta}\big)}\right| \\
&\quad \times\Bigg|\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q\big(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(k_j,\mathrm{RL}\mathcal{I}^{\sigma}p_n(k_j))-Y_{p_n}(k_j)\big) \\
&\quad -\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q\big(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(k_j,\mathrm{RL}\mathcal{I}^{\sigma}p(k_j))-Y_p(k_j)\big)\Bigg|,
\end{aligned}
$$

where

$$
\begin{cases}
Y_{p_n}(t)=\dfrac{\big(\varphi_0(t)\big)^{\delta-1}}{\big(\varphi_0(a)\big)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h\big(a,\mathrm{RL}\mathcal{I}^{\sigma}p_n(a)\big),\\[2ex]
Y_p(t)=\dfrac{\big(\varphi_0(t)\big)^{\delta-1}}{\big(\varphi_0(a)\big)^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(a,\mathrm{RL}\mathcal{I}^{\sigma}p(a)).
\end{cases}
$$

Thanks to the continuity of the function h and the assumptions $(\mathcal{A}_2)$, we can deduce that

$$
\left|\mathcal{B}(p_n)(t)-\mathcal{B}(p)(t)\right|\le 2\Omega_2(q-1)\|p_n-p\|,
$$

thus, by the dominated convergence $\left\|\mathcal{B}(p_n)-\mathcal{B}(p)\right\|_Z\longrightarrow 0$. Therefore, $\mathcal{B}$ is continuous. Let $t_1,t_2\in \mathrm{J}_a$ such that $t_1<t_2$ and $v\in\mathcal{A}_\theta$. We obtain

$$
\begin{aligned}
&\left|\mathcal{B}v(t_2)-\mathcal{B}v(t_1)\right| \\
&\le \frac{1}{\Gamma(\eta_2+\eta_3+1)}\bigg|\int_0^{t_2}\varphi'(s)\big(\varphi_s(t_2)\big)^{\eta_2+\eta_3}\psi_q\big(\mathcal{I}_{0^+}^{\eta_1;\varphi}h(s,\mathcal{I}^{\sigma}v(s))-Y_v(s)\big)\,\mathrm{d}s \\
&\quad -\frac{1}{\Gamma(\eta_2+\eta_3+1)}\int_0^{t_1}\varphi'(s)\big(\varphi_s(t_1)\big)^{\eta_2+\eta_3}\psi_q\big(\mathcal{I}_{0^+}^{\eta_1;\varphi}h(s,\mathcal{I}^{\sigma}v(s))-Y_v(s)\big)\,\mathrm{d}s\bigg| \\
&\quad +\frac{\Gamma\big(\sigma+\widetilde{\delta}\big)}{|\Delta|\Gamma\big(\widetilde{\delta}\big)}\left|\big(\varphi_0(t_2)\big)^{\widetilde{\delta}-1}-\big(\varphi_0(t_1)\big)^{\widetilde{\delta}-1}\right| \\
&\quad \times\left|\sum_{j=1}^{m}\mathrm{RL}\mathcal{I}_{0^+}^{\sigma+\eta_2+\eta_3;\varphi}\psi_q\big(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(k_j,\mathrm{RL}\mathcal{I}^{\sigma}v(k_j))-Y_v(k_j)\big)\right| \\
&\le \frac{1}{\Gamma(\eta_2+\eta_3+1)}\int_0^{t_1}\varphi'(s)\left|\big(\varphi_s(t_2)\big)^{\eta_2+\eta_3}-\big(\varphi_s(t_1)\big)^{\eta_2+\eta_3}\right| \\
&\quad \times\psi_q\left|\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(s,\mathrm{RL}\mathcal{I}^{\sigma}v(s))-Y_v(s)\right|\mathrm{d}s+\frac{1}{\Gamma(\eta_2+\eta_3+1)} \\
&\quad \int_{t_1}^{t_2}\varphi'(s)\left|\big(\varphi_s(t_2)\big)^{\eta_2+\eta_3}\psi_q\big(\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;\varphi}h(s,\mathrm{RL}\mathcal{I}^{\sigma}v(s))-Y_v(s)\big)\right|\mathrm{d}s \\
&\quad +\frac{\Gamma\big(\sigma+\widetilde{\delta}\big)}{|\Delta|\Gamma\big(\widetilde{\delta}\big)}\left|\big(\varphi_0(t_2)\big)^{\widetilde{\delta}-1}-\big(\varphi_0(t_1)\big)^{\widetilde{\delta}-1}\right|\times 2^{q-1}\left[\sum_{j=1}^{m}\frac{\big(\varphi_0(k_j)\big)^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)}\right]\widetilde{\Omega}_1.
\end{aligned}
$$

On the other hand, one can find a constant C satisfying

$$
\left|\big(\varphi_0(t_2)\big)^{\widetilde{\delta}-1}-\big(\varphi_0(t_1)\big)^{\widetilde{\delta}-1}\right|\le C\big(\varphi_0(t_2)\big)^{\widetilde{\delta}-1}\longrightarrow 0,\quad t_2\longrightarrow t_1,
$$

then

$$
\begin{aligned}
&\left|\mathcal{B}v(t_2)-\mathcal{B}v(t_1)\right| \\
&\le 2^{q-1}\Bigg[\frac{\big(\varphi_s(t_2)\big)^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)}\left|\big(\varphi_s(t_2)\big)^{\eta_2+\eta_3}-\big(\varphi_s(t_1)\big)^{\eta_2+\eta_3}\right| \\
&\quad +\frac{\left|\varphi_{t_1}(t_2)\right|^{\eta_2+\eta_3}}{\Gamma(\eta_2+\eta_3+1)}+\frac{\Gamma\big(\sigma+\widetilde{\delta}\big)}{|\Delta|\Gamma\big(\widetilde{\delta}\big)}C\big(\varphi_{t_1}(t_2)\big)^{\widetilde{\delta}-1}\left[\sum_{j=1}^{m}\frac{\big(\varphi_0(k_j)\big)^{\sigma+\eta_2+\eta_3}}{\Gamma(\sigma+\eta_2+\eta_3+1)}\right]\Bigg]\widetilde{\Omega}_1.
\end{aligned}
$$

Clearly, $\left|\mathcal{B}v(t_2)-\mathcal{B}v(t_1)\right|\longrightarrow 0, t_2\longrightarrow t_1$. The Arzelà–Ascoli theorem confirms that $\mathcal{B}(\mathcal{A}_\theta)$ is relatively compact. It is evident that the set $\Sigma(\mathcal{B})=(v\in Z, v=\mu\mathcal{B}(v);\mu\in[0,1])$ is bounded.

By Schauder's theorem, $\mathcal{B}$ has at least one FP.

4 | Applications

Example 1. We consider the $H\mathbb{FDE}$:

$$
\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_1,12;t}\psi_3\Big(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;14;t}\Big(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;59;t}v(t)\Big)\Big)=\frac{\mathrm{RL}\mathcal{I}^1(v(t))}{t^2+\sqrt{3}},\quad (20)
$$

for $t\in \mathrm{J}_a$, $a=1$, with

$$
\begin{cases}
\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;59;t}v(0)=0,\ \ \mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;59;t}v(1)=0,\\
\mathrm{RL}\mathcal{I}_{0^+}^{1;t}(1)=\mathrm{RL}\mathcal{I}_{0^+}^{1;t}v\left(\dfrac{1}{\sqrt{3}}\right)+\mathrm{RL}\mathcal{I}_{0^+}^{1;t}v\left(\dfrac{1}{\pi}\right),\\
v(0)=0,\\
\quad\begin{cases}
\psi_3\Big(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;14;t}\Big(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;59;t}v(0)\Big)\Big)=0,\\
\psi_3\Big(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_2;14;t}\Big(\mathrm{Hr}\mathcal{D}_{0^+}^{\eta_3;59;t}v(1)\Big)\Big)=0.
\end{cases}
\end{cases}
$$

Here, $p=3$, $\mu_1=1/2$, $\mu_2=1/4$, $\mu_3=5/9$, $\sigma=1$, and $k_1=1/\sqrt{3}$, $k_2=1/\pi$. The continuity of $h(t,u)=\big(1/t^2+\sqrt{3}\big)u$, and $\varphi(t)=t$ in three cases is as follow for $\eta_i\in(1,2)$:

$$
\begin{aligned}
&\text{Case 1.}\ \ \eta_1\in\left\{\frac{3}{2},\frac{5}{3},\frac{9}{5}\right\},\eta_2=\frac{4}{3},\eta_3=\frac{13}{12},\\
&\text{Case 2.}\ \ \eta_2\in\left\{\frac{4}{3},\frac{10}{7},\frac{12}{7}\right\},\eta_3=\frac{13}{12},\eta_1=\frac{3}{2},\\
&\text{Case 3.}\ \ \eta_3\in\left\{\frac{13}{12},\frac{14}{11},\frac{13}{9}\right\},\eta_1=\frac{3}{2},\eta_2=\frac{4}{3}.
\end{aligned}
$$

Further, for $u,v\in\mathbb{R}$,

$$
|h(t,u)-h(t,v)|=\left|\frac{1}{t^2+\sqrt{3}}u-\frac{1}{t^2+\sqrt{3}}v\right|=\frac{1}{t^2+\sqrt{3}}|u-v|\le\mathcal{M}|u-v|,
$$

where $\mathcal{M}=1/\sqrt{3}$.

In all three cases, the computations confirm that the conditions ensuring the existence of a solution are satisfied. Consequently, Assumptions (A1) and (A2) are fulfilled. For different predicted states, we review the challenges in order to confirm the existence of a solution.

Case 1. From the given data in $\mathbb{BVP}$ (Equation 20), thanks to Equation (14), we find that $\delta = 1.5 \longrightarrow 2$, and

$$\begin{aligned}\left|Y_v(t)-Y_u(t)\right| &= \left|\frac{(\varphi_0(t))^{\delta-1}}{(\varphi_0(1))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;t}h(1,\mathrm{RL}\mathcal{I}^{\sigma}v(1))\right.\\ &\quad\left.-\frac{(\varphi_0(t))^{\delta-1}}{(\varphi_0(1))^{\delta-1}}\mathrm{RL}\mathcal{I}_{0^+}^{\eta_1;t}h(1,\mathrm{RL}\mathcal{I}^{\sigma}u(1))\right|\\ &\le \left|\frac{(\varphi_0(t))^{\delta-1}}{(\varphi_0(1))^{\delta-1}}\right| \le \Omega_1\|v-u\|,\end{aligned}$$

where $\Omega_1 \simeq 0.577 \longrightarrow 0.289$. Also, $\widetilde{\delta} = \eta_3 + \mu_3(2-\eta_3) \simeq 1.5925$, and from Equation (8), we get $\Delta \simeq 1.5784 \neq 0$. We plot the changes of Ω_1 in Figure 1. These estimated results are shown in Table 1.

Additionally, $\varsigma = \eta_2 + \mu_2(2-\eta_2) = 1.5$. By considering $\mathcal{C} = 0.15 > 0$ and $q = 1.25$, from Equation (18), we obtain $\Omega_2 \simeq 0.830 \longrightarrow 0.415$ and $(q-1)\Omega_2 \simeq 0.207 \longrightarrow 0.104$. We plot the changes of Ω_2 and $(q-1)\Omega_2 < 1/2$ in Figure 2. These estimated results are shown in Table 1.

On the other hand,

$$|h(t,u)| = \left|\frac{1}{t^2+\sqrt{3}}u\right| < \widetilde{\mathcal{M}}, \quad \widetilde{\mathcal{M}} = \frac{1}{\sqrt{3}},$$

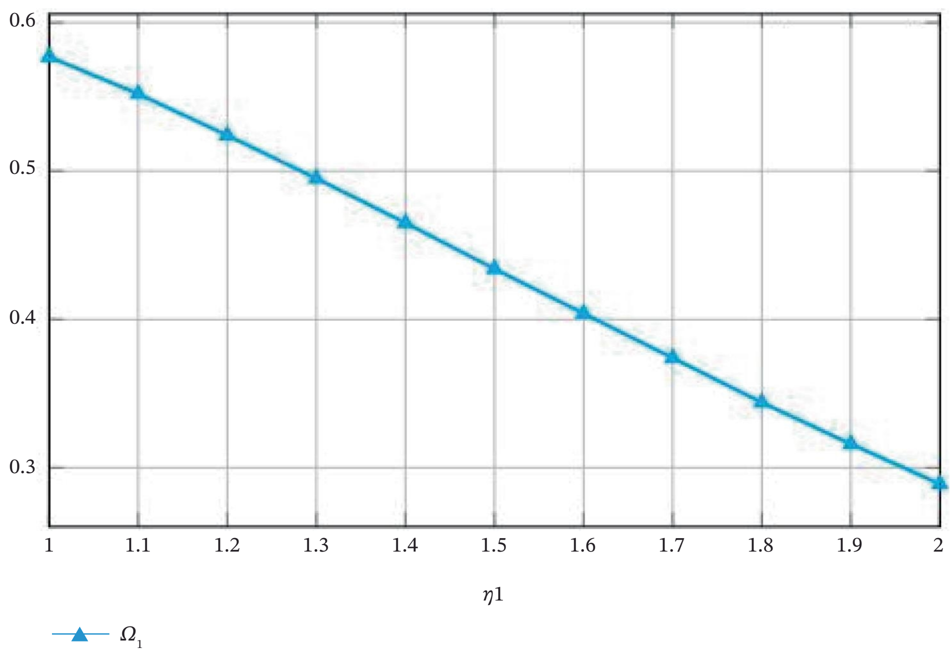

FIGURE 1 | 2D plots of Ω_1 for $\mathbb{BVP}$ (Equation 20) with $\eta_1 : 1 \longrightarrow 2$ in Case 1.

TABLE 1 | Estimated results of $\mathbb{BVP}$ (Equation 20) for $\eta_1 : 1 \longrightarrow 2$ in Case 1.

$\boldsymbol{\eta_1}$	$1 \xrightarrow{\eta_1} 2$							
	$\boldsymbol{\delta}$	$\boldsymbol{\Omega_1}$	$\boldsymbol{\Delta \neq 0}$	$\boldsymbol{\Omega_2}$	$\boldsymbol{(q-1)\Omega_2 < \frac{1}{2}}$	$\boldsymbol{\widetilde{\Omega}_1}$	$\boldsymbol{\Omega_3}$	$\boldsymbol{\theta \geq \cdots}$
1.00	1.500	0.577	1.578	0.830	0.207	0.872	0.302	0.359
1.10	1.550	0.552	1.578	0.793	0.198	0.862	0.298	0.355
1.20	1.600	0.524	1.578	0.753	0.188	0.851	0.295	0.350
1.30	1.650	0.495	1.578	0.711	0.178	0.839	0.290	0.345
1.40	1.700	0.465	1.578	0.668	0.167	0.826	0.286	0.340
1.50	1.750	0.434	1.578	0.624	0.156	0.812	0.281	0.334
1.60	1.800	0.404	1.578	0.580	0.145	0.797	0.276	0.328
1.70	1.850	0.374	1.578	0.537	0.134	0.782	0.271	0.322
1.80	1.900	0.344	1.578	0.495	0.124	0.766	0.265	0.316
1.90	1.950	0.316	1.578	0.454	0.113	0.750	0.260	0.309
2.00	2.000	0.289	1.578	0.415	0.104	0.733	0.254	0.302

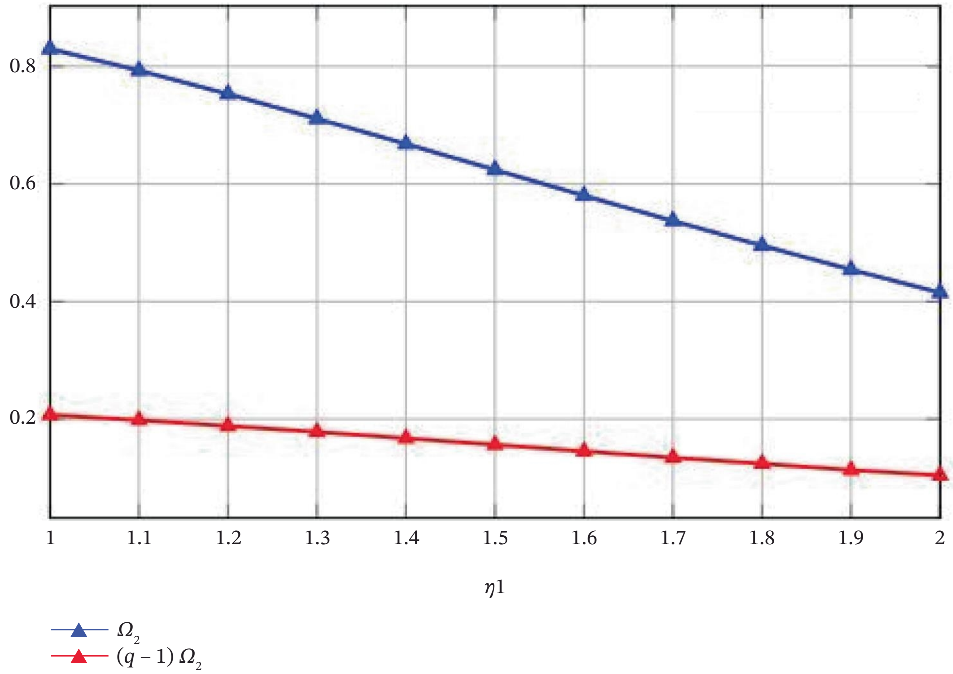

FIGURE 2 | 2D plots of Ω_2 and $(q-1)\Omega_2$ for $\mathbb{BVP}$ (Equation 20) with $\eta_1 : 1 \longrightarrow 2$ in Case 1.

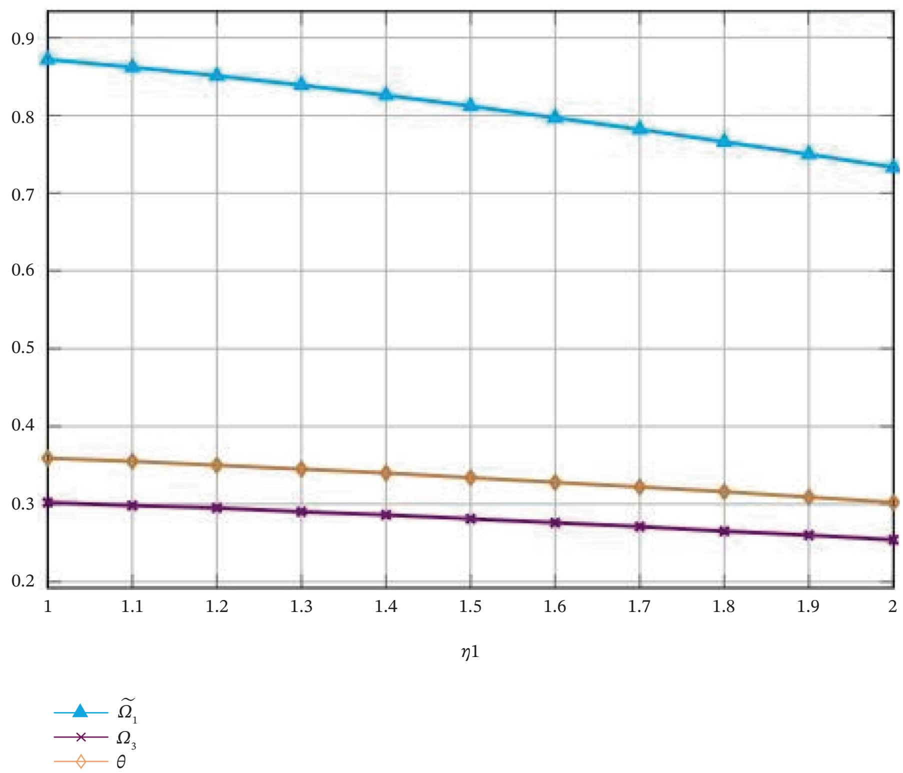

FIGURE 3 | 2D plots of $\widetilde{\Omega}_1$, Ω_3 and suitable value of θ for $\mathbb{BVP}$ (Equation 20) with $\eta_1 : 1 \longrightarrow 2$ in Case 1.

and so from Equations (15) and (19), we find $\widetilde{\Omega}_1 \simeq 0.872 \longrightarrow 0.733$ and $\Omega_3 \simeq 0.302 \longrightarrow 0.254$, which implies that $\theta \geq 2^{q-1}\Omega_3 \simeq 0.359$. These numerical results are shown in Table Table 1. We plot the changes of $\widetilde{\Omega}_1$, Ω_3 and suitable values of θ in Figure 3.

Therefore, thanks to Theorem 3.4, $\mathbb{BVP}$ (Equation 20) has at least one solution in Z for Case 1.

Case 2. This case verifies the validity results for $\mathbb{BVP}$ (Equation 20) with the various values of η_2. With the given data from this case, thanks to Equation (14), we get $\delta = \eta_1 + \mu_1(2-\eta_1) = 1.75$, $\Omega_1 \simeq 0.4343$, $\widetilde{\delta} = \eta_3 + \mu_3(2-\eta_3) \simeq 1.5925$, and from Equation (8), we get $\Delta \simeq 1.5784 \neq 0$. These estimated results are shown in Table 2. Additionally, $\varsigma = \eta_2 + \mu_2(2-\eta_2) = 1.25 \longrightarrow 2.0$. By considering $C = 0.15 > 0$ and $q = 1.25$, from Equation (18), we obtain

TABLE 2 | Estimated results of $\mathbb{BVP}$ (Equation 20) for $\eta_2 : 1 \longrightarrow 2$ in Case 2.

η_2	$1 \xrightarrow{\eta_2} 2$							
	ς	Ω_1	$\Delta \neq 0$	Ω_2	$(q-1)\Omega_2 < \frac{1}{2}$	$\widetilde{\Omega}_1$	Ω_3	$\theta \geq \cdots$
1.00	1.250	0.434	1.578	0.891	0.223	0.812	0.402	0.478
1.10	1.325	0.434	1.578	0.804	0.201	0.812	0.362	0.431
1.20	1.400	0.434	1.578	0.723	0.181	0.812	0.326	0.387
1.30	1.475	0.434	1.578	0.648	0.162	0.812	0.292	0.347
1.40	1.550	0.434	1.578	0.579	0.145	0.812	0.261	0.310
1.50	1.625	0.434	1.578	0.516	0.129	0.812	0.232	0.276
1.60	1.700	0.434	1.578	0.458	0.115	0.812	0.206	0.246
1.70	1.775	0.434	1.578	0.406	0.102	0.812	0.183	0.218
1.80	1.850	0.434	1.578	0.359	0.090	0.812	0.162	0.192
1.90	1.925	0.434	1.578	0.316	0.079	0.812	0.142	0.169
2.00	2.000	0.434	1.578	0.278	0.069	0.812	0.125	0.149

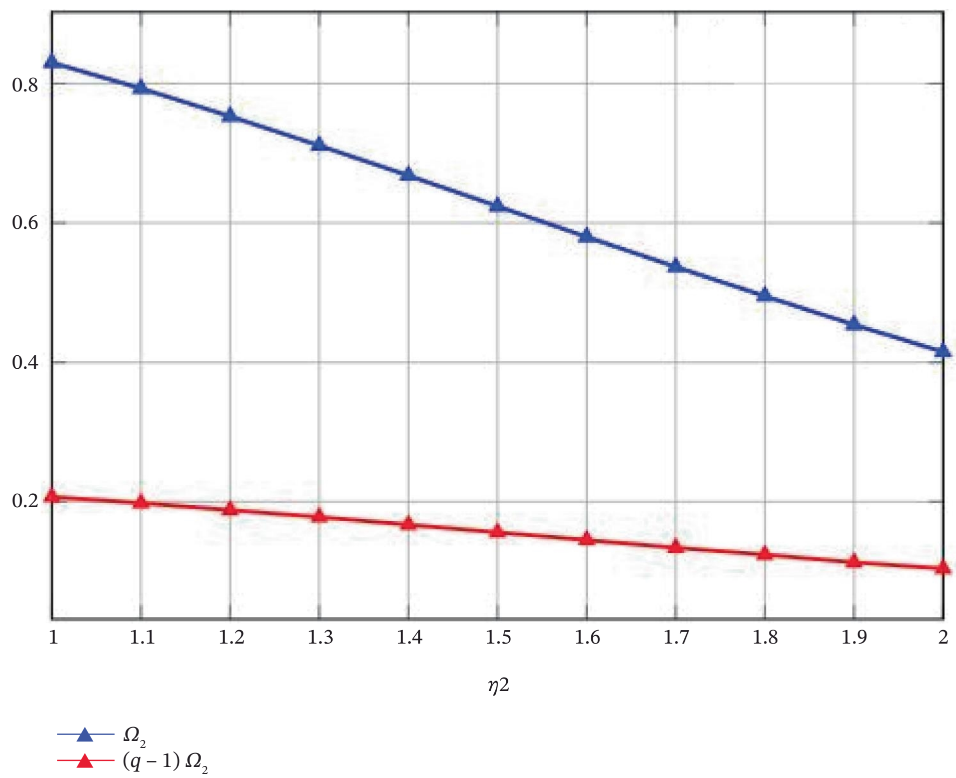

FIGURE 4 | 2D plots of Ω_2 and $(q-1)\Omega_2$ for $\mathbb{BVP}$ (Equation 20) with $\eta_2 : 1 \longrightarrow 2$ in Case 2.

$\Omega_2 \simeq 0.8912 \longrightarrow 0.2779$ and $(q-1)\Omega_2 \simeq 0.2228 \longrightarrow 0.0694 < 1/2$. We plot the changes of Ω_2 and $(q-1)\Omega_2 < 1/2$ in Figure 4. These estimated results are shown in Table Table 2.

On the other hand, $|h(t,u)| < \widetilde{\mathcal{M}} = 1/\sqrt{3}$, and so from Equations (15) and (19), we find $\widetilde{\Omega}_1 \simeq 0.8118$ and $\Omega_3 \simeq 0.4015 \longrightarrow 0.1252$, which implies that $\theta \geq 2^{q-1}\Omega_3 \simeq 0.4775 \longrightarrow 0.1488$. These numerical results are shown in Table 2. We plot the changes of $\widetilde{\Omega}_1$, Ω_3 and suitable values of θ in Figure 5.

Therefore, thanks to Theorem 3.4, $\mathbb{BVP}$ (Equation 20) has at least one solution in Z for Case 2.

Case 3. In this case, we consider $\mathbb{BVP}$ (Equation 20) with the different values for the derivative order η_3. With the given data from this case, thanks to Equation (14), we get $\delta = \eta_1 + \mu_1(2-\eta_1) = 1.75$, $\Omega_1 \simeq 0.4343$, and $\widetilde{\delta} = \eta_3 + \mu_3(2-\eta_3) \simeq 1.5555 \longrightarrow 2.0$, and from Equation (8), we get $\Delta \simeq 1.5940 \longrightarrow 1.4346 \neq 0$. These estimated results are

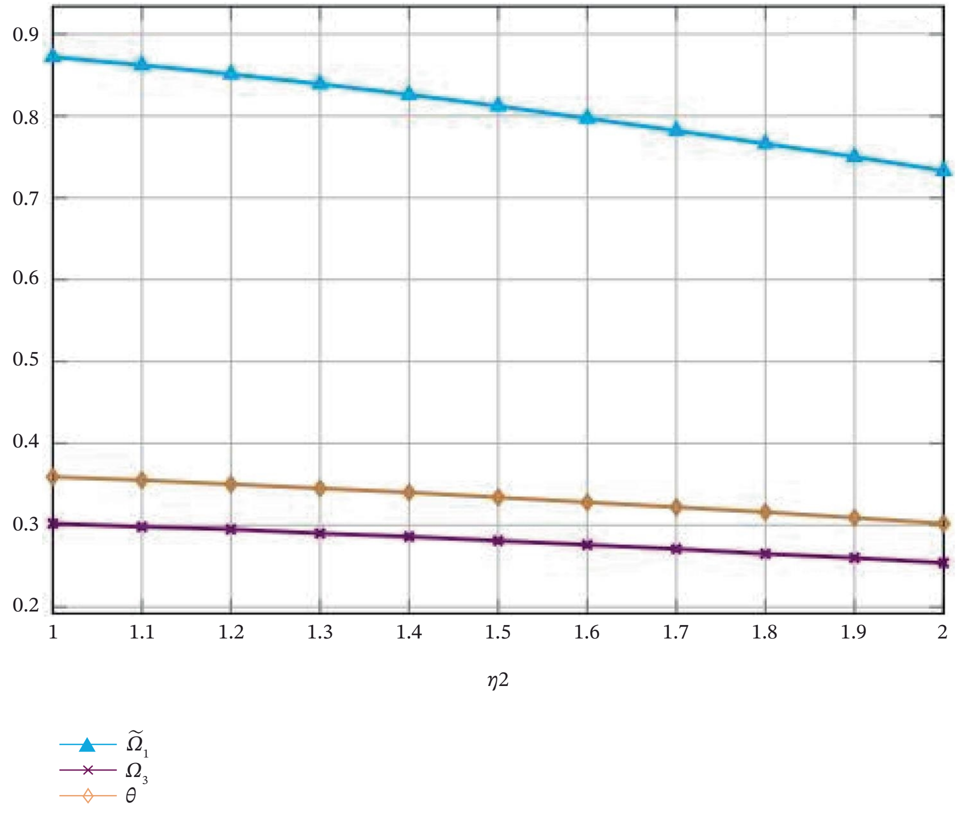

FIGURE 5 | 2D plots of $\widetilde{\Omega}_1$, Ω_3 and suitable value of θ for $\mathbb{BVP}$ (Equation 20) with η_2: $1 \longrightarrow 2$ in Case 2.

TABLE 3 | Estimated results of $\mathbb{BVP}$ (Equation 20) for η_3: $1 \longrightarrow 2$ in Case 3.

η_3	$1 \xrightarrow{\eta_3} 2$							
	$\widetilde{\delta}$	Ω_1	$\Delta \neq 0$	Ω_2	$(q-1)\Omega_2 < \frac{1}{2}$	$\widetilde{\Omega}_1$	Ω_3	$\theta \geq \cdots$
1.00	1.556	0.434	1.594	0.683	0.171	0.812	0.308	0.366
1.10	1.600	0.434	1.575	0.613	0.153	0.812	0.276	0.328
1.20	1.644	0.434	1.557	0.548	0.137	0.812	0.247	0.293
1.30	1.689	0.434	1.540	0.488	0.122	0.812	0.220	0.261
1.40	1.733	0.434	1.523	0.434	0.108	0.812	0.195	0.232
1.50	1.778	0.434	1.507	0.384	0.096	0.812	0.173	0.206
1.60	1.822	0.434	1.492	0.339	0.085	0.812	0.153	0.182
1.70	1.867	0.434	1.477	0.299	0.075	0.812	0.135	0.160
1.80	1.911	0.434	1.462	0.262	0.066	0.812	0.118	0.141
1.90	1.956	0.434	1.448	0.230	0.057	0.812	0.104	0.123
2.00	2.000	0.434	1.435	0.201	0.050	0.812	0.090	0.108

shown in Table 3. Further, $\varsigma = \eta_2 + \mu_2(2-\eta_2) = 1.5$. By considering $C = 0.15 > 0$ and $q = 1.25$, from Equation (18), we obtain $\Omega_2 \simeq 0.6832 \longrightarrow 0.2008$ and $(q-1)\Omega_2 \simeq 0.1708 \longrightarrow 0.0502 < 1/2$. We plot the changes of Ω_2 and $(q-1)\Omega_2 < 1/2$ in Figure 6. These estimated results are shown in Table Table 3.

On the other hand,

$$|h(t,u)| < \widetilde{\mathcal{M}} = \frac{1}{\sqrt{3}},$$

and so from Equations (15) and (19), we find $\widetilde{\Omega}_1 \simeq 0.8118$ and $\Omega_3 \simeq 0.3078 \longrightarrow 0.0904$, which implies that $\theta \geq 2^{q-1}\Omega_3 \simeq 0.3660 \longrightarrow 0.1075$. These numerical results are

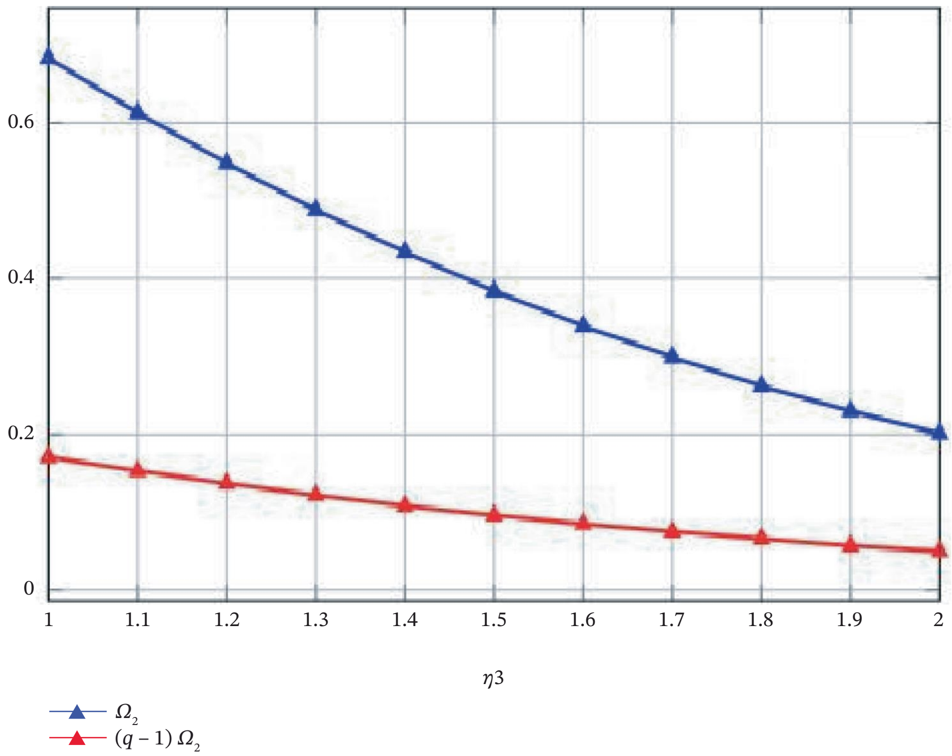

FIGURE 6 | 2D plots of Ω_2 and $(q-1)\Omega_2$ for $\mathbb{BVP}$ (Equation 20) with $\eta_3: 1 \longrightarrow 2$ in Case 3.

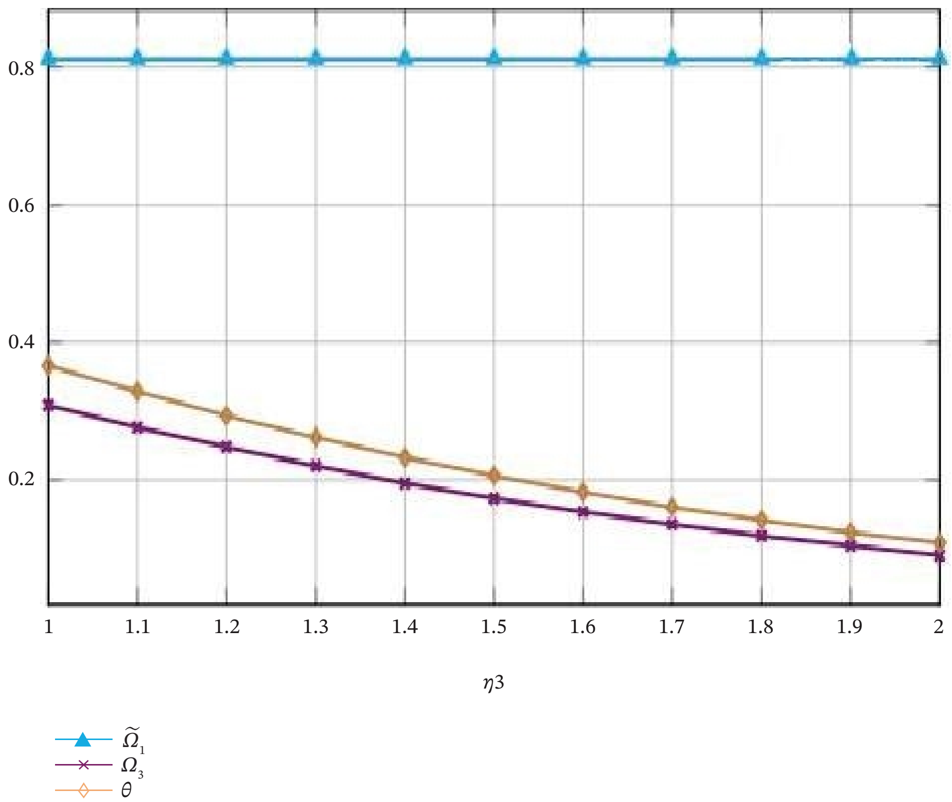

FIGURE 7 | 2D plots of $\widetilde{\Omega}_1$, Ω_3 and suitable value of θ for $\mathbb{BVP}$ (Equation 20) with $\eta_3: 1 \longrightarrow 2$ in Case 3.

shown in Table 3. We plot the changes of $\widetilde{\Omega}_1$, Ω_3 and suitable values of θ in Figure 7.

Therefore, thanks to Theorem 3.4, $\mathbb{BVP}$ (Equation 20) has at least one solution in Z for Case 3.

5 | Conclusion

In this study, we investigated a class of sequential $\mathbb{HFDE}$s. By combining the analytical properties of the Hilfer derivative with the FP method, we established rigorous results concerning the existence and uniqueness of solutions. The application of Banach's FP principle allowed us to prove the uniqueness of solutions under contraction conditions, whereas the use of Schauder's FPT ensured the existence of at least one solution in more general settings, assuming continuity and compactness. As a future direction, it would be interesting to develop numerical methods specifically adapted to the Hilfer derivative and to compare the obtained results with those derived from other types of fractional derivatives. Such investigations could contribute to a deeper theoretical and practical understanding of sequential fractional models in applied sciences.

Nomenclature

$\mathbb{DE}$	differential equation
$\mathbb{FE}$	fractional equation
$\mathbb{FDE}$	fractional differential equation
$\mathbb{HFDE}$	Hilfer fractional differential equation
RL	Riemann–Liouville
$\mathbb{FBVP}$	fractional boundary value problem
FPT	fixed-point theorem

Author Contributions

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Disclosure

All authors read and approved the final presented manuscript.

Ethics Statement

The authors have nothing to report.

Conflicts of Interest

The authors declare no conflicts of interests.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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