RAIRO-Oper. Res. 60 (2026) 2923–2932
https://doi.org/10.1051/ro/2026080

RAIRO Operations Research
www.rairo-ro.org

CONVEXITY IN TERMS OF CAPUTO FRACTIONAL DERIVATIVES

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Abstract. The first criterion of convexity states that a differentiable function defined on an open interval is convex if and only if its first derivative is increasing. The second criterion of convexity states that a twice differential function is convex if and only if its second derivative is positive. In this work, by using a change of variable transforming the interval $[a,t]$ into $[0,1]$, we are able to get a new formulation of Caputo derivative. Then, we use this new version of Caputo derivative to investigate convexity and obtain generalizations to the first and second criteria of convexity depending on fractional orders $0<\alpha<1$ and $1<\alpha<2$, respectively. As a further application of the new representation of the Caputo derivative $(D_a^\alpha f)(x)$ of a function f defined on an interval $[a,b]$ and of order $n-1<\alpha<n$ for some $n\in\{1,2,\ldots\}$, we obtain the estimation, for all $x\in[a;b]$:

$$f^{(n-1)}(x)\leq f^{(n-1)}(a)+(x-a)^{\alpha-(n-1)}\Gamma(n-\alpha)D_a^\alpha f(x),$$

provided that $f^{(n)}(x)\geq 0$ for all $x\in[a,b]$. From which some geometric interpretations can follow. For the sake of better clarification to our results, we give several illustrative examples.

Mathematics Subject Classification. 26D15, 26D20, 26A51.

Received August 31, 2024. Accepted July 7, 2026.

1. Introduction

The fractional analysis is a branch of mathematical analysis that studies the possibility of defining non-integer powers of the derivation and integration operators. It was invented, independently by Isaac Newton and Gottfried Leibniz, and it was understood that the notion of the derivative of nth order, that is, applying the differentiation operation n times in succession, was meaningful. It was introduced after 1695 as a simply academic generalization of integer derivative [6,7,13,15,17].

A fractional derivative generalizes the order of differentiation from set of natural numbers $\mathbb{N}$ to Set of real numbers $\mathbb{R}$ [5,6]. One of the axiomatical questions that appears in Mathematical is how to generalize certain geometrical concepts from ordinary calculus to fractional calculus. Convexity plays an important role in mathematical analysis and optimization. Our aim in this work is characterize convexity by means of the so-called

Keywords. Fractional calculus, fractional derivative, fractional integral, Caputo fractional derivative, fractional convex analysis.

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Caputo fractional derivatives. Up to our knowledge very few works have discussed about convexity in fractional calculus. Most of the works care about generalizing certain inequalities of convex functions to the fractional case (see [10, 18]). For the analysis of convexity geometrically in fractional calculus we mention [3], where the author considered it in a minor way. A discussion and a comparison to this reference is to be given in the conclusion section. This article is organised as follows. In Section 2 some preliminary results about fractional calculus and convexity will be given. In Section 3 we have our main results, in which we start it by an interesting tool representation to Caputo derivatives with some clarification examples. We then divide Section 3 into three subsections. The first subsection is devoted to the characterization of convexity by means of Caputo fractional derivatives of order $1 < \alpha < 2$ (generalization of the second convexity criterion to the fractional case). In the second subsection, we discuss the generalization of the first convexity principle on open intervals to the fractional case. In the third section, as an application of the new proven representation of Caputo fractional derivatives, we shall prove a useful inequality, by which we prove some useful geometrical results. Finally, in Section 4 we have our concluding remarks where we propose an open problem and make a comparison to reference [3]. In fact, the sufficient condition for convexity depends on the starting point $r \in [a, b]$ which is different from what was proven in [3]. For a recent work on convexity in the fractional calculus we refer to [4] as well.

2. Preliminaries

In this section, we present some useful definitions and lemmas [6–9, 16].

Definition 1. A function $f : I \to \mathbb{R}$ is said to be convex on an interval I , if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Lemma 1 (First criterion of convexity [16])**.** *Let I be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. Then $f(t)$ is convex if and only if the first derivative $f^{(1)}(t)$ is increasing on I.*

Lemma 2 (Second criterion of convexity [16])**.** *Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. Suppose $f(t)$ is twice differentiable on I . Then $f(t)$ is convex if and only if the second derivative $f^{(2)}(t) \geq 0$ for all $t \in I$.*

Definition 2. A function $f : I \to \mathbb{R}$ is said to be midpoint convexity on I where I is an real interval , if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{2}$$

holds for all $x, y \in I$.

Lemma 3 (Third criterion: midpoint convexity and continuous [16])**.** *Let I be an open interval and let $f : I \to \mathbb{R}$ be continuous. Then $f(t)$ is convex if and only if for all $x, y \in I$*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}. \tag{3}$$

Definition 3. Euler's Gamma function is defined by the following integral:

$$\Gamma(x) = \int_0^{+\infty} \exp(-t)t^{x-1} \, \mathrm{d}t \quad x \in \mathbb{R}_+. \tag{4}$$

The important property of the Gamma function $\Gamma(x)$ is the following recurrence relation:

$$\Gamma(x+1) = x\Gamma(x) \quad x \in \mathbb{R}_+. \tag{5}$$

As a consequence of this property, we have if $x \in \mathbb{N}$ then:

$$\Gamma(x+1) = x! \tag{6}$$

This allows us to say that the Gamma function generalizes the notion of factorial. We have

$$\Gamma(1) = 1,\, \Gamma(0_+) = +\infty,\, \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

Definition 4 (The Riemann–Liouville fractional integral)**.** Let α be a positive real and $f : [a;b] \to \mathbb{R}$ a continuous function. The Riemann–Liouville fractional integral of order α of f is defined by integral:

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{(\alpha-1)} f(t)\,\mathrm{d}t. \tag{7}$$

For any continuous function f, we have:

$$I_a^\alpha (I_a^\beta f(x)) = I_a^{\alpha+\beta} f(x) \quad \alpha, \beta > 0 \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(I_a^\alpha f(x)) = I_a^{\alpha-1} f(x). \tag{9}$$

There are several definitions of fractional derivatives, unfortunately they are not all equivalent. In this work, we shall use on Capotu type derivatives

Definition 5 (The Caputo fractional derivative [7])**.** Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$ and f a function of class $\mathrm{AC}^n([a;b])$. The derivative of order α in Caputo's sense of order α of the function f on the interval $[a;b]$ is defined by

$$D_a^\alpha f(x) = I_a^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)\right) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{(n-\alpha-1)} f^{(n)}(t)\,\mathrm{d}t, \tag{10}$$

where $\mathrm{AC}^n([a,b]) = \{f : [a;b] \to \mathbb{R} : f^{(n-1)}(x)$ is absolutely continuous$\}$.

A function $f(x)$ is absolutely continuous ($f(x) \in \mathrm{AC}[a,b]$) if and only if $f(x) = c + \int_a^x l(t)\mathrm{d}t,\ (l(x) \in L(a,b))$. For the properties of Caputo fractional derivatives, we refer to [6,7]. Notice that since an absolutely continuous function has an integrable derivative, then $C^n([a;b]) \subset \mathrm{AC}^n([a;b])$. The book [6] deals with Caputo derivatives in $C^n([a;b]) = \{f : [a;b] \to \mathbb{R} : f^{(n)}(x)$ is continuous$\}$.

3. Main results

We start by proposing the following tool lemma, which gives a new equivalent definition of Caputo derivatives by using a change of variable leading to the interval $[0,1]$. Hence, this equivalent form will be useful in convex analysis.

Lemma 4. *Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$ and f be a function of class $C^n([a;b])$. Then, the derivative of order α in Caputo's sense of the function f can be represented by*

$$D_a^\alpha f(x) = \frac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} f^{(n)}(a+(x-a)\theta)\,\mathrm{d}\theta; \quad \text{for all } x \in [a;b].$$

Proof. Let $t = a + (x-a)\theta; 0 \leq \theta < 1$ in Definition 5. Then,

$$\begin{aligned} D_a^\alpha f(x) &= \tfrac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{(n-\alpha-1)} \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\,\mathrm{d}t \\ &= \tfrac{(x-a)^{(n-\alpha-1)}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} f^{(n)}(a+(x-a)\theta)(x-a)\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} f^{(n)}(a+(x-a)\theta)\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} f^{(n)}(a+(x-a)\theta)\,\mathrm{d}\theta. \end{aligned}$$

This completes the proof. □

Below, we discuss some illustrative examples for Lemma 4.

Example 6. By substituting $f(x) = x^2$ in the Lemma 4, we obtain for all $x, a \in \mathbb{R}$ with $x > a$

$$D_a^\alpha f(x) = \begin{cases} 2\tfrac{(x-a)^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-\theta)^{-\alpha}(a+(x-a)\theta)\,\mathrm{d}\theta & \text{if } 0 < \alpha < 1 \\ 2\tfrac{(x-a)^{2-\alpha}}{\Gamma(2-\alpha)} \int_0^1 (1-\theta)^{1-\alpha}\,\mathrm{d}\theta & \text{if } 1 < \alpha < 2 \\ 0 & \text{if } n-1 < \alpha < n \text{ and } n \geq 3. \end{cases}$$

Then, by the help of Beta function $\beta(x,y) = \int_0^1 \theta^{x-1}(1-\theta)^{y-1}$, and the identity:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we reach at

$$D_a^\alpha f(x) = \begin{cases} \tfrac{2(x-a)^{2-\alpha}}{\Gamma(3-\alpha)} + \tfrac{2a(2-\alpha)(x-a)^{1-\alpha}}{\Gamma(3-\alpha)} & \text{if } 0 < \alpha < 1 \\ \tfrac{2(x-a)^{2-\alpha}}{\Gamma(3-\alpha)} & \text{if } 1 < \alpha < 2 \\ 0 & \text{if } n-1 < \alpha < n \text{ and } n \geq 3. \end{cases}$$

Remark 1. The change of variable above was before used in [14] to calculate the Riemann–Liouville fractional derivatives of power functions of the form $(x-a)^r$. In fact, the following formula was proved:

For $(0 \leq m \leq \alpha < m+1, r > m)$ or $(\alpha < 0, r > -1)$ we have

$$^{R}D_a^\alpha (x-a)^r = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}(x-a)^{r-\alpha}. \tag{11}$$

Example 7. By substituting $f(x) = e^x$ in the Lemma 4, we obtain for all $x, a \in \mathbb{R}$ with as $x > a$

$$\begin{aligned} D_a^\alpha f(x) &= \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} f^{(n)}(a+(x-a)\theta)\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} e^{a+(x-a)\theta}\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{n-\alpha}e^x}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} e^{(x-a)(\theta-1)}\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{n-\alpha}e^x}{\Gamma(n-\alpha)} \int_0^1 \theta^{n-\alpha-1} e^{-(x-a)\theta}\,\mathrm{d}\theta. \end{aligned}$$

Hence, for $0 < \alpha < 1$, we have

$$\begin{aligned} D_a^\alpha f(x) &= \tfrac{e^a(x-a)^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-\theta)^{-\alpha} e^{(x-a)\theta}\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{1-\alpha}e^x}{\Gamma(1-\alpha)} \int_0^1 \theta^{-\alpha} e^{-(x-a)\theta}\,\mathrm{d}\theta, \end{aligned} \tag{12}$$

and for $n = 2$ or $1 < \alpha < 2$ we have

$$\begin{aligned} D_a^\alpha f(x) &= \tfrac{e^a (x-a)^{2-\alpha}}{\Gamma(2-\alpha)} \int_0^1 (1-\theta)^{1-\alpha} e^{(x-a)\theta}\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{2-\alpha} e^x}{\Gamma(2-\alpha)} \int_0^1 \theta^{1-\alpha} e^{-(x-a)\theta}\,\mathrm{d}\theta. \end{aligned} \tag{13}$$

3.1. Convex characterization by Caputo fractional derivatives of order $1 < \alpha < 2$

In this section we prove Caputo-type fractional second criterion for convexity without further restrictions.

Theorem 8. *Assume that f is a function of class $C^2([a;b])$. Then,*

(1) *If f is convex, then the Caputo derivative $(D_r^\alpha f)(x)$ of any order $1 < \alpha < 2$ of the function f, starting at any $r \in [a,b)$, is positive on $[r,b]$ and for all $x, y \in [a;b]$ with $x > y$ we have*

$$D_a^\alpha f(x) - D_a^\alpha f(y) \le D_y^\alpha f(x).$$

(2) *If for some $\alpha \in (1,2)$ and all $r \in [a,b)$ the Caputo derivative $(D_r^\alpha f)(x)$ is positive on $[r;d]$, then f is convex.*

Proof. **Proof of (1)**: The Caputo derivative of order α starting at r satisfies

$$D_r^\alpha f(x) = I_r^{2-\alpha}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)\right) = \frac{1}{\Gamma(2-\alpha)} \int_r^x (x-t)^{(1-\alpha)} f^{(2)}(t)\,\mathrm{d}t \ge 0, \tag{14}$$

since f is a convex function of class $C^2([a;b])$ if only if $f^{(2)}(t) \ge 0$ for all $t \in [a;b]$.

Further, let $x, y \in [a;b]$ and $x > y$, then we have

$$\begin{aligned} D_a^\alpha f(x) - D_a^\alpha f(y) &= \tfrac{1}{\Gamma(2-\alpha)} \left[\int_a^x (x-t)^{(1-\alpha)} f^{(2)}(t)\,\mathrm{d}t - \int_a^y (y-t)^{(1-\alpha)} f^{(2)}(t)\,\mathrm{d}t\right] \\ &\le \tfrac{1}{\Gamma(2-\alpha)} \left[\int_a^x (x-t)^{(1-\alpha)} f^{(2)}(t)\,\mathrm{d}t - \int_a^y (x-t)^{(1-\alpha)} f^{(2)}(t)\,\mathrm{d}t\right] \\ &= \tfrac{1}{\Gamma(2-\alpha)} \int_y^x (x-t)^{(1-\alpha)} f^{(2)}(t)\,\mathrm{d}t \\ &= D_y^\alpha f(x). \end{aligned}$$

Proof of (2): We prove this part by contradiction. Suppose for any $r \in [a,b)$ the Caputo derivative $(D_r^\alpha f)(x)$ is positive on $[r;b]$ and f is not a convex function. Then, from second criteria of convexity,

$$\exists t_0 \in [a;b] \text{ such that } f^{(2)}(t_0) < 0.$$

Since f is a function of class $C^2([a;b])$ then $f^{(2)}(t)$ is continuous function, and hence there exists a subinterval $[r;d] \subset [a;b]$ and $r < d$ such that

$$t_0 \in [r,d] \;\text{ and } f^{(2)}(t) < 0 \text{ for all } t \in [r;d].$$

Then, $M = \max_{t\in[r;d]} f^{(2)}(t) < 0$, and hence for every $x \in [r,d] \subset [r,b]$, we have

$$\begin{aligned} D_r^\alpha f(x) &= \frac{1}{\Gamma(2-\alpha)} \int_r^x (x-t)^{(1-\alpha)} f^{(2)}(t)\ \mathrm{d}t \\ &< \frac{M}{\Gamma(2-\alpha)} \int_r^x (x-t)^{(1-\alpha)}\ \mathrm{d}t \\ &= \frac{M}{\Gamma(2-\alpha)} \left[-\frac{(x-t)^{(2-\alpha)}}{(2-\alpha)}\right]_r^x \\ &= \frac{M}{\Gamma(2-\alpha)} \left[\frac{(x-r)^{(2-\alpha)}}{(2-\alpha)}\right] \\ &< 0. \end{aligned} \tag{15}$$

Which is a contradiction. This completes the proof. $\square$

In the next examples, we discuss the applications of Theorem 8 for some specific functions.

Example 9. The function $f(x) = x^2$ is convex on any interval $[a, b]$, $a \in \mathbb{R}$. From Example 6, we recall that for all $x > a$ and $1 < \alpha < 2$, we have

$$D_a^\alpha f(x) = \frac{2(x-a)^{2-\alpha}}{\Gamma(3-\alpha)} \geq 0. \tag{16}$$

Example 10. The function $f(x) = e^x$ is convex on any interval $[a, b]$, $a, b \in \mathbb{R}$. From Example 7, we recall that for any $x > a$ and $1 < \alpha < 2$, we have

$$\begin{aligned} D_a^\alpha f(x) &= \tfrac{e^a (x-a)^{2-\alpha}}{\Gamma(2-\alpha)} \int_0^1 (1-\theta)^{1-\alpha} e^{(x-a)\theta} \, \mathrm{d}\theta \\ &= \tfrac{(x-a)^{2-\alpha} e^x}{\Gamma(2-\alpha)} \int_0^1 \theta^{1-\alpha} e^{-(x-a)\theta} \, \mathrm{d}\theta. \end{aligned} \tag{17}$$

Upon Theorem 8, we must have $D_a^\alpha f(x) \geq 0$ for all $x > a$, which can be clearly verified from above.

Theorem 11. *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^2(\mathbb{R}^n)$. Let $a \in \mathbb{R}$ and $1 < \alpha < 2$. Then, the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by:*

$$\psi(\beta) = \frac{1}{\Gamma(2-\alpha)} \int_a^\beta (\beta - t)^{(1-\alpha)} \langle \nabla^2 f(x + t(y-x))(y-x), (y-x) \rangle \, \mathrm{d}t \tag{18}$$

is positive for all $\beta \in [a, +\infty[$

Proof. The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by:

$$\phi(t) = f(x + t(y-x)) \tag{19}$$

is convex of class $C^2(\mathbb{R}^n)$ since f is convex of class $C^2(\mathbb{R}^n)$.

By applying Theorem 8 to the convex function ϕ, for $1 < \alpha < 2$, we conclude that

$$\begin{aligned} D_a^\alpha \phi(\beta) &= \tfrac{1}{\Gamma(2-\alpha)} \int_a^\beta (\beta - t)^{(1-\alpha)} \langle \nabla^2 f(x + t(y-x))(y-x), (y-x) \rangle \, \mathrm{d}t \\ &= \psi(\beta) \end{aligned} \tag{20}$$

is increasing for all $\beta \in [a, +\infty[$. This completes the proof. ☐

In the next example, we discuss the application of Theorem 11 for a specific function.

Example 12. Applying Theorem 11 to the convex function $f(x) = \sum_{i=1}^n x_i^2$ of class $C^2(\mathbb{R}^n)$, for $\beta \geq a$, we must have

$$\begin{aligned} \psi(\beta) &= \tfrac{1}{\Gamma(2-\alpha)} \int_a^\beta (\beta - t)^{(1-\alpha)} \langle \nabla^2 f(x + t(y-x))(y-x), (y-x) \rangle \, \mathrm{d}t \\ &= \tfrac{2\|y-x\|^2}{\Gamma(2-\alpha)} \int_a^\beta (\beta - t)^{(1-\alpha)} \, \mathrm{d}t \\ &= \tfrac{2\|y-x\|^2}{\Gamma(2-\alpha)} \left[-\tfrac{(\beta-t)^{(2-\alpha)}}{2-\alpha} \right]_a^\beta \\ &= \tfrac{2\|y-x\|^2}{\Gamma(2-\alpha)} \left[\tfrac{(\beta-a)^{(2-\alpha)}}{2-\alpha} \right] \\ &= \tfrac{2\|y-x\|^2}{\Gamma(3-\alpha)} (\beta - a)^{(2-\alpha)} \geq 0. \end{aligned}$$

3.2. Convex characterization by Caputo fractional derivatives of order $0 < \alpha < 1$

The following example shows that the first criterion of convexity can not be generalized fractionally for $0 < \alpha < 1$ without further restrictions.

Example 13. If we differentiate in Example 6, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} D_a^\alpha f(x) &= \tfrac{2(2-\alpha)(x-a)^{1-\alpha}}{\Gamma(3-\alpha)} + \tfrac{2a(2-\alpha)(1-\alpha)(x-a)^{-\alpha}}{\Gamma(3-\alpha)} && \text{if } 0<\alpha<1\\
&= \tfrac{2(2-\alpha)(x-a)^{1-\alpha}}{\Gamma(3-\alpha)} + \tfrac{2(2-a)(1-\alpha)(x-a)^{-\alpha}}{\Gamma(3-\alpha)} && \text{if } 0<\alpha<1\\
&= ((x-a)+a(1-\alpha))\tfrac{2(2-\alpha)(x-a)^{-\alpha}}{\Gamma(3-\alpha)} && \text{if } 0<\alpha<1\\
&= ((x-a)+a(1-\alpha))\tfrac{2(2-\alpha)(x-a)^{-\alpha}}{\Gamma(3-\alpha)} < 0 && \text{if } 0<\alpha<1 \text{ and } a<x<0.
\end{aligned}$$

Therefore, $f(x) = x^2$ is a convex function with $D_a^\alpha f(x)$ is not increasing when $a < x < 0$. But clearly $f(x)$ is decreasing on $(-\infty, 0)$.

The proof of following lemma which we shall use is straightforward.

Lemma 5. *Assume $h(x)$ and $g(x)$ are differentiable functions on an interval $I = [a, b]$ such that $g^{(1)}(x) < 0$ and both $h, h^{(1)}$ are positive on I. If $g(x) \le \frac{-h(x)g^{(1)}(x)}{h^{(1)}(x)}$, for all $x \in I$, then the function $T(x) = h(x)g(x)$ is strictly decreasing on I. In particular, if $g(x) < 0$ for all $x \in I$, then $T(x)$ is strictly decreasing.*

Theorem 14. *For a real valued function f defined on an interval $[a, b]$, we have*

(i) *If f is an increasing convex function in $C^1([a; b])$, then for each $\alpha \in (0, 1)$ the Caputo fractional derivative $(D_r^\alpha f)(x)$, starting at any $c \in [a, b)$, is increasing on $[c, b)$.*
(ii) *If for some $\alpha \in (0, 1)$ and for all $r \in [a, b)$ the Caputo fractional derivative $(D_r^\alpha f)(x)$, starting at r, is increasing on $(c, b]$, and if $f \in C^2([a; b])$ is strictly decreasing, then f is convex.*

Proof. $(\Rightarrow)$ Since $f(x)$ is convex and increasing then $f^{(1)}(x)$ is increasing and positive. Let $x, y \in [a; b]$ and $x > y$. Then, Upon the Caputo representation in Lemma 4 , we have

$$\begin{aligned}
D_a^\alpha f(x) - D_a^\alpha f(y) &= \tfrac{(x-a)^{(1-\alpha)}}{\Gamma(1-\alpha)} \int_0^1 (1-\theta)^{(-\alpha)} [f^{(1)}(a+(x-a)\theta) - f^{(1)}(a+(y-a)\theta)] \,\mathrm{d}\theta\\
&\quad + \tfrac{(x-a)^{(1-\alpha)} - (y-a)^{(1-\alpha)}}{\Gamma(1-\alpha)} \int_0^1 (1-\theta)^{(-\alpha)} [f^{(1)}(a+(y-a)\theta)] \,\mathrm{d}\theta\\
&\ge 0.
\end{aligned}$$

Above, we subtracted and added the term

$$\frac{(x-a)^{(1-\alpha)}}{\Gamma(1-\alpha)} \int_0^1 (1-\theta)^{(-\alpha)} f^{(1)}(a+(y-a)\theta).$$

$(\Leftarrow)$ We use the proof by absurd. Suppose that the derivative of order α in Caputo's sense of order α of the function $f(x)$ is increasing in any subinterval $[c; d] \subset [a; b]$, and $f(x)$ is not a convex function of class $C^2([a; b])$. Then, $f^{(2)}(x_0) < 0$ for some $x_0 \in [a, b]$. By continuity of $f^{(2)}$ there exists a subinterval $[c; d]$ with $x_0 \in [c, d]$ such that

$$f^{(2)}(x) < 0 \text{ or } f^{(1)}(x) \text{ is strictly decreasing on } [c; d].$$

Let $z_1, z_2 \in [c; d]$ and $z_2 < z_1$. Then, by applying Lemma 5 with $h(x) = (x-c)^{1-\alpha}$ and $g(x) = f^{(1)}(c+(x-c)\theta)$, we have

$$\begin{aligned}
D_c^\alpha f(z_1) - D_c^\alpha f(z_2) &= \tfrac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\theta)^{(-\alpha)} [(z_1-c)^{(1-\alpha)} f^{(1)}(c+(z_1-c)\theta)\\
&\quad - (z_2-c)^{(1-\alpha)} f^{(1)}(c+(z_2-c)\theta)] \,\mathrm{d}\theta\\
&\le 0.
\end{aligned}$$

Which is a contradiction. This completes the proof. □

Remark 2. Positivity of $f^{(1)}$ can not be dropped from necessity of Theorem 14 as Example 13 shows. However, if we consider the function $f(x) = x^2$ in Example 13 on the interval $(0, \infty)$, then f is increase, convex and hence it is clear that its fractional derivative on any subinterval is increasing and hence this verifies the necessity in Theorem 14.

Example 15. In Example 2 the function $f(x) = e^x$ is convex and increasing. It can be also easily verified that the Caputo derivative of f is positive and hence is increasing. This verifies the necessity of Theorem 14.

3.3. A fractional inequality and some geometric consequences

In this section as an application of Lemma 4, we further prove an inequality with a fractional Caputo derivative as an upper bound.

Theorem 16. *Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, f a function of class $C^n([a;b])$ and $f^{(n)}(x) \geq 0$ for all $x \in [a;b]$. The derivative of order α in Caputo's sense of order α of the function f on the interval $[a;b]$ verification*

$$f^{(n-1)}(x) \leq f^{(n-1)}(a) + (x-a)^{\alpha-(n-1)}\Gamma(n-\alpha) D_a^\alpha f(x) \text{ for all } x \in [a;b].$$

Proof. Using the Lemma 4 and let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, f a function of class $C^n([a;b])$ and $f^{(n)}(x) \geq 0$ for all $x \in [a;b]$. We have

$$(1-\theta)^{(n-\alpha-1)} \geq 1 \text{ for all } \theta \in [0;1[$$

so

$$\begin{aligned} D_a^\alpha f(x) &= \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-\theta)^{(n-\alpha-1)} f^{(n)}(a+(x-a)\theta)\,\mathrm{d}\theta \\ &\geq \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 f^{(n)}(a+(x-a)\theta)\,\mathrm{d}\theta \\ &= \tfrac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (x-a)^{-1} df^{(n-1)}(a+(x-a)\theta) \\ &= \tfrac{(x-a)^{n-\alpha-1}}{\Gamma(n-\alpha)} [f^{(n-1)}(a+(x-a)\theta)]_0^1 \\ &= \tfrac{(x-a)^{n-\alpha-1}}{\Gamma(n-\alpha)} [f^{(n-1)}(x) - f^{(n-1)}(a)] \end{aligned}$$

then

$$f^{(n-1)}(x) \leq f^{(n-1)}(a) + (x-a)^{\alpha-(n-1)}\Gamma(n-\alpha) D_a^\alpha f(x) \text{ for all } x \in [a;b].$$

This completes the proof. □

Corollary 17. *Assume $0 < \alpha < 1$ and f is a function of class $C^1([a;b])$ and $f^{(1)}(x) \geq 0$ for all $x \in [a;b]$. Then,*

(1) *The slope of the secant line connecting the point $(a, f(a))$ and $(x, f(x))$ for any $x > a$ is bounded above by $(x-a)^{\alpha-1}\Gamma(1-\alpha) D_a^\alpha f(x)$.*
(2) *For all $x > a$, there exists $a < c < x$ such that the of the tangent line of f at $(c, f(c))$ is bounded above by $f(c) + (x-a)^\alpha \Gamma(1-\alpha) D_a^\alpha f(x)$.*

Proof. (1) Applying Theorem 16 with $n = 1$ leads to

$$f(x) - f(a) \leq (x-a)^\alpha \Gamma(1-\alpha) D_a^\alpha f(x). \tag{21}$$

Then, the assertion follows by dividing the above inequality by $(x-a)$.

(2) Assume $x > a$. By applying the mean-value theorem to f on an interval $[a, x]$, we find $a < c < x$ such that $f^{(1)}(c)(x-a) = f(x) - f(a)$. Then, by substituting this in (21), it follows that

$$f^{(1)}(c)(x-c) \leq f^{(1)}(c)(x-a) \leq (x-a)^\alpha \Gamma(1-\alpha) D_a^\alpha f(x). \tag{22}$$

Then, the assertion follows by adding $f(c)$ to both of the outer sides of the inequality (22).

□

4. Conclusion

We summarize our concluding points as follow:

- We have used a change of variable transforming the interval domain of the Caputo derivative to the interval $[0,1]$ to get a proper representation for the investigation of convexity through Caputo derivatives of orders in $(0,1) \cup (1,2)$.
- We have characterized convexity by means of Caputo derivatives of order $1 < \alpha < 2$ and for functions of class C^2. This generalize the second criteria of convexity to the fractional case. WE have shown that a function of class C^2 is convex if and only if its Caputo fractional derivative of order $1 < \alpha < 2$ is positive of any subinterval.
- Regarding the generalization of the first convexity criteria we have shown that the Caputo derivative of order $0 < \alpha < 1$ of an increasing function of class C^1 is increasing on each subinterval. However, conversely we have shown that if the Caputo derivative of a C^2 decreasing function is increasing on each subinterval then this function is convex. Unfortunately, it was not enough to guarantee the convexity by considering a C^1 function. In this regard we present the following **open problem**: Which kind of C^1 functions of which their Caputo fractional derivatives of order $0 < \alpha < 1$ are increasing on each subinterval, can be convex? Are increasing functions of class C^1 of which their Caputo fractional derivatives of order $0 < \alpha < 1$ are increasing on each subinterval, convex? Is it possible to find an increasing (or decreasing) C^1 function which is not C^2 of which its Caputo fractional derivative is increasing on each subinterval but it is not convex?
- The $(0,1)$ interval representation of the Caputo derivative has been used to get some geometric facts as well (see Sect. 3.3).
- We have given several examples, depending the $(0,1)$ Caputo representation, to illustrate our findings. For instance, we have shown that the Caputo derivative of order $0 < \alpha < 1$ of the convex function $f(x) = x^2$ of class C^1 is not increasing on in each subinterval. This is due to the fact that $f(x) = x^2$ is not increasing on the negative part of the real line. We also have given examples confirming our theorems.
- A general version of the fractional second convexity criterion has been proven for the $\mathbb{R}^n$ domains.
- Using the fact that for $1 < \alpha < 2$ we have $\lim_{\alpha \to 2^-} D_a^\alpha f(x) = f^{(2)}(x)$ [14], the author in [3] proved a one side different convexity result. He worked on functions in the space of twice weighted continuously functions $C_\mu^2(a,b] = \{f : f^{(2)}(x) \in C_\mu\}$, where $C_\mu = \{f : (x-a)^\mu f(x) \text{is continuous}\}$,. For general function spaces valid to prove existence and uniqueness theorems of fractional Cauchy problems we refer to [7]. Our convexity results in this article are different. However, it is to be noted that our results in article can be done in spaces C_μ^1 and $C_\mu^2,\ 0 \le \mu \le 1$. This is obvious since if a function $f \in C_\mu$ is negative at a point t_0 then it is negative on an interval containing t_0.
- The discrete analysis for convexity of different fractional difference operators and their investigation theoretically and numerically has been recently extensively studied [1, 2, 11, 12]. We believe that this work will be useful to continue investigation in the discrete counterpart case.

Acknowledgments

The authors are very grateful and would like to thank the Editor-in-Chief and the anonymous referees for their suggestions and helpful comments which significantly improved the presentation of this paper.

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