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Reduction Analysis and Solitary Wave Solutions of the (2+1)-D Kadomtsev-Petviashvili-Benjamin-Bona-Mahony Equation

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Abstract

In this study, the Lie symmetries of the $(2+1)$-D Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation, which is considerable extension of the KP equation with applications in water waves, fluid dynamics, nonlinear optics, and mathematical physics were investigated. This study has also focused on the exact solutions of this model. We systematically identify the infinitesimal generators and obtain symmetry reductions that convert the equation into lower-dimensional forms using Lie group analysis. The findings shed information on the solution space of the equation and demonstrate how particular symmetries affect its structure. Moreover, exact solutions describing wave propagation behavior are made possible by the simplified equations. The $\exp(-\Psi(\omega))$-expansion approach yields innovative traveling wave solutions after

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considerable investigation. Solving the nonlinear evolution equations using this analytical method yields rational, hyperbolic, and trigonometric functions. The research reveals new solutions to the suggested problem using the extremely effective proposed method. The stability of the system is explored by computing stability gains using a linearization technique, revealing solution behavior. 3D, 2D, and contour graphs illustrate the dynamics of the obtained solutions, enhancing our knowledge to suggested problem.

Keywords (2+1)-D Kadomtsev-Petviashvili-Benjamin-Bona-Mahony · Lie symmetries · $\exp(-\Psi(\omega))$-expansion method · Exact solutions · Stability analysis

1 Introduction

Numerous fields, including physics, hydrodynamics, mathematics, science, engineering, and biology, depend heavily on non-linear partial differential equations (NLPDEs) [1, 2]. Owing to recent advances in mathematical research and theoretical physics, exact solutions to these equations can now be achieved using a range of reliable approaches [3]. Analytical methods include exp-function [4], Lie group analysis [5, 6, 6], variational iteration [7], improved F-expansion [8], generalized exponential rational function [9], Bäcklund transformation [10], bifurcation analysis [11], extended $\left(\frac{G'}{G^2}\right)$-expansion [12–14], modified $\left(\frac{G'}{G^2}\right)$-expansion [15], Jacobi elliptic function expansion [16], extended Jacobi's elliptic [17], modified sub-equation [18, 19], inverse scattering transform [20], sub-equation [21], Bernouli sub-equation function [22], fractional function method [23], modified auxiliary equation [24], Sine-Gordon expansion [25], new extended direct algebraic [26], tanh-coth [27], the extended $\left(\frac{G'}{G}\right)$-exapnsion [28], truncated expansion [29], fractional simple equation [30], $exp(-\phi(\zeta))$-expansion [31–33], fractional unified solver method [34, 35], extended trial [36], fractal semi inverse method [37], the Riccati-sub-ODE Bernoulli's [38], extended tanh [39], Hamiltonian method [40], first integral method [41], KP hierarchy reduction [42], modified simple equation [43], Darboux transformation [44], first integral [45], modified extended mapping method [46], Nucci's reduction method [47] the Ansatz [48], the sine-cosine [49], new Kudryashov method [50, 51], the auxiliary equation [52], new generalized exponential differential function approach [53], Trial equation [54], and Hirota bilinear [55, 56] methods. These methods are crucial because they can provide practical solutions to a wide range of scientific issues as well as theoretical understanding. However, compared to linear wave equations, dealing with nonlinear wave equations is more challenging. To investigate the solutions and applications, a variety of methods for solving nonlinear wave equations have been devised and put into practice.

Boris Kadomtsev and Vladimir Petviashvili first presented the Kadomtsev-Petviashvili equation (KPE) [57] in 1970 as a model for weakly dispersive and weakly nonlinear long waves in specific types of media, mostly plasmas. The Benjamin-Bona-Mahony (BBM) equation is the long-wave propagation model with nonlinear dispersion considered herein. A modification of the KdV equation resulted in the KPE. Using the BBM and KdV equations allows one to also define long wavelengths in liquids and other materials. The two equations taken together provide the KP-BBM equation. A water wave model is included in the KP-BBM system, which controls fluid flow. These model equations clarify the water wave surface with two directions of propagation. Ali and Alam [58] recently found the traveling wave solutions of the KP-BBM problem using the generalized $(\frac{G'}{G})$-expansion. Rajib *et al.* obtained the traveling wave solution of the KP-BBM equation using the new analytical method [59]. The exact solution (ES) of the compact and non-compact forms of the KP-BBM equation were covered in Wazwaz [60]. Lu *et al.* [61] investigated the proposed equation using a lump and strip soliton mix. It is observed that a certain analytical method could not be able to present a possible wave pattern for a model under investigation. This offers the incentive value for the deployment of several strong techniques for additional investigation. The main goal of this work is to find the ES of (2 + 1)-D KP-BBM equation by utilizing $\exp(-\Psi(\omega))$-expansion approach (EEA) [62].

In this work, we examine the exact solutions and symmetry reductions of the (2+1)-D KP-BBM equation.

$$g_{xt} + g_{xx} + \eta(g^2)_{xx} + \Lambda g_{xxt} + \Theta g_{yy} = 0. \tag{1}$$

In addition to analytical computation, the utilization of Lie group theory for symmetry analysis is the most prominent method for addressing nonlinear problems [63, 64]. This method can be used to identify symmetries in any model of differential equations. Understanding these symmetries assists in the analysis of physical phenomena governed by the equations. To decrease similarity, one can employ non-classical symmetry or Lie symmetry group approaches. The determining mechanisms for converting nonlinear partial differential equations (NPDEs) into nonlinear ordinary differential equations (NODEs). However, if this conversion cannot be achieved directly, additional strategies must be considered. This article focuses on a NPDEs in the field of optical communication. (2 + 1)-D KP-BBM is being investigated for the first time with EEA. For the extraction of analytical solutions of nonlinear evolution equations (NLEEs), the proposed approach is efficient, dependable, and competent. Among the closed-form solutions found by the proposed method were rational, hyperbolic, and trigonometric functions. Akbar and Khan [65] lately used the proposed method to solve the Vakhnenko-Parkes equation. Noufe [66] investigated the Boussinesq classical equation. The generalized Hirota-Satsuma pair KdV system was covered in Khater [67]. Using EEA, Islam et al. [68] found travelling wave solutions of two different equation. Sadaf et al [69] apply the proposed method to derive ECBS equation traveling wave dynamics. Suggested model can be systematically reduced using the Lie symmetry analysis described in this work, producing lower-dimensional forms that are easier to solve exactly. In contrast to numerical techniques that necessitate discretization and could result in approximation mistakes, our solution maintains the models analytical structure. Furthermore, symmetry reductions offer a methodical approach to categorizing solution families, in contrast to direct integration techniques, which might not necessarily produce closed-form solutions. The Lie symmetry approach does have a drawback, though, in that it depends on the presence of adequate symmetries, and certain nonlinear PDEs might not allow reductions to simpler forms. In order to confirm and expand on the conclusions achieved, future research could investigate a hybrid technique that combines symmetry analysis with numerical simulations.

This work presents a novel contribution by methodically examining the $(2+1)$-dimensional KP-BBM equation utilizing a blend of Lie symmetry analysis and the $\exp(-\Psi(\omega))$-expansion technique, a combination that has not been documented in the current literature. Differing from earlier works that applied different approaches or considered the model in lower dimensions, this framework provides the means to derive new families of exact solutions, which consist of rational, hyperbolic, and trigonometric forms, along with an array of new soliton patterns including bright, dark, periodic, and M-shaped solitons. Additionally, the application of linear stability analysis affirms the physical validity and strength of the solutions achieved, an aspect that has not been examined in prior related literature. The graphical illustrations in $3D$ surface plots, contour graphs, and line plots further underscore the dynamic aspects of the solutions and yield a deeper physical comprehension of nonlinear wave propagation. Consequently, this work contributes in a novel way by expanding the KP-BBM model's solution space and offering physically significant and mathematically valid results that enhance current nonlinear wave theory research.

The remaining parts of the paper are outlined as. Sect. 2 explain Lie symmetry of suggested problem. In Sect. 3 the transformation of NLPDE to NLODE. Sects. 4 employ the recommended strategy to obtain the ES. In Sect. 5 stability analysis of suggested model has discussed. Sect. 6 presents graphical explanation of the obtained results. Concluding remarks are given in Sect. 7.

2 Lie Symmetries of (2 + 1)-D KP-BBM Equation

Theorem 1 *If a one-parameter Lie group of symmetries is admitted by (2+1)-D KPBBM, then the solution space can be shrunk by one dimension. A simplified version of the solution can be obtained by reducing the system by two dimensions if it admits a two-parameter group.*

Proof We take into consideration the one-point Lie group of infinitesimal transformations acting on the given equation's independent and dependent variables. We may determine the symmetries of the (2+1)-D KPBBM equation using this method. The following is a definition of these transformations:

$$\bar{x} = x + \hbar\aleph^1(x, t, y, g) + O(\hbar^2). \tag{2}$$

$$\bar{t} = t + \hbar\aleph^2(x, t, y, g) + O(\hbar^2). \tag{3}$$

$$\bar{y} = y + \hbar\aleph^3(x, t, y, g) + O(\hbar^2). \tag{4}$$

$$\bar{g} = g + \hbar\wp(x, t, y, g) + O(\hbar^2). \tag{5}$$

whereas the group parameter is indicated by $\hbar$. The Lie algebra of the considered equation is generated by the vector field in the following form:

$$V = \aleph^1(x,t,y,g)\frac{\partial}{\partial x} + \aleph^2(x,t,y,g)\frac{\partial}{\partial t} + \aleph^3(x,t,y,g)\frac{\partial}{\partial y} + \wp(x,t,y,g)\frac{\partial}{\partial g}. \quad (6)$$

A first extension of the generator incorporates derivatives of g concerning the variables x, y, and t.

$$V^{[1]} = V + P^t\frac{\partial}{\partial g_t} + P^x\frac{\partial}{\partial g_x} + P^y\frac{\partial}{\partial g_y}. \quad (7)$$

A second extension of the generator incorporates second-order derivatives.

$$V^{[2]} = V^{[1]} + P^{tt}\frac{\partial}{\partial g_{tt}} + P^{xt}\frac{\partial}{\partial g_{xt}} + P^{ty}\frac{\partial}{\partial g_{ty}} + P^{xx}\frac{\partial}{\partial g_{xx}} + P^{yy}\frac{\partial}{\partial g_{yy}} + P^{xy}\frac{\partial}{\partial g_{xy}}. \quad (8)$$

Eq.(1) has a Lie point symmetry with the vector field Eq.(6) if and only if

$$V^{[3]}\big(g_{xt} + g_{xx} + \eta(g^2)_{xx} + \Lambda g_{xxt} + \Theta g_{yy}\big)| = 0 \quad (9)$$

whereas:

$$V^{[3]} = V + P^{xt}\frac{\partial}{\partial g_{xt}} + P^{xx}\frac{\partial}{\partial g_{xx}} + P^{xxt}\frac{\partial}{\partial g_{xxt}} + P^{yy}\frac{\partial}{\partial g_{yy}}. \quad (10)$$

where the above equation is third prolongation for Eq.(1). This results in an overdetermined system of linear PDEs when Eq.(9) is expanded and divided according to the derivatives of g. The values of $\aleph^1$, $\aleph^2$, $\aleph^3$, and $\wp$ are then obtained by solving this system which gives the following translation symmetries:

$$V_1 = \frac{\partial}{\partial y}c_1, \quad V_2 = \frac{\partial}{\partial t}c_2, \quad V_3 = \frac{\partial}{\partial x}c_3,$$
$$V_4 = y\left(\frac{\partial}{\partial y}c_1\right) + 2t\left(\frac{\partial}{\partial t}c_2\right) + \frac{(-2\eta g - 1)\left(\frac{\partial}{\partial g}c_3\right)}{\eta},$$

In this instance, c_1, c_2, and c_3 represent arbitrary functions of t.

In order to simplify the (2+1)-D KP-BBM equation to a PDE involving three independent variables, we first take into consideration the linear combination of translation symmetries of V_1, V_2, and V_3 which is given by

$$V = V_1 + V_2 + \mho V_3,$$

where an arbitrary constant is $\mho$. With $c_1 = 1$, V can be expressed as follows:

$$V = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial y}.$$

Three invariants are obtained from the associated Lagrangian system for symmetry V:

$$C_1 = x,\ C_2 = \frac{t}{y^2},\ \omega = -\left(-g - \frac{1}{2\eta}\right) y^2 \tag{11}$$

At this point, ω is treated as a new dependent variable, whereas C_1 and C_2 are considered new independent variables.

When we enter these expressions into the model under consideration with dependent variable transformation $\omega(C_1, C_2, y) = y^{\alpha}\tilde{\omega}(C_1, C_2)$, we get:

$$(1+\eta)\tilde{\omega}_{C_1C_1} + \tilde{\omega}_{C_1C_2} + \Lambda\tilde{\omega}_{C_1C_1C_2} + (4\Theta - 6\Theta + 8\Theta)\tilde{\omega} - 2\eta\tilde{\omega}\tilde{\omega}_{C_1C_1} = 0.$$

In the above equation we have consider $\alpha = 2$. After simplification and considering dependent variable transformation, reduced equation is:

$$(1+\eta)\tilde{\omega}_{C_1C_1} + \tilde{\omega}_{C_1C_2} + \Lambda\tilde{\omega}_{C_1C_1C_2} + 4\Theta\tilde{\omega} - 2\eta\tilde{\omega}\tilde{\omega}_{C_1C_1} = 0. \tag{12}$$

It is less complicated than the initial version. The reduction procedure is repeated to further reduce the number of independent variables if a two-parameter symmetry group is admitted. The theorem is thus proven. □

3 Transformation NLPDE to ODE

Theorem 2 *An NLPDEs solutions can be written in terms of invariant variables, which lowers the number of independent variables, if it admits a one-parameter Lie group of symmetries. As a result, similarity solutions that solve a simplified differential equation with fewer independent variables are constructed.*

Proof Let's consider the suggested equation:

$$(1+\eta)\tilde{\omega}_{C_1C_1} + \tilde{\omega}_{C_1C_2} + \Lambda\tilde{\omega}_{C_1C_1C_2} + 4\Theta\tilde{\omega} - 2\eta\tilde{\omega}\tilde{\omega}_{C_1C_1} = 0, \tag{13}$$

where η, Λ, and Θ are arbitrary constants, and $\tilde{\omega}(C_1, C_2)$ is the dependent variable. The traveling wave theory considers these changes in variables:

$$\tilde{\omega}(C_1, C_2) = v(\Pi), \quad \Pi = n \cdot C_1 + m \cdot C_2 - p \cdot t, \tag{14}$$

here m, n, and p are integers. Adding Eq.(14) to Eq.(13) yields the following connection:

$$\Lambda p n^3 v'''' + (pn - \Theta m^2 - n^2)v'' - 2\eta n^2(vv'' + (v')^2) = 0. \tag{15}$$

Taking $\eta = pn - \Theta m^2 - n^2$, Eq.(15) can be rewritten as:

$$\Lambda p n^3 v'''' + \eta v'' - 2\eta n^2\left[vv'' + (v')^2\right] = 0. \tag{16}$$

Integrating Eq.(16) twice yields the following equation:

$$\Lambda p n^3 v'' + \eta v - \eta n^2 v^2 = 0. \tag{17}$$

Similarity variables limit the number of independent variables and enable the formulation of exact solutions, proving the theorem. □

4 Mathematical Analysis of $\exp(-\Psi(\tilde{\omega}))$-Expansion Method

Theorem 3 *The structure of the general solution of Eq. (17) derived from Eq. (13) can expressed in following form*

$$v(\Pi) = \sum_{j=0}^{r} \Big[\Gamma_j (\exp(-\Psi(\Pi))^j) \Big]. \tag{18}$$

Here, the ODE is satisfied by $\Psi(\Pi)$.

$$\Psi'(\Pi) = \kappa + \lambda \cdot \exp(\Psi(\Pi)) + \exp(-\Psi(\Pi)), \tag{19}$$

where λ *and* κ *are integers.*

The Eq.(19) can be transformed into a Riccati equation for $y = \exp(-\Psi)$.

Suppose $\kappa^2 - 4\lambda > 0$ *and* $\lambda \neq 0$.

$$\Psi(\Pi) = \ln\left(\frac{-\sqrt{\kappa^2 - 4\lambda} \cdot \tanh(\frac{\sqrt{\kappa^2-4\lambda}}{2}(\Pi + e)) - \kappa}{2\lambda} \right). \tag{20}$$

If $\kappa^2 - 4\lambda > 0$, $\lambda = 0$, *and* $\kappa \neq 0$,

$$\Psi(\Pi) = -\ln\left(\frac{\kappa}{\cosh(\kappa(\Pi + e)) + \sinh(\kappa(\Pi + e)) - 1} \right). \tag{21}$$

If $\kappa^2 - 4\lambda = 0$, $\kappa \neq 0$ *and* $\lambda \neq 0$, *then*

$$\Psi(\Pi) = \ln\left(-\frac{2(\kappa(\Pi + e) + 2)}{\left(\kappa^2 (\Pi + e)\right)} \right). \tag{22}$$

If $\kappa^2 - 4\lambda = 0$,$\kappa = 0$ *and* $\lambda = 0$, *then*

$$\Psi(\Pi) = \ln(\Pi + e). \tag{23}$$

If $\kappa^2 - 4\lambda < 0$ *and* $\lambda \neq 0$, *then*

$$\Psi(\Pi) = \ln\left(\frac{\sqrt{4\cdot\lambda-\kappa^2}\cdot\tan\left(\frac{\sqrt{4\cdot\lambda-\kappa^2}}{2}(\Pi+e)\right)-\kappa}{2\cdot\lambda}\right). \tag{24}$$

□

Proof The proof follows from the application of the suggested method to reduced the ode as Eq.(17). The homogenous balancing principle produces $r = 2$. The expansion of Eq.(18) is a rational function of v, and therefore, the result of the manuscript provides a special application of the transformed rational function method.

$$v(\Pi) = \Gamma_0 + \Gamma_1(\exp(-\Psi(\Pi))) + \Gamma_2(\exp(-\Psi(\Pi))^2). \tag{25}$$

To calculate the different powers of $\exp(-\Psi(\Pi))$, insert Eq.(25) into Eq.(17) and put in Eq.(19). A system of equations are created by setting the coefficients of different powers of $\exp(-\Psi(\Pi))$ to zero.

$$(\exp(-4\Psi(\Pi))) : 6\Lambda n^3 p\Gamma_2 - \eta n^2\Gamma_2^2 = 0, \tag{26}$$

$$(\exp(-3\Psi(\Pi))) : 10\Lambda n^3 p\Gamma_2\kappa + 2\Lambda p n^3\Gamma_1 - 2\eta n^2\Gamma_1\Gamma_2 = 0, \tag{27}$$

$$(\exp(-2\Psi(\Pi))) : 4\Lambda p n^3\Gamma_2\kappa^2 + 3\Lambda n^3 p\Gamma_1\kappa + 8\Lambda p n^3\Gamma_2\lambda - 2\eta n^2\Gamma_0\Gamma_2 - \eta n^2\Gamma_1^2 + \eta\Gamma_2 = 0, \tag{28}$$

$$(\exp(-\Psi(\Pi))) : \Lambda p n^3\Gamma_1\kappa^2 + 6\Lambda n^3 p\Gamma_2\kappa\lambda + 2\Lambda p n^3\Gamma_1\lambda - 2\eta n^2\Gamma_0\Gamma_1 + \eta\Gamma_1 = 0, \tag{29}$$

$$(\exp(0\Psi(\Pi))) : \Lambda n^3 p\lambda\Gamma_1\kappa + 2\Lambda n^3 p\Gamma_2\lambda^2 - \eta n^2\Gamma_0^2 + \eta\Gamma_0 = 0. \tag{30}$$

Eqs.(26)-(30) are solved simultaneously to yield the unknown constants values. As a result, the wave solution families mentioned below relate to the solution collections for these equations.

$$\Lambda = \frac{\eta\,\Gamma_2\sqrt{6}}{36\sqrt{\frac{1}{\kappa^2\Gamma_2-4\Gamma_2\lambda}}\,p}, \quad n = \sqrt{6}\sqrt{\frac{1}{\kappa^2\Gamma_2-4\Gamma_2\lambda}}, \quad \Gamma_0 = \frac{(\kappa^2+2\lambda)\Gamma_2}{6},$$
$$\Gamma_1 = \Gamma_2\kappa, \quad \Gamma_2 = \Gamma_2. \tag{31}$$

The following wave solution in terms of hyperbolic function is derived for $\kappa^2 - 4\Pi > 0$ and $\Pi \neq 0$.

$$v_1(C_1, C_2) = \frac{\left(\left(\kappa^2+2\lambda\right)\tanh\left(\frac{\sqrt{\kappa^2-4\lambda}\,(\Pi+e)}{2}\right)^2 + 2\sqrt{\kappa^2-4\lambda}\,\tanh\left(\frac{\sqrt{\kappa^2-4\lambda}\,(\Pi+e)}{2}\right)\kappa + \kappa^2 - 6\lambda\right)\left(\kappa^2-4\lambda\right)\Gamma_2}{6\left(\sqrt{\kappa^2-4\lambda}\,\tanh\left(\frac{\sqrt{\kappa^2-4\lambda}\,(\Pi+e)}{2}\right)+\kappa\right)^2}. \tag{32}$$

Hyperbolic function solution is found in cases $\kappa^2 - 4\Pi > 0$, $\Pi = 0$, and $\kappa \neq 0$.

$$v_2(C_1, C_2) = \frac{\Gamma_2\left(\left(\kappa^2+2\lambda\right)\cosh(\kappa(\Pi+e)) + 2\kappa^2 - 2\lambda\right)}{6\cosh(\kappa(\Pi+e)) - 6}. \tag{33}$$

The rational function solution is derived for $\kappa^2 - 4\Pi = 0$, $\kappa \neq 0$, and $\Pi \neq 0$.

$$v_3(C_1, C_2) = \frac{\left(\kappa^2 + 2\lambda\right)\Gamma_2}{6} - \frac{\kappa\kappa(\Pi + e)^2\,\Gamma_2}{2\kappa(\Pi + e) + 4} + \frac{\Gamma_2\kappa(\Pi + e)^4}{4\,(\kappa(\Pi + e) + 2)^2}. \tag{34}$$

Given the conditions $\kappa^2 - 4\Pi = 0$, $\kappa = 0$, and $\Pi = 0$, the solution of the rational function is found as

$$v_4(C_1, C_2) = \frac{\Gamma_2\left(\left(\kappa^2 + 2\lambda\right)\Pi^2 + \left(\left(2\kappa^2 + 4\lambda\right)e + 6\kappa\right)\Pi + 6 + \left(\kappa^2 + 2\lambda\right)e^2 + 6\kappa e\right)}{6\,(\Pi + e)^2}. \tag{35}$$

The following wave solution is derived in terms of trigonometric function in the case of $\kappa^2 - 4\Pi < 0$ and $\Pi \neq 0$.

$$v_5(C_1, C_2) = -\frac{\left(\left(\kappa^2 + 2\lambda\right)\tan\left(\frac{\sqrt{-\kappa^2+4\lambda}\,(\Pi+e)}{2}\right)^2 + 2\sqrt{-\kappa^2 + 4\lambda}\,\tan\left(\frac{\sqrt{-\kappa^2+4\lambda}\,(\Pi+e)}{2}\right)\kappa - \kappa^2 + 6\lambda\right)\left(\kappa^2 - 4\lambda\right)\Gamma_2}{6\left(-\sqrt{-\kappa^2 + 4\lambda}\,\tan\left(\frac{\sqrt{-\kappa^2+4\lambda}\,(\Pi+e)}{2}\right) + \kappa\right)^2}. \tag{36}$$

Family 2:

$$\Lambda = \frac{\eta\Gamma_2}{6\sqrt{-\frac{6}{\kappa^2\Gamma_2 - 4\Gamma_2\lambda}}\,p}, \quad n = \sqrt{-\frac{6}{\kappa^2\Gamma_2 - 4\Gamma_2\lambda}}, \quad \Gamma_0 = \Gamma_2\lambda, \quad \Gamma_1 = \Gamma_2\kappa, \quad \Gamma_2 = \Gamma_2. \tag{37}$$

For $\kappa^2 - 4\Pi > 0$ and $\Pi \neq 0$, the following wave response is derived by using hyperbolic function.

$$v_6(C_1, C_2) = \frac{\Gamma_2\lambda\left(\tanh\left(\frac{\sqrt{\kappa^2-4\lambda}\,(\Pi+e)}{2}\right) - 1\right)\left(\tanh\left(\frac{\sqrt{\kappa^2-4\lambda}\,(\Pi+e)}{2}\right) + 1\right)\left(\kappa^2 - 4\lambda\right)}{\left(\sqrt{\kappa^2 - 4\lambda}\,\tanh\left(\frac{\sqrt{\kappa^2-4\lambda}\,(\Pi+e)}{2}\right) + \kappa\right)^2}. \tag{38}$$

The response to the hyperbolic function is identified when $\kappa^2 - 4\Pi > 0$, $\Pi = 0$, and $\kappa \neq 0$.

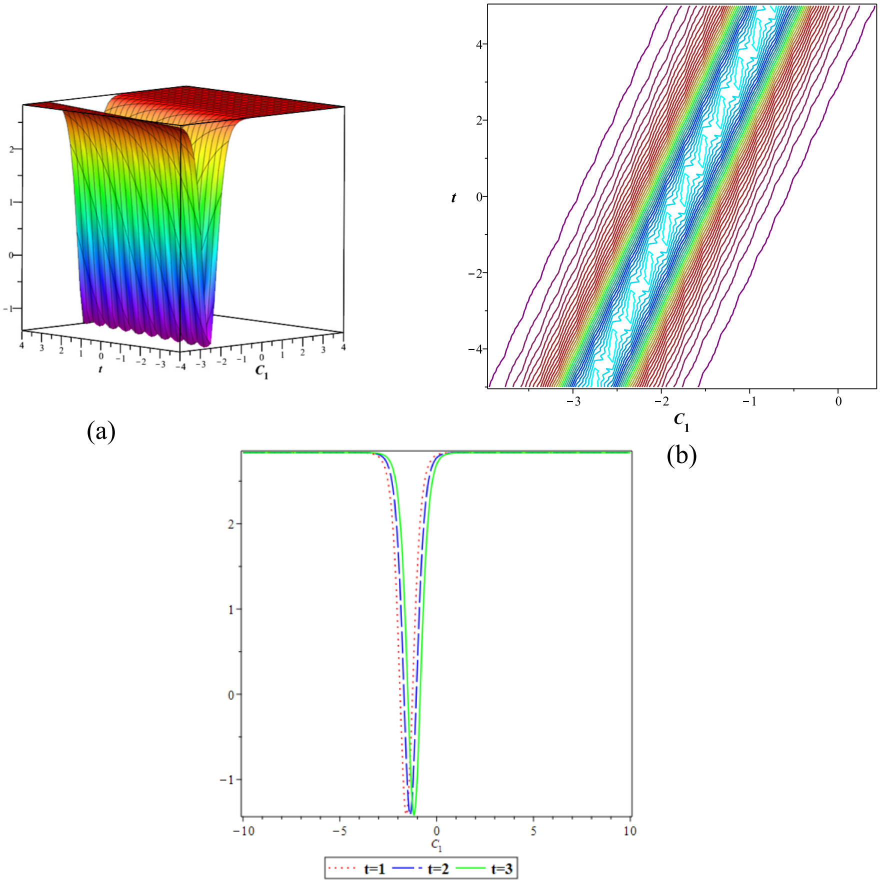
$$v_7(C_1, C_2) = \frac{\left(2\lambda\cosh(\kappa(\Pi + e)) + \kappa^2 - 2\lambda\right)\Gamma_2}{2\cosh(\kappa(\Pi + e)) - 2}. \tag{39}$$

The rational function solution can be obtained for $\kappa^2 - 4\Pi = 0$, $\kappa \neq 0$, and $\Pi \neq 0$.

$$v_8(C_1, C_2) = \Gamma_2\lambda - \frac{\Gamma_2\kappa\kappa(\Pi + e)^2}{2\kappa(\Pi + e) + 4} + \frac{\Gamma_2\kappa(\Pi + e)^4}{4\,(\kappa(\Pi + e) + 2)^2}. \tag{40}$$

The rational function solution appears in case $\kappa^2 - 4\Pi = 0$, $\kappa = 0$, and $\Pi = 0$.

$$v_9(C_1, C_2) = \frac{\Gamma_2\left(\Pi^2\lambda + (2\lambda e + \kappa)\,\Pi + e^2\lambda + \kappa e + 1\right)}{(\Pi + e)^2}. \tag{41}$$



(a)

(b)

(c)

Fig. 1 Present $v_1(C_1, C_2)$ for $\kappa = 5, \lambda = 2, e = 1, \Gamma_2 = 1, n = 1, m = 1, p = 0.2, C_2 = 0.2$

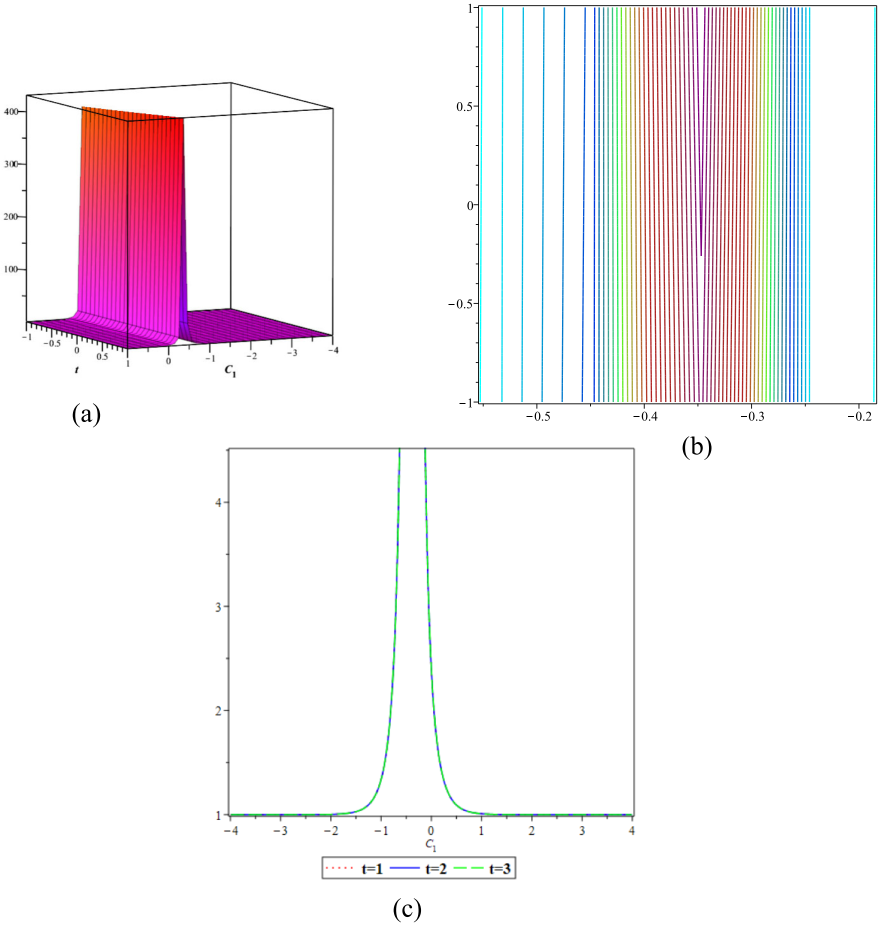
The following wave solution is computed in terms of trigonometric function provided $\kappa^2 - 4\Pi < 0$ and $\Pi < 0$ (Figs. 1, 2, 3, 4, 5, 6 and 7).

$$v_{10}(C_1, C_2) = -\frac{\Gamma_2\lambda\left(\tan\left(\frac{\sqrt{-\kappa^2+4\lambda}\,(\Pi+e)}{2}\right)^2+1\right)\left(\kappa^2-4\lambda\right)}{\left(-\sqrt{-\kappa^2+4\lambda}\,\tan\left(\frac{\sqrt{-\kappa^2+4\lambda}\,(\Pi+e)}{2}\right)+\kappa\right)^2}. \tag{42}$$

□

5 Stability Analysis

In this section, we will investigate the stability of Eq.(12) using the linear stability analysis approach. Following more carefully the revised answer from the model, we find [18]:



(a)

(b)

(c)

Fig. 2 Present $v_2(C_1, C_2)$ for $\kappa = 2, \lambda = 0, e = 0.9, \Gamma_2 = 1.5, n = 2.4, m = 1, p = 0.001, C_2 = 0.01$

$$\tilde{\omega}(C_1, C_2) = \rho + \aleph v(C_1, C_2). \tag{43}$$

It is possible to establish the stability of the preceding Eq.(12) for any constant value of ρ. Substituting Eq.(43) into Eq.(12) yields

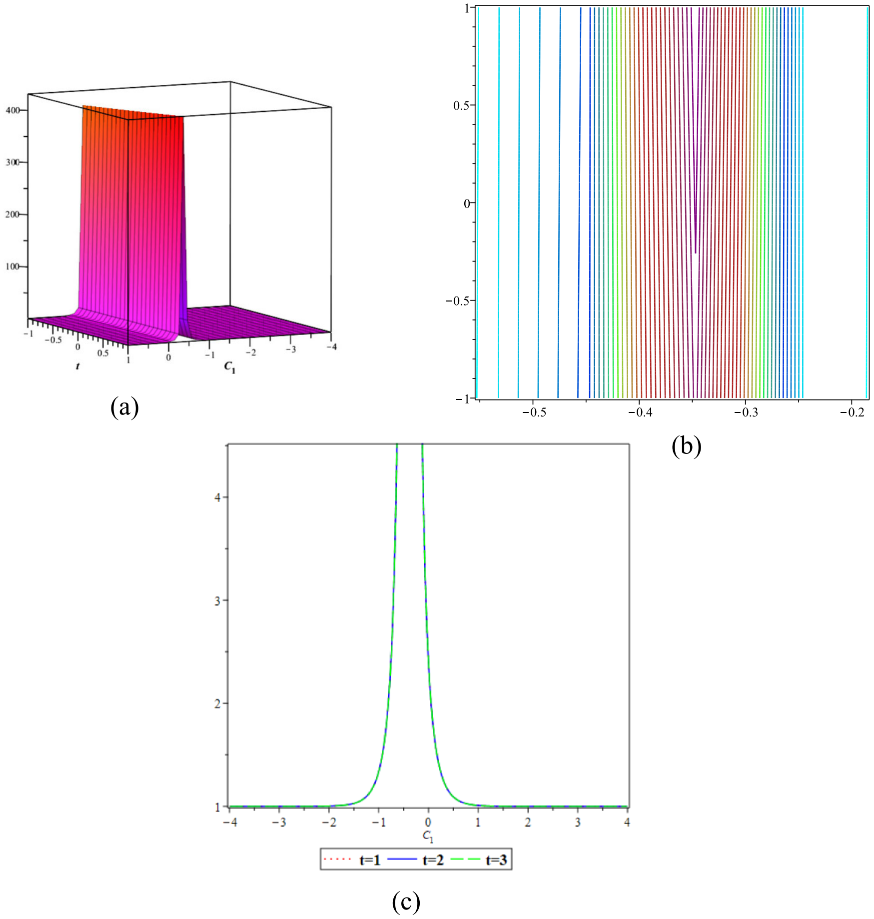
$$\aleph v_{C_1C_1} + \eta\aleph v_{C_1C_2} + \aleph v_{C_1C_2} + \Lambda\aleph v_{C_1C_1C_2} + 4\Theta\rho + 4\aleph\Theta v - 2\eta\rho\aleph v_{C_1C_2} - 2\eta\aleph^2 v v_{C_1C_2} = 0. \tag{44}$$

The given equation can be expressed through the use of the variable v to make it linear.

$$\aleph v_{C_1C_1} + \eta\aleph v_{C_1C_2} + \aleph v_{C_1C_2} + \Lambda\aleph v_{C_1C_1C_2} + 4\Theta\rho + 4\aleph\Theta v - 2\eta\rho\aleph v_{C_1C_2} - 2\eta = 0. \tag{45}$$

Let us suppose that the aforementioned issue has a solution, which can be phrased as follows:

$$v(x, y, t) = \exp^{\iota(kC_1 + \delta C_2)}. \tag{46}$$



(a)

(b)

(c)

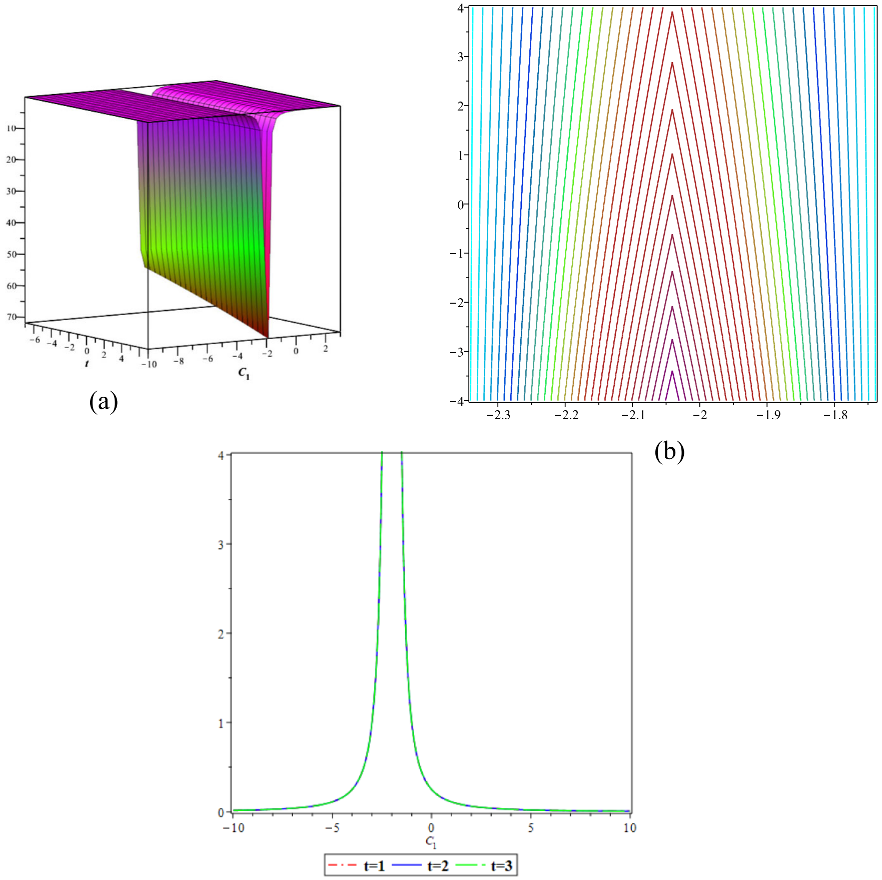
Fig. 3 Pictorial representation of $v_3(C_1, C_2)$ for $\kappa = 2, \lambda = 1, e = 1, \Gamma_2 = 1, n = 1, m = 0, p = 0.001, C_2 = 1$

Eq.(46) is replaced by Eq.(45), where k is the normalized wave number and ρ represents the dispersion relation. The results are as follows:

$$\delta = \frac{4\Theta\rho}{\ln\left(2\aleph + \eta\aleph + 4\Lambda\aleph\Theta - 2\eta\rho\aleph\right) C_2} - K \cdot \frac{C_1}{C_2} \tag{47}$$

6 Results and Discussion

The accumulated results show a variety of nonlinear wave shapes, such as periodic solutions, rational waveforms, and solitary waves. Parameters that affect the waves' amplitude, width, and stability govern these various waveforms. An effective toolkit for investigating these nonlinear behaviours is provided by the precise analytical forms



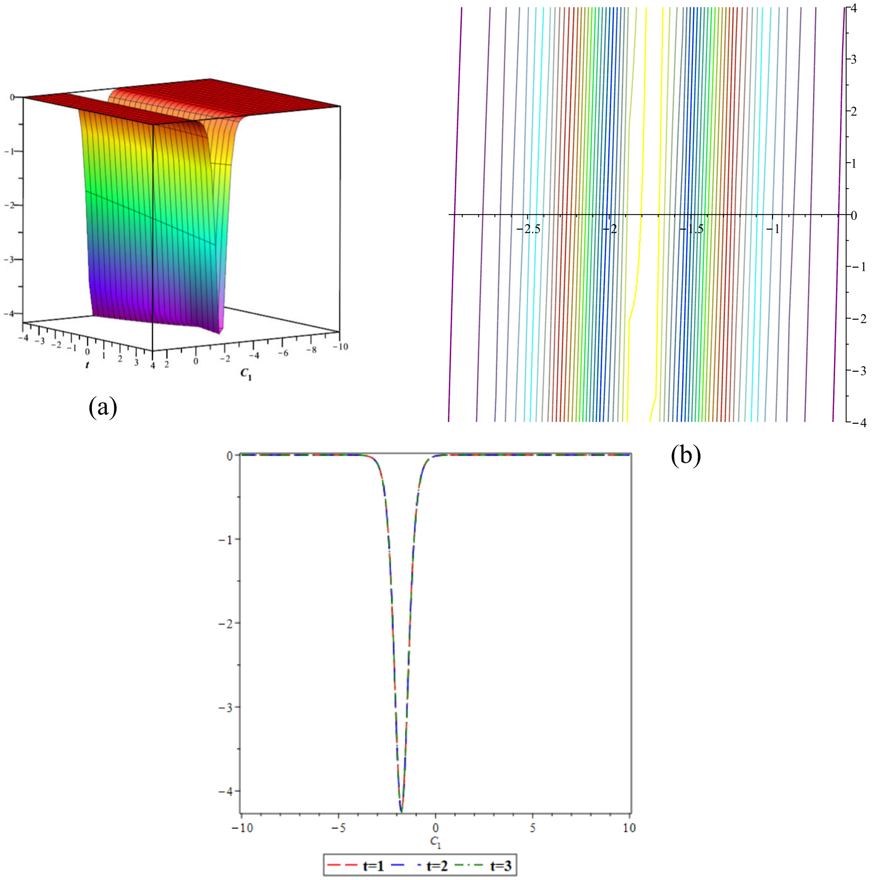
(a)

(b)

(c)

Fig. 4 Visual representation of $v_5(C_1, C_2)$ for $\kappa = 1, \lambda = 1, e = 1, \Gamma_2 = 1, n = 1, m = 1, p = 0.002$, $C_2 = 0.2$

obtained using the EEA. Importantly, the stability analysis highlights the mathematical validity of several wave profiles by confirming their durability under minor perturbations. These results not only improve the KP-BBM model's solution space but also shed light on how these equations control intricate fluid and optical systems in higher dimensions. The graphical representation of the derived solutions reveals the dynamic features of the $(2 + 1)$-D KP-BBM equation equation. Figures 1-8 shows traveling waves and soliton patterns along with 3D, 2D and contour profiles, showing the resulting solutions under suitable choice of parametric values. Graphically observing the temporal development of the wave generation with varied C_1 and C_2 is not possible as the examined KP-BBM equation is $(2 + 1)$-dimensional. The extracted solutions demonstrate the acquisition of a wide range of traveling wave solutions by the EEA. Some fascinating graphical simulations exhibit the dynamics wave pattern for the $(2+1)$-D KP-BBM equation. These effective applications demonstrate to the stability



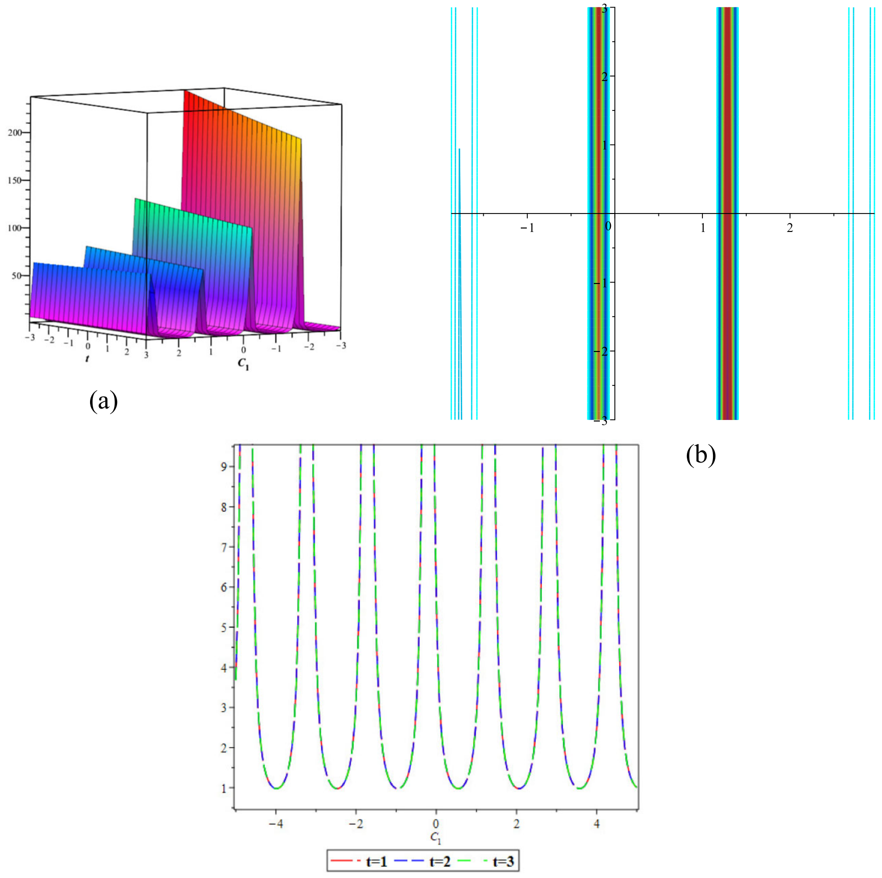
(a)

(b)

(c)

Fig. 5 Graph of $v_6(C_1, C_2)$ based on $\kappa = 5, \lambda = 2, e = 1, \Gamma_2 = 1, n = 1, m = 1, p = 0.01, C_2 = 0.2$

and importance of the suggested methods. These methods are especially useful for exactly solving nonlinear NPDEs that are difficult to solve using conventional methods. The recommended analytical approaches have some limitations, however; they may not always be able to identify a non-trivial solution or work for certain NLEE. The discovered solutions include bright-singular soliton, bell soliton, dark soliton, M-shaped soliton and periodic soliton solutions. Although singular solutions have no practical use, they have surfaced in our theoretical inquiry as a component of the spectrum of traveling wave patterns found for the $(2+1)$-D KP-BBM equation under consideration. The results obtained are compared with the findings in Alam [58], Jalil et al. [70], Rajib et al. [59], Wazwaz [60] and Lu et al. [61] prove the validity of the proposed techniques and demonstrate that the findings presented include some unique traveling wave patterns found for the $(2+1)$-D KP-BBM equation. To address the study's shortcomings, we admit that the exact solutions derived for the $(2+1)$-



(a)

(b)

(c)

Fig. 6 Graph of $v_{10}(C_1, C_2)$ for $\kappa = 1, \lambda = 1, e = 1, \Gamma_2 = 1.3, n = 2.4, m = 1, p = 0.001, C_2 = 0.1$

dimensional KP-BBM problem are subject to particular constraints. The efficacy of such solutions is dependent on precise parameter selections, and not all values of the parameters produce physically significant results. Inappropriate choice may result in singularities or nonexistent wave forms. In real life, external perturbations, dissipation, and higher-order nonlinear processes can all have an impact on wave behavior. Furthermore, while our theoretical conclusions are consistent with previous research, experimental confirmation remains an open area of study. Our recent works have made major advances to the study of integrable systems and nonlocal differential equations in parallel with the current research. Understanding whether a certain wave pattern in fluid flow will endure, fade, or alter over time is made easier by the stability analysis carried out in this work [71].

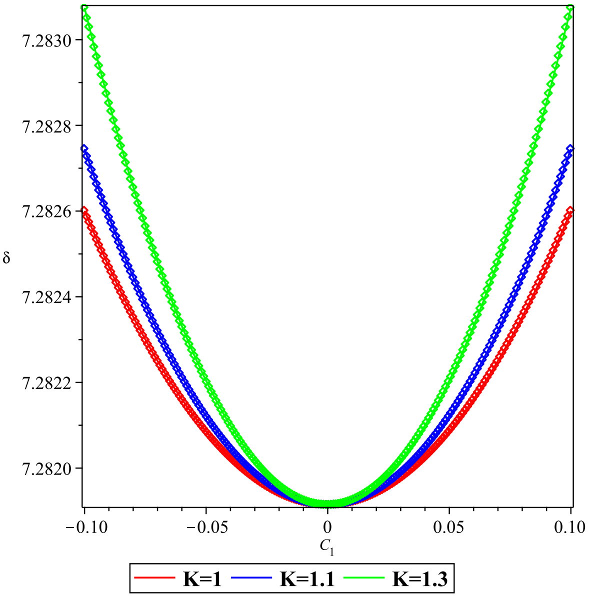


Fig. 7 Stability Analysis using parameters $\Theta = 1$, $\eta = 1$, $\aleph = 1$, $\rho = 2$, $C_2 = 1$, $\Lambda = 1$

7 Conclusion

In this study, we have conducted a comprehensive analytical investigation of the (2+1)-dimensional KP-BBM equation by employing Lie symmetry analysis in conjunction with the $\exp(-\Psi(\omega))$-expansion method. The Lie symmetry approach enabled the systematic reduction of the governing nonlinear partial differential equation into lower-dimensional forms, facilitating the construction of group-invariant solutions. Subsequently, the adopted analytical framework produced a rich class of exact traveling wave solutions, including rational, hyperbolic, and trigonometric structures, corresponding to physically meaningful wave profiles such as bright, dark, periodic, and M-shaped solitons. Moreover, the implementation of linear stability analysis confirmed the robustness and reliability of the obtained solutions under small perturbations. The graphical visualizations, including 2D, 3D, and contour plots, further illustrated the dynamical characteristics and propagation behavior of the nonlinear wave structures. The obtained results significantly extend the solution space of the KP-BBM model and demonstrate the effectiveness of combining symmetry-based reductions with analytical solution techniques for solving complex nonlinear evolution equations. Future work may focus on integrating the present analytical framework with advanced numerical and data-driven techniques to explore more complex nonlinear regimes. Additionally, extending the model to fractional and variable-coefficient forms may provide deeper insights into realistic physical systems. Overall, this work provides a solid mathematical framework for analyzing higher-dimensional nonlinear wave models and contributes to the advancement of theoretical studies in fluid dynamics, nonlinear optics, and related applied fields.

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Author Contributions MABI participated in the conceptualization, investigation, validation, visualization, data curation, review, and writing the original draft. MZR participated in the conceptualization, investigation, validation, visualization. MS participated in the conceptualization, supervision, administration, validation, visualization and writing of the manuscript. GA participated in the review, supervision, administration, and editing of the manuscript. AK participated in the conceptualization, investigation, and validation. MY participated in the conceptualization, visualization, investigation, and validation. BR participated in the investigation, and validation. TA participated in the conceptualization, investigation, and validation. All authors read and approved the final manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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