KYUNGPOOK Math. J. 66(2026), 279-293
https://doi.org/10.5666/KMJ.2026.66.2.279
pISSN 1225-6951 eISSN 0454-8124

On Close-to-Convex Harmonic Univalent Functions Associated with Poisson and Pascal Distribution Series

Serkan Çakmak
Management Information Systems Department, İstanbul Gelişim University, Avcılar, 34310, İstanbul, Türkiye
e-mail: secakmak@gelisim.edu.tr

Yildiz Aydin*
Management Information Systems Department, İstanbul Gelişim University, Avcılar, 34310, İstanbul, Türkiye
e-mail: yaydin@gelisim.edu.tr

Abstract. In this study, relations are presented between subclasses of harmonic univalent functions via convolution operators involving Pascal and Poisson distribution series. Focusing on close-to-convex harmonic functions in the open unit disk $\mathbb{U}$, inclusion statements are investigated and sufficient conditions are derived to ensure that these operators preserve geometric properties.

1. Introduction

In the complex plane $\mathbb{C}$, let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk. In [7], a characterization of a harmonic function

$$f = u + \overline{v},$$

to be locally univalent and sense-preserving in $\mathbb{U}$ is given by the condition

$$|v'(z)| < |u'(z)|,$$

where u and v are analytic in $\mathbb{U}$, for all $z \in \mathbb{U}$.

In the open disk $\mathbb{U}$, consider the normalization of harmonic, univalent, and sense-preserving functions satisfying the conditions $f(0) = 0$ and $f_z(0) = 1$. We denote

* Corresponding Author.
Received August 9, 2025; revised September 9, 2025; accepted October 27, 2025.
2020 Mathematics Subject Classification: 30C20, 30C45, 30C55.
Key words and phrases: Harmonic univalent functions, Pascal distribution series, Poisson distribution series, close-to-convex functions.

this class by $\mathcal{S}_{\mathcal{H}}$. The functions $f \in \mathcal{S}_{\mathcal{H}}$ satisfying $v'(0) = b_1 = 0$ form a subclass denoted by $\mathcal{S}_{\mathcal{H}}^0$. Here, u and v can be expressed as

$$u(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu, \quad v(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu. \tag{1}$$

If the co-analytic component $v(z)$ is identically zero, then any function $f \in \mathcal{S}_{\mathcal{H}}$ coincides with a function in the class $\mathcal{S}$ of normalized univalent analytic functions defined on $\mathbb{U}$. Clearly, the inclusions

$$\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}$$

hold.

The subclasses $\mathcal{K}$ and $\mathcal{S}^*$ of $\mathcal{S}$ consist of functions that map $\mathbb{U}$ onto convex and starlike domains, respectively. Similarly, the subclasses $\mathcal{K}_{\mathcal{H}}^0$ and $\mathcal{S}_{\mathcal{H}}^{0,*}$ of harmonic functions in $\mathcal{S}_{\mathcal{H}}^0$ correspond to convex and starlike harmonic mappings, respectively. For further details, see [15, 3].

Before presenting the main results, we state some useful lemmas that will be used frequently in the sequel.

Lemma 1.1 ([7]). *Let $f = u + \overline{v} \in \mathcal{K}_{\mathcal{H}}^0$, where u and v have the form given in (1) with $b_1 = 0$. Then, for $\nu \geq 2$,*

$$|a_\nu| \leq \frac{\nu+1}{2}, \quad |b_\nu| \leq \frac{\nu-1}{2}.$$

Lemma 1.2 ([7]). *Let u and v be given by (1) with the condition $b_1 = 0$. If $f = u + \overline{v} \in \mathcal{S}_{\mathcal{H}}^{0,*}$, then for $\nu \geq 2$ we have*

$$|a_\nu| \leq \frac{(2\nu+1)(\nu+1)}{6}, \quad |b_\nu| \leq \frac{(2\nu-1)(\nu-1)}{6}.$$

Building on previous work on harmonic functions, in this paper we consider the class of harmonic close-to-convex functions defined by Çakmak et al. in [6]. The aim of this paper is to bring together the inclusion relations of $\mathcal{A}_{\mathcal{H}}^0(\sigma, \tau, \xi)$ using Poisson and Pascal distribution series, respectively.

2. The Class $\mathcal{A}_{\mathcal{H}}^0(\sigma, \tau, \xi)$

In this section, we introduce the class $\mathcal{A}_{\mathcal{H}}^0(\sigma, \tau, \xi)$ of harmonic functions defined by an inequality involving parameters σ, τ, and ξ. We review some of the standard facts on this special class, including important lemmas regarding coefficient bounds and inclusion criteria that will be useful in our subsequent analysis.

Definition 2.1 ([6]). Denote by $\mathcal{A}_{\mathcal{H}}^0(\sigma, \tau, \xi)$ the class of functions $f = u + \overline{v} \in \mathcal{H}^0$ providing

$$\operatorname{Re}\big[\sigma u'(z) + \tau z u''(z) - \xi\big] > |\sigma v'(z) + \tau z v''(z)|, \tag{2}$$

where $\sigma > \xi \geq 0$ and $\tau \geq 0$.

Lemma 2.2 (Theorem 2.2, [6]). *The functions in the class $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$ are close-to-convex in $\mathbb{U}$.*

Lemma 2.3 (Theorem 2.3, [6]). *Let $f = u + \overline{v} \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$. Then for $\nu \geq 2$,*

$$|b_\nu| \leq \frac{\sigma-\xi}{\nu[\sigma+\tau(\nu-1)]}.$$

In the case

$$f(z) = z + \frac{\sigma-\xi}{\nu[\sigma+\tau(\nu-1)]}\overline{z}^\nu,$$

the equality holds.

Lemma 2.4 (Theorem 2.4, [6]). *Let $f = u + \overline{v} \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$. Then for $\nu \geq 2$,*

$$(i) \quad |a_\nu| + |b_\nu| \leq \frac{2(\sigma-\xi)}{\nu[\sigma+\tau(\nu-1)]},$$

$$(ii) \quad \big||a_\nu| - |b_\nu|\big| \leq \frac{2(\sigma-\xi)}{\nu[\sigma+\tau(\nu-1)]},$$

$$(iii) \quad |a_\nu| \leq \frac{2(\sigma-\xi)}{\nu[\sigma+\tau(\nu-1)]}.$$

All these results are sharp and we obtain that the equality holds when

$$f(z) = z + \sum_{\nu=2}^{\infty} \frac{2(\sigma-\xi)}{\nu[\sigma+\tau(\nu-1)]} z^\nu.$$

Lemma 2.5 (Theorem 2.5, [6]). *Let $f = u + \overline{v} \in \mathcal{H}^0$ satisfy*

$$\sum_{\nu=2}^{\infty} \nu[\sigma+\tau(\nu-1)]\big(|a_\nu| + |b_\nu|\big) \leq \sigma - \xi.$$

Then, $f \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$.

3. Applications to Poisson Distribution Series

The Poisson distribution series has recently attracted considerable attention in Geometric Function Theory due to its effectiveness in describing various subclasses of analytic and harmonic univalent functions (see, e.g., [1, 2, 13, 11, 14, 12]). Motivated by earlier studies utilizing hypergeometric or special function based convolution operators to derive inclusion relations, we establish several new results concerning the class $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$ by applying a convolution operator associated with the Poisson distribution series. Our main results provide inclusion relations involving the classes $\mathcal{K}^0_{\mathcal{H}}$, $\mathcal{S}^{0,*}_{\mathcal{H}}$, and $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, based on the coefficient bounds obtained in the previous section.

The Poisson distribution with parameter $\alpha > 0$ leads to the power series

$$\Pi_\alpha(z) = z + \sum_{\nu=2}^{\infty} \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} z^\nu, \tag{3}$$

which was introduced by Porwal [11]. Later, Porwal and Srivastava [14] defined the corresponding convolution operator

$$\Pi_{\alpha,\beta}(f)(z) = \Pi_\alpha(z) * u(z) + \overline{\Pi_\beta(z) * v(z)} = \mathfrak{U}(z) + \overline{\mathfrak{V}(z)}, \tag{4}$$

where

$$\begin{aligned} \mathfrak{U}(z) &= z + \sum_{\nu=2}^{\infty} \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} a_\nu z^\nu, \\ \mathfrak{V}(z) &= b_1 z + \sum_{\nu=2}^{\infty} \frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!} b_\nu z^\nu. \end{aligned} \tag{5}$$

The next theorem expresses the inclusion of class of $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$ and the image of the class $\mathcal{K}^0_{\mathcal{H}}$ under the function given in (3).

Theorem 3.1. *Let $0 \le \alpha, \beta < 1$, $\sigma > \xi \ge 0$, and $\tau \ge 0$. If the inequality*

$$\tau\alpha^3+(6\tau+\sigma)\alpha^2+(6\tau+4\sigma)\alpha+2\sigma(1-e^{-\alpha})+\tau\beta^3+(4\tau+\sigma)\beta^2+2(\tau+\sigma)\beta \le 2(\sigma-\xi) \tag{6}$$

is satisfied, then $\Pi_{\alpha,\beta}(\mathcal{K}^0_{\mathcal{H}}) \subset \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$.

Proof. Assume u and v are of the form (1) with $b_1 = 0$ and $f = u + \overline{v} \in \mathcal{K}^0_{\mathcal{H}}$. It suffices to show that $\Pi_{\alpha,\beta}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions in $\mathbb{U}$ defined as in (5) with $b_1 = 0$.

In light of Lemma 2.5, first we need to verify that $\Phi_1 \le \sigma - \xi$, where

$$\Phi_1 = \sum_{\nu=2}^{\infty} \nu[\sigma+\tau(\nu-1)] \left| \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} a_\nu \right| + \sum_{\nu=2}^{\infty} \nu[\sigma+\tau(\nu-1)] \left| \frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!} b_\nu \right|. \tag{7}$$

Using Lemma 1.1, we get

$$\begin{aligned} \Phi_1 \le \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} [\sigma+\tau(\nu-1)]\nu(\nu+1) \cdot \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} \\ &+ \sum_{\nu=2}^{\infty} [\sigma+\tau(\nu-1)]\nu(\nu-1) \cdot \frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!} \Bigg\} \\ = \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \left[\tau\nu^3 + \sigma\nu^2 + (\sigma-\tau)\nu\right] \cdot \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} \\ &+ \sum_{\nu=2}^{\infty} \left[\tau\nu^3 + (\sigma-2\tau)\nu^2 - (\sigma-\tau)\nu\right] \cdot \frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!} \Bigg\} \\ = \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \big[\tau(\nu-1)(\nu-2)(\nu-3) + (6\tau+\sigma)(\nu-1)(\nu-2) \\ &\quad + (6\tau+4\sigma)(\nu-1) + 2\sigma\big] \cdot \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} \end{aligned}$$

$$+\sum_{\nu=2}^{\infty}\big[\tau(\nu-1)(\nu-2)(\nu-3)+(4\tau+\sigma)(\nu-1)(\nu-2)$$
$$+2(\tau+\sigma)(\nu-1)\big]\cdot\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}.$$

Applying the shift of index and the exponential series expansion,

$$\begin{aligned}\Phi_1 \le \frac{1}{2}\Bigg\{&\tau e^{-\alpha}\alpha^3\sum_{\nu=4}^{\infty}\frac{\alpha^{\nu-4}}{(\nu-4)!}+(6\tau+\sigma)e^{-\alpha}\alpha^2\sum_{\nu=3}^{\infty}\frac{\alpha^{\nu-3}}{(\nu-3)!}\\
&+(6\tau+4\sigma)e^{-\alpha}\alpha\sum_{\nu=2}^{\infty}\frac{\alpha^{\nu-2}}{(\nu-2)!}+2\sigma e^{-\alpha}\sum_{\nu=2}^{\infty}\frac{\alpha^{\nu-1}}{(\nu-1)!}\\
&+\tau e^{-\beta}\beta^3\sum_{\nu=4}^{\infty}\frac{\beta^{\nu-4}}{(\nu-4)!}+(4\tau+\sigma)e^{-\beta}\beta^2\sum_{\nu=3}^{\infty}\frac{\beta^{\nu-3}}{(\nu-3)!}\\
&+2(\tau+\sigma)e^{-\beta}\beta\sum_{\nu=2}^{\infty}\frac{\beta^{\nu-2}}{(\nu-2)!}\Bigg\}\\
=\frac{1}{2}\Bigg\{&\tau\alpha^3+(6\tau+\sigma)\alpha^2+(6\tau+4\sigma)\alpha+2\sigma(1-e^{-\alpha})\\
&+\tau\beta^3+(4\tau+\sigma)\beta^2+2(\tau+\sigma)\beta\Bigg\}.\end{aligned}$$

The last expression is bounded above by $\sigma-\xi$ according to (6). Therefore, the desired result follows. □

In the following, the inclusion between the class of $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$ and the image of the class $\mathcal{S}^{0,*}_{\mathcal{H}}$ under the function given in (3), is presented.

Theorem 3.2. *Let $0\le\alpha,\beta<1$, $\sigma>\xi\ge0$ and $\tau\ge0$. If the inequality*

$$\begin{aligned}&2\tau\alpha^4+(21\tau+2\sigma)\alpha^3+(54\tau+15\sigma)\alpha^2+(30\tau+24\sigma)\alpha\\
&\quad+6\sigma(1-e^{-\alpha})+2\tau\beta^4+(15\tau+2\sigma)\beta^3+(24\tau+9\sigma)\beta^2+6(\tau+\sigma)\beta\\
&\quad\le 6(\sigma-\xi)\end{aligned}\tag{8}$$

is satisfied, then $\Pi_{\alpha,\beta}(\mathcal{S}^{0,*}_{\mathcal{H}})\subset\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$.

Proof. Assume u and v are of the form (1) with $b_1=0$ and $f=u+\overline{v}\in\mathcal{S}^{0,*}_{\mathcal{H}}$. It suffices to show that $\Pi_{\alpha,\beta}(f)=\mathfrak{U}+\overline{\mathfrak{V}}\in\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions in $\mathbb{U}$ defined as in (5) with $b_1=0$.

By Lemma 2.5, we need to verify that $\Phi_1\le\sigma-\xi$, where Φ_1 is given in (7).

Using Lemma 1.2, we have

$$
\begin{aligned}
\Phi_1 \le \frac{1}{6}\Bigg\{ &\sum_{\nu=2}^{\infty}[\sigma+\tau(\nu-1)]\nu(2\nu+1)(\nu+1)\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} \\
&+\sum_{\nu=2}^{\infty}[\sigma+\tau(\nu-1)]\nu(2\nu-1)(\nu-1)\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\} \\
= \frac{1}{6}\Bigg\{ &\sum_{\nu=2}^{\infty}\big[2\tau\nu^4+(2\sigma+\tau)\nu^3+(3\sigma-2\tau)\nu^2+(\sigma-\tau)\nu\big]\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} \\
&+\sum_{\nu=2}^{\infty}\big[2\tau\nu^4+(2\sigma-5\tau)\nu^3+(4\tau-3\sigma)\nu^2+(\sigma-\tau)\nu\big]\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}.
\end{aligned}
$$

Expanding the binomial terms:

$$
\begin{aligned}
\Phi_1 \le \frac{1}{6}\Bigg\{ &\sum_{\nu=2}^{\infty}\Big[2\tau(\nu-1)(\nu-2)(\nu-3)(\nu-4)+(21\tau+2\sigma)(\nu-1)(\nu-2)(\nu-3) \\
&+(54\tau+15\sigma)(\nu-1)(\nu-2)+(30\tau+24\sigma)(\nu-1)+6\sigma\Big]\cdot\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} \\
&+\sum_{\nu=2}^{\infty}\Big[2\tau(\nu-1)(\nu-2)(\nu-3)(\nu-4)+(15\tau+2\sigma)(\nu-1)(\nu-2)(\nu-3) \\
&+(24\tau+9\sigma)(\nu-1)(\nu-2)+6(\tau+\sigma)(\nu-1)\Big]\cdot\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}
\end{aligned}
$$

Rearranging sums via exponential generating functions:

$$
\begin{aligned}
\Phi_1 \le \frac{1}{6}\Bigg\{ &2\tau\, e^{-\alpha}\alpha^4\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}+(21\tau+2\sigma)\, e^{-\alpha}\alpha^3\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!} \\
&+(54\tau+15\sigma)\, e^{-\alpha}\alpha^2\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}+(30\tau+24\sigma)\, e^{-\alpha}\alpha\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!} \\
&+6\sigma\, e^{-\alpha}\sum_{\nu=1}^{\infty}\frac{\alpha^\nu}{\nu!} \\
&+2\tau\, e^{-\beta}\beta^4\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}+(15\tau+2\sigma)\, e^{-\beta}\beta^3\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!} \\
&+(24\tau+9\sigma)\, e^{-\beta}\beta^2\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}+6(\tau+\sigma)\, e^{-\beta}\beta\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}\Bigg\}
\end{aligned}
$$

$$= \frac{1}{6}\Big\{2\tau\alpha^4 + (21\tau + 2\sigma)\alpha^3 + (54\tau + 15\sigma)\alpha^2 + (30\tau + 24\sigma)\alpha + 6\sigma + 2\tau\beta^4 + (15\tau + 2\sigma)\beta^3 + (24\tau + 9\sigma)\beta^2 + 6(\tau + \sigma)\beta\Big\}.$$

Using inequality (8), this quantity is bounded above by $\sigma - \xi$, and the proof is complete. □

Theorem 3.3. *Let $0 \le \alpha, \beta < 1$, $\sigma > \xi \ge 0$, and $\tau \ge 0$. If the inequality*

$$2e^{-\alpha} + e^{-\beta} \ge 2 \tag{9}$$

is satisfied then $\Pi_{\alpha,\beta}(\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)) \subset \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$.

Proof. Assume u and v are of the form (1) with $b_1 = 0$ and $f = u + \overline{v} \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$. It suffices to show that $\Pi_{\alpha,\beta}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions in $\mathbb{U}$ defined as in (5) with $b_1 = 0$.

By Lemma 2.5, it is enough to verify that $\Phi_1 \le \sigma - \xi$, where Φ_1 is given by (7).

Using Lemma 2.3 and Lemma 2.4, we estimate

$$\begin{aligned}\Phi_1 &\le 2(\sigma - \xi)\sum_{\nu=2}^{\infty}\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} + (\sigma - \xi)\sum_{\nu=2}^{\infty}\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\\ &= 2(\sigma - \xi)\big(1 - e^{-\alpha}\big) + (\sigma - \xi)\big(1 - e^{-\beta}\big).\end{aligned}$$

By inequality (9), we have

$$2(1 - e^{-\alpha}) + (1 - e^{-\beta}) \le 1,$$

which implies

$$\Phi_1 \le \sigma - \xi.$$

Therefore, $\Pi_{\alpha,\beta}(f) \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, completing the proof. □

4. Applications to Pascal Distribution Series

The Pascal distribution series, as well as its effect on various subclasses of analytic and harmonic single-valued functions such as the Poisson distribution series, has given rise to some important work in Geometric Function Theory (see, e.g., [4, 5, 8, 9, 10, 17, 16]). In this section, we apply a convolution operator associated with the Pascal distribution series to obtain inclusion relations and, drawing inspiration from previous works using hypergeometric or special function based convolution operators, derive several new results concerning the class $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$. The theorems given below provide inclusion relations involving the classes $\mathcal{K}^0_{\mathcal{H}}$, $\mathcal{S}^{0,*}_{\mathcal{H}}$, and $\mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, based on the coefficient bounds obtained in the previous section.

The Pascal distribution with parameters $\kappa \ge 1$ and $0 \le \theta \le 1$ has the probability mass function,

$$\mathbb{P}(\mathcal{X} = \nu) = \binom{\nu + \kappa - 1}{\kappa - 1}\theta^{\nu}(1-\theta)^{\kappa}, \quad \nu \in \{0, 1, 2, \ldots\}. \tag{10}$$

This yields the power series,

$$\mathbb{P}_\theta^\kappa(z) = z + \sum_{\nu=2}^{\infty} \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^\kappa z^\nu, \tag{11}$$

which has an infinite radius of convergence. Based on this series, Yaşar et al. [16] defined the convolution operator,

$$\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(f)(z) = \mathbb{P}_\theta^\kappa(z) * u(z) + \overline{\mathbb{P}_\phi^\mu(z) * v(z)} = \mathfrak{U}(z) + \overline{\mathfrak{V}(z)}, \tag{12}$$

where

$$\begin{aligned} \mathfrak{U}(z) &= z + \sum_{\nu=2}^{\infty} \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^\kappa a_\nu z^\nu, \\ \mathfrak{V}(z) &= b_1 z + \sum_{\nu=2}^{\infty} \binom{\nu+\mu-2}{\mu-1} \phi^{\nu-1}(1-\phi)^\mu b_\nu z^\nu, \end{aligned} \tag{13}$$

and "$*$" denotes the convolution (Hadamard product) of power series.

Lemma 4.1. *For integer $k \geq 1$ and $0 \leq \theta < 1$, the following generating function identities hold:*

$$\begin{aligned} \sum_{\nu=0}^{\infty} \binom{\nu+k-1}{k-1} \theta^\nu &= \frac{1}{(1-\theta)^k}, \\ \sum_{\nu=0}^{\infty} \binom{\nu+k-2}{k-2} \theta^\nu &= \frac{1}{(1-\theta)^{k-1}}, \\ \sum_{\nu=0}^{\infty} \binom{\nu+k}{k} \theta^\nu &= \frac{1}{(1-\theta)^{k+1}}, \\ \sum_{\nu=0}^{\infty} \binom{\nu+k+1}{k+1} \theta^\nu &= \frac{1}{(1-\theta)^{k+2}}, \\ \sum_{\nu=0}^{\infty} \binom{\nu+k+2}{k+2} \theta^\nu &= \frac{1}{(1-\theta)^{k+3}}. \end{aligned}$$

The theorem below determines the inclusion between the class of $\mathcal{A}_{\mathcal{H}}^0(\sigma,\tau,\xi)$ and the image of the class $\mathcal{K}_{\mathcal{H}}^0$ under the function given in (11).

Theorem 4.2. *Let $0 \leq \theta, \phi < 1$, $\sigma > \xi \geq 0$ and $\tau \geq 0$. If the inequality*

$$\begin{aligned} &\frac{\tau\kappa(\kappa+1)(\kappa+2)\theta^3}{(1-\theta)^3} + \frac{(6\tau+\sigma)\kappa(\kappa+1)\theta^2}{(1-\theta)^2} + \frac{(6\tau+4\sigma)\kappa\theta}{1-\theta} + 2\sigma\big[1-(1-\theta)^\kappa\big] \\ &\quad + \frac{\tau\mu(\mu+1)(\mu+2)\phi^3}{(1-\phi)^3} + \frac{(4\tau+\sigma)\mu(\mu+1)\phi^2}{(1-\phi)^2} + \frac{2(\tau+\sigma)\mu\phi}{1-\phi} \leq 2(\sigma-\xi). \end{aligned} \tag{14}$$

then $\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(\mathcal{K}_{\mathcal{H}}^0) \subset \mathcal{A}_{\mathcal{H}}^0(\sigma,\tau,\xi)$.

Proof. Assume u and v are of the form (1) while $b_1 = 0$ and $f = u + \overline{v} \in K^0_{\mathcal{H}}$. It is enough to show that $\mathbb{P}^{\kappa,\mu}_{\theta,\phi}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{A}^0_{\mathcal{H}}(\sigma,\tau,\xi)$, where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions on $\mathbb{U}$ defined as in (5) when $b_1 = 0$. In light of Lemma 2.5, first we will see that $\Phi_2 \le \sigma - \xi$, where

$$\begin{aligned}\Phi_2 = &\sum_{\nu=2}^{\infty} \nu\big[\sigma + \tau(\nu-1)\big] \left| \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^{\kappa} a_\nu \right| \\ &+ \sum_{\nu=2}^{\infty} \nu\big[\sigma + \tau(\nu-1)\big] \left| \binom{\nu+\mu-2}{\mu-1} \phi^{\nu-1}(1-\phi)^{\mu} b_\nu \right|.\end{aligned} \tag{15}$$

Considering Lemma 1.1 and Lemma 4.1, we get

$$\begin{aligned}\Phi_2 \le \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} [\sigma+\tau(\nu-1)]\nu(\nu+1)\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa} \\ &+ \sum_{\nu=2}^{\infty} [\sigma+\tau(\nu-1)]\nu(\nu-1)\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\mu} \Bigg\}\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \left[\tau\nu^3 + \sigma\nu^2 + (\sigma-\tau)\nu\right] \binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa} \\ &+ \sum_{\nu=2}^{\infty} \left[\tau\nu^3 + (\sigma-2\tau)\nu^2 - (\sigma-\tau)\nu\right] \binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\mu} \Bigg\}\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \Big[\tau(\nu-1)(\nu-2)(\nu-3) + (6\tau+\sigma)(\nu-1)(\nu-2) \\ &+ (6\tau+4\sigma)(\nu-1) + 2\sigma\Big] \binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa} \\ &+ \sum_{\nu=2}^{\infty} \Big[\tau(\nu-1)(\nu-2)(\nu-3) + (4\tau+\sigma)(\nu-1)(\nu-2) \\ &+ 2(\tau+\sigma)(\nu-1)\Big] \binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\mu} \Bigg\}\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}\Bigg\{ &\tau\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^3 \sum_{\nu=4}^{\infty} \binom{\nu+\kappa+2}{\kappa+2}\theta^{\nu-4} \\ &+ (6\tau+\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^2 \sum_{\nu=3}^{\infty} \binom{\nu+\kappa-2}{\kappa+1}\theta^{\nu-3} \\ &+ (6\tau+4\sigma)\kappa(1-\theta)^{\kappa}\theta \sum_{\nu=2}^{\infty} \binom{\nu+\kappa-2}{\kappa}\theta^{\nu-2}\end{aligned}$$

$$+ 2\sigma(1-\theta)^{\kappa} \sum_{\nu=1}^{\infty} \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}$$
$$+ \tau\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3} \sum_{\nu=4}^{\infty} \binom{\nu+\mu+2}{\mu+2} \phi^{\nu-4}$$
$$+ (4\tau+\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2} \sum_{\nu=3}^{\infty} \binom{\nu+\mu-2}{\mu+1} \phi^{\nu-3}$$
$$+ 2(\tau+\sigma)\mu(1-\phi)^{\mu}\phi \sum_{\nu=2}^{\infty} \binom{\nu+\mu-2}{\mu} \phi^{\nu-2} \Bigg\}$$

$$= \frac{1}{2}\Bigg\{ \tau\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3} \sum_{\nu=0}^{\infty} \binom{\nu+\kappa+2}{\kappa+2} \theta^{\nu}$$
$$+ (6\tau+\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2} \sum_{\nu=0}^{\infty} \binom{\nu+\kappa+1}{\kappa+1} \theta^{\nu}$$
$$+ (6\tau+4\sigma)\kappa(1-\theta)^{\kappa}\theta \sum_{\nu=0}^{\infty} \binom{\nu+\kappa}{\kappa} \theta^{\nu}$$
$$+ 2\sigma(1-\theta)^{\kappa} \sum_{\nu=0}^{\infty} \binom{\nu+\kappa-1}{\kappa-1} \theta^{\nu} - 2\sigma(1-\theta)^{\kappa}$$
$$+ \tau\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3} \sum_{\nu=0}^{\infty} \binom{\nu+\mu+2}{\mu+2} \phi^{\nu}$$
$$+ (4\tau+\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2} \sum_{\nu=0}^{\infty} \binom{\nu+\mu+1}{\mu+1} \phi^{\nu}$$
$$+ 2(\tau+\sigma)\mu(1-\phi)^{\mu}\phi \sum_{\nu=0}^{\infty} \binom{\nu+\mu}{\mu} \phi^{\nu} \Bigg\}$$

$$= \frac{1}{2}\Bigg\{ \frac{\tau\kappa(\kappa+1)(\kappa+2)\theta^{3}}{(1-\theta)^{3}} + \frac{(6\tau+\sigma)\kappa(\kappa+1)\theta^{2}}{(1-\theta)^{2}} + \frac{(6\tau+4\sigma)\kappa\theta}{1-\theta}$$
$$+ 2\sigma - 2\sigma(1-\theta)^{\kappa} + \frac{\tau\mu(\mu+1)(\mu+2)\phi^{3}}{(1-\phi)^{3}} + \frac{(4\tau+\sigma)\mu(\mu+1)\phi^{2}}{(1-\phi)^{2}} + \frac{2(\tau+\sigma)\mu\phi}{1-\phi} \Bigg\}$$
$$\leq \sigma - \xi.$$

□

Finally, the last inclusion appears between the class of $\mathcal{A}^{0}_{\mathcal{H}}(\sigma,\tau,\xi)$ and the image of the class $\mathcal{S}^{0,*}_{\mathcal{H}}$ under the function given in (11), by the next theorem.

Theorem 4.3. *Let $0 \leq \theta, \phi < 1$, $\sigma > \xi \geq 0$ and $\tau \geq 0$. If the inequality*

$$\begin{aligned}
&\frac{2\tau\kappa(\kappa+1)(\kappa+2)(\kappa+3)\theta^4}{(1-\theta)^4} + \frac{(21\tau+2\sigma)\kappa(\kappa+1)(\kappa+2)\theta^3}{(1-\theta)^3} \\
&+ \frac{(54\tau+15\sigma)\kappa(\kappa+1)\theta^2}{(1-\theta)^2} + \frac{(30\tau+24\sigma)\kappa\theta}{(1-\theta)} + 6\sigma - 6\sigma(1-\theta)^\kappa \\
&+ \frac{2\tau\mu(\mu+1)(\mu+2)(\mu+3)\phi^4}{(1-\phi)^4} + \frac{(15\tau+2\sigma)\mu(\mu+1)(\mu+2)\phi^3}{(1-\phi)^3} \\
&+ \frac{(24\tau+9\sigma)\mu(\mu+1)\phi^2}{(1-\phi)^2} + \frac{6(\tau+\sigma)\mu\phi}{(1-\phi)} \leq 12(\sigma-\xi)
\end{aligned} \tag{16}$$

is satisfied, then $\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(\mathcal{S}_{\mathcal{H}}^{0,*}) \subset \mathcal{A}_{\mathcal{H}}^{0}(\sigma,\tau,\xi)$.

Proof. Assume u and v are of the form 1 while $b_1 = 0$ and $f = u + \overline{v} \in \mathcal{S}_{\mathcal{H}}^{0,*}$. It is enough to show that $\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{A}_{\mathcal{H}}^{0}(\sigma,\tau,\xi)$, where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions in $\mathbb{U}$ defined in (5) when $b_1 = 0$. In light of Lemma 2.5, first we will see that $\Phi_2 \leq \sigma - \xi$, where Φ_2 is defined as in (15). Considering Lemma 1.2 and Lemma 4.1, we get

$$\begin{aligned}
\Phi_2 \leq \frac{1}{6}\Bigg\{&\sum_{\nu=2}^{\infty}\big[\sigma+\tau(\nu-1)\big]\nu(2\nu+1)(\nu+1)\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^\kappa \\
&+\sum_{\nu=2}^{\infty}\big[\sigma+\tau(\nu-1)\big]\nu(2\nu-1)(\nu-1)\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^\mu\Bigg\}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{6}\Bigg\{&\sum_{\nu=2}^{\infty}\big[2\tau\nu^4+(2\sigma+\tau)\nu^3+(3\sigma-2\tau)\nu^2+(\sigma-\tau)\nu\big]\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^\kappa \\
&+\sum_{\nu=2}^{\infty}\big[2\tau\nu^4+(2\sigma-5\tau)\nu^3+(4\tau-3\sigma)\nu^2+(\sigma-\tau)\nu\big]\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^\mu\Bigg\}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{6}\Bigg\{&\sum_{\nu=2}^{\infty}\big[2\tau(\nu-1)(\nu-2)(\nu-3)(\nu-4)+(21\tau+2\sigma)(\nu-1)(\nu-2)(\nu-3) \\
&+(54\tau+15\sigma)(\nu-1)(\nu-2)+(30\tau+24\sigma)(\nu-1)+6\sigma\big]\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^\kappa \\
&+\sum_{\nu=2}^{\infty}\big[2\tau(\nu-1)(\nu-2)(\nu-3)(\nu-4)+(15\tau+2\sigma)(\nu-1)(\nu-2)(\nu-3) \\
&+(24\tau+9\sigma)(\nu-1)(\nu-2)+6(\tau+\sigma)(\nu-1)\big]\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^\mu\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\Bigg\{2\tau\kappa(\kappa+1)(\kappa+2)(\kappa+3)(1-\theta)^{\kappa}\theta^{4}\sum_{\nu=5}^{\infty}\binom{\nu+\kappa-2}{\kappa+3}\theta^{\nu-5}\\
&\quad+(21\tau+2\sigma)\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3}\sum_{\nu=4}^{\infty}\binom{\nu+\kappa-2}{\kappa+2}\theta^{\nu-4}\\
&\quad+(54\tau+15\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2}\sum_{\nu=3}^{\infty}\binom{\nu+\kappa-2}{\kappa+1}\theta^{\nu-3}\\
&\quad+(30\tau+24\sigma)\kappa(1-\theta)^{\kappa}\theta\sum_{\nu=2}^{\infty}\binom{\nu+\kappa-2}{\kappa}\theta^{\nu-2}\\
&\quad+6\sigma(1-\theta)^{\kappa}\sum_{\nu=1}^{\infty}\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{6}\Bigg\{2\tau\mu(\mu+1)(\mu+2)(\mu+3)(1-\phi)^{\mu}\phi^{4}\sum_{\nu=5}^{\infty}\binom{\nu+\mu-2}{\mu+3}\phi^{\nu-5}\\
&+(15\tau+2\sigma)\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3}\sum_{\nu=4}^{\infty}\binom{\nu+\mu-2}{\mu+2}\phi^{\nu-4}\\
&+(24\tau+9\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2}\sum_{\nu=3}^{\infty}\binom{\nu+\mu-2}{\mu+1}\phi^{\nu-3}\\
&+6(\tau+\sigma)\mu(1-\phi)^{\mu}\phi\sum_{\nu=2}^{\infty}\binom{\nu+\mu-2}{\mu}\phi^{\nu-2}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{6}\Bigg\{2\tau\kappa(\kappa+1)(\kappa+2)(\kappa+3)(1-\theta)^{\kappa}\theta^{4}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+3}{\kappa+3}\theta^{\nu}\\
&\quad+(21\tau+2\sigma)\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+2}{\kappa+2}\theta^{\nu}\\
&\quad+(54\tau+15\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+1}{\kappa+1}\theta^{\nu}\\
&\quad+(30\tau+24\sigma)\kappa(1-\theta)^{\kappa}\theta\sum_{\nu=0}^{\infty}\binom{\nu+\kappa}{\kappa}\theta^{\nu}\\
&\quad+6\sigma(1-\theta)^{\kappa}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa-1}{\kappa-1}\theta^{\nu}\Bigg\}
\end{aligned}
$$

$$+\frac{1}{6}\Bigg\{2\tau\mu(\mu+1)(\mu+2)(\mu+3)(1-\phi)^{\mu}\phi^{4}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+3}{\mu+3}\phi^{\nu}$$
$$+(15\tau+2\sigma)\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+2}{\mu+2}\phi^{\nu}$$
$$+(24\tau+9\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+1}{\mu+1}\phi^{\nu}$$
$$+6(\tau+\sigma)\mu(1-\phi)^{\mu}\phi\sum_{\nu=0}^{\infty}\binom{\nu+\mu}{\mu}\phi^{\nu}\Bigg\}$$

Finally, by the binomial series expansion, the above equals

$$=\frac{1}{6}\Bigg\{\frac{2\tau\kappa(\kappa+1)(\kappa+2)(\kappa+3)\theta^{4}}{(1-\theta)^{4}}+\frac{(21\tau+2\sigma)\kappa(\kappa+1)(\kappa+2)\theta^{3}}{(1-\theta)^{3}}$$
$$+\frac{(54\tau+15\sigma)\kappa(\kappa+1)\theta^{2}}{(1-\theta)^{2}}+\frac{(30\tau+24\sigma)\kappa\theta}{(1-\theta)}+6\sigma-6\sigma(1-\theta)^{\kappa}$$
$$+\frac{2\tau\mu(\mu+1)(\mu+2)(\mu+3)\phi^{4}}{(1-\phi)^{4}}+\frac{(15\tau+2\sigma)\mu(\mu+1)(\mu+2)\phi^{3}}{(1-\phi)^{3}}$$
$$+\frac{(24\tau+9\sigma)\mu(\mu+1)\phi^{2}}{(1-\phi)^{2}}+\frac{6(\tau+\sigma)\mu\phi}{(1-\phi)}\Bigg\}$$

Last algebraic expression is bounded above by $2(\sigma-\xi)$ by (16). Hence, the proof is complete. □

Theorem 4.4. *Let $0\le\theta,\phi<1$, $\sigma>\xi\ge 0$ and $\tau\ge 0$. If the inequality*

$$2(1-\theta)^{\kappa}+(1-\phi)^{\mu}\ge 2 \tag{17}$$

is satisfied then $\mathbb{P}^{\kappa,\mu}_{\theta,\phi}(\mathcal{A}^{0}_{\mathcal{H}}(\sigma,\tau,\xi))\subset\mathcal{A}^{0}_{\mathcal{H}}(\sigma,\tau,\xi)$.

Proof. Let $f=u+\overline{v}\in\mathcal{A}^{0}_{\mathcal{H}}(\sigma,\tau,\xi)$ where u and v are of the form 1 with $b_1=0$. We need to show that $\mathbb{P}^{\kappa,\mu}_{\theta,\phi}(f)=\mathfrak{U}+\overline{\mathfrak{V}}\in\mathcal{A}^{0}_{\mathcal{H}}(\sigma,\tau,\xi)$, where $\mathfrak{U}$ and $\mathfrak{V}$ defined by (5) with $b_1=0$ are analytic functions in $\mathbb{U}$. In view of Lemma 2.5, we need to prove that $\Phi_2\le\sigma-\xi$, where Φ_2 is defined as in Equation (15). Considering Lemma 2.3 and Lemma 2.4 we get

$$\Phi_2\le 2(\sigma-\xi)\sum_{\nu=2}^{\infty}\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa}+(\sigma-\xi)\sum_{\nu=2}^{\infty}\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\mu}$$
$$=2(\sigma-\xi)(1-\theta)^{\kappa}\left\{\sum_{\nu=1}^{\infty}\binom{\nu+\kappa-1}{\kappa-1}\theta^{\nu}\right\}+(\sigma-\xi)(1-\phi)^{\mu}\sum_{\nu=1}^{\infty}\binom{\nu+\mu-1}{\mu-1}\phi^{\nu}$$
$$=2(\sigma-\xi)(1-(1-\theta)^{\kappa})+(\sigma-\xi)(1-(1-\phi)^{\mu})$$
$$=(\sigma-\xi)\left[3-2(1-\theta)^{\kappa}-(1-\phi)^{\mu}\right]$$

The last expression is bounded above by $\sigma - \xi$ by the given condition (17). Thus, the proof of Theorem 4.4 is complete. □

Acknowledgements

The authors would like to thank the referee for careful reading.

References

[1] Ş. Altınkaya and S. Yalçın, *Poisson distribution series for certain subclasses of starlike functions with negative coefficients*, An. Univ. Oradea Fasc. Mat., **24(2)**(2017), 5–8.

[2] Ş. Altınkaya and S. Yalçın, *Poisson distribution series for analytic univalent functions*, Complex Anal. Oper. Theory, **12(5)**(2018), 1315–1319. DOI: 10.1007/s11785-018-0764-y

[3] Y. Avcı and E. Złotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **44**(1990), 1–7.

[4] S. Çakmak, S. Yalçın, and Ş. Altınkaya, *An application of the power series distribution for certain analytic univalent function classes*, Lib. Math. (N.S.), **39(1)**(2019), 95–101. DOI: 10.14510/lm-ns.v39i1.1441

[5] S. Çakmak, S. Yalçın, and Ş. Altınkaya, *Some connections between various classes of analytic functions associated with the power series distribution*, Sakarya University Journal of Science, **23(5)**(2019), 982–985. DOI: 10.16984/saufenbilder.552957

[6] S. Çakmak, E. Yaşar, and S. Yalçın, *Some basic properties of a subclass of close-to-convex harmonic mappings*, TWMS J. Pure Appl. Math., **15(2)**(2024), 163–173. DOI: 10.30546/2219-1259.15.2.2024.01163

[7] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., **9**(1984), 3–25. DOI: 10.5186/aasfm.1984.0905

[8] S. M. El-Deeb, T. Bulboacă, and J. Dziok, *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Math. J., **59(2)**(2019), 301–314. DOI: 10.5666/KMJ.2019.59.2.301

[9] B. A. Frasin, G. Murugusundaramoorthy, and S. Yalçın, *Subclass of analytic functions associated with Pascal distribution series*, Bull. Transilv. Univ. Braşov Ser. III, **13(62)**(2020), 521–528. DOI: 10.31926/but.mif.2020.13.62.2.11

[10] M. Gangadharan, S. Yalçın, E. Yaşar, and S. Çakmak, *An application of the power series distribution for univalent function classes with positive coefficients*, Int. J. Nonlinear Anal. Appl., **14(5)**(2023), 259–265. DOI: 10.22075/ijnaa.2022.22768.2414

[11] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., **2014(1)**(2014), 984135. DOI: 10.1155/2014/984135

[12] S. Porwal, Ş. Altınkaya, and S. Yalçın, *The Poisson distribution series of general subclasses of univalent functions*, Acta Univ. Apulensis Math. Inform., **58**(2019), 45–52; DOI: 10.17114/j.aua.2019.58.04

[13] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat., **27(5-6)**(2016), 1021–1027. DOI: 10.1007/s13370-016-0398-z

[14] S. Porwal and D. Srivastava, *Some connections between various subclasses of planar harmonic mappings involving Poisson distribution series*, Electron. J. Math. Anal. Appl., **6(2)**(2018), 163–171.

[15] H. Silverman and E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math., **28(2)**(1999), 275–284.

[16] E. Yaşar, S. Çakmak, S. Yalçın, and Ş. Altınkaya, *Some connections between various subclasses of harmonic univalent functions involving Pascal distribution series*, Acta Univ. Apulensis Math. Inform., **66**(2021), 133–147.

[17] S. Yalçın, G. Murugusundaramoorthy, and K. Vijaya, *Inclusion results on subclasses of harmonic univalent functions associated with Pascal distribution series*, Palest. J. Math., **11(3)**(2022), 267–275.