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Generalized Convergence of Sequences of Fuzzy Numbers by Means of Modulus Functions

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ABSTRACT

In this paper, we extend the concepts of statistical convergence and strong summability for the sequences of fuzzy numbers using modulus functions. By introducing appropriate conditions on the modulus functions, we generalize and refine existing notions of convergence within the fuzzy setting. Additionally, we establish several interrelationships between these extended concepts, thereby contributing to the deeper understanding of summability and convergence behavior in the sequences of fuzzy numbers.

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1 Introduction

The idea of statistical convergence, originally formulated for sequences of real numbers, was introduced independently by Fast [1] and Steinhaus [2] in the same year. This foundational concept offered a broader framework than classical convergence by allowing the convergence of sequences in

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a statistical sense, even when they fail to converge in the usual sense. Schoenberg [3] later revisited the idea and independently contributed to its further theoretical development.

Building on the above foundation, statistical convergence was generalized to the setting of fuzzy numbers in [4], where the authors extended the classical notions to accommodate the uncertainty inherent in fuzzy sequences. In continuation of this line of research, statistical summability methods were explored in the fuzzy context by the authors of [5], providing valuable insights into approximation processes and convergence behavior in fuzzy environments.

Moreover, the work in [6] systematically investigated the interplay between various types of convergence and summability for sequences of fuzzy numbers, highlighting both similarities and distinctions between these concepts. For a broader and more comprehensive overview of the development and applications of statistical convergence and summability in fuzzy settings, readers may consult the surveys and discussions presented in [7–10].

A fuzzy set $\mathfrak{w}$ on the real line $\mathbb{R}$ is called a fuzzy number if it meets certain essential properties. First, there must exist at least one point $\mathfrak{v} \in \mathbb{R}$ at which the membership function $\mathfrak{w}$ attains the value 1. Additionally, for any two real numbers $\mathfrak{v}$ and $\mathfrak{y}$, and for every $\lambda \in [0, 1]$, the membership grade at the convex combination $\lambda\mathfrak{v} + (1-\lambda)\mathfrak{y}$ should not be less than the minimum of the grades at $\mathfrak{v}$ and $\mathfrak{y}$. The function $\mathfrak{w}$ is also required to be upper semicontinuous across its domain. Moreover, the set of all real numbers where $\mathfrak{w}(\mathfrak{v})$ is strictly positive must be bounded and closed, ensuring that its support forms a compact subset of $\mathbb{R}$.

The $\sigma-$level set corresponding to a fuzzy number $\mathfrak{w}$, represented as $[\mathfrak{w}]^{\sigma}$, is given by:

$$[\mathfrak{w}]^{\sigma} = \begin{cases} \operatorname{supp}\mathfrak{w}, & \sigma = 0, \\ \{\mathfrak{v} \in \mathbb{R} : \mathfrak{w}(\mathfrak{v}) \geq \sigma\}, & \sigma\ (0, 1]. \end{cases}$$

It is evident that $\mathfrak{w}$ qualifies as a fuzzy number if and only if $[\mathfrak{w}]^{\sigma}$ forms a closed interval for every $\sigma \in [0, 1]$ and $[\mathfrak{w}]^{1} \neq \varnothing$.

A real number t can be expressed as a fuzzy number $\bar{t}$, where its membership function is defined as:

$$\bar{t}(\mathfrak{v}) = \begin{cases} 0, & \mathfrak{v} \neq t, \\ 1, & \mathfrak{v} = t. \end{cases}$$

In this study, the notation $\mathcal{F}(\mathbb{R})$ represents the collection of all fuzzy numbers. Specifically, if $\mathfrak{w} \in \mathcal{F}(\mathbb{R})$, then $\mathfrak{w}$ is referred to as a fuzzy number.

To compute the distance between two fuzzy numbers $\mathfrak{w}$ and $\mathfrak{w}^*$, the following metric is utilized:

$$d(\mathfrak{w}, \mathfrak{w}^*) = \sup_{\sigma\in[0,1]} d_H\left([\mathfrak{w}]^{\sigma}, [\mathfrak{w}^*]^{\sigma}\right),$$

where, d_H represents the Hausdorff metric, given as:

$$d_H\left([\mathfrak{w}]^{\sigma}, [\mathfrak{w}^*]^{\sigma}\right) = \max\left\{\left|\underline{\mathfrak{w}}^{\sigma} - \underline{\mathfrak{w}}^{*\sigma}\right|, \left|\overline{\mathfrak{w}}^{\sigma} - \overline{\mathfrak{w}^*}^{\sigma}\right|\right\}.$$

It has been established that d is a valid metric on $\mathcal{F}(\mathbb{R})$, and thus the space $(\mathcal{F}(\mathbb{R}), d)$ forms a complete metric space, as demonstrated in [11].

A sequence $(\mathfrak{D}_k)$ of fuzzy numbers is said to converge to a fuzzy number $\mathfrak{D}_0$ if for each $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that $d(\mathfrak{D}_k, \mathfrak{D}_0) < \varepsilon$ whenever $k > k_0$. Throughout the study, $\mathfrak{c}(\mathcal{F}(\mathbb{R}))$ denotes the space of all convergent sequences of fuzzy numbers.

A sequence $(\mathfrak{D}_k)$ of fuzzy numbers is called statistically convergent to a fuzzy number $\mathfrak{D}_0$ (see [4]) if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : d(\mathfrak{D}_k, \mathfrak{D}_0) \ge \varepsilon\}| = 0.$$

Colak [12] introduced a generalized notion of density for subsets of natural numbers, referred to as the ρ-density, where $\rho \in (0, 1]$. Given a subset $\Upsilon \subset \mathbb{N}$, the ρ-density of Υ, denoted by $\delta^{\rho}(\Upsilon)$, is defined (when the limit exists) by

$$\delta^{\rho}(\Upsilon) = \lim_{n\to\infty} \frac{1}{n^{\rho}} |\{a \in \Upsilon : a \le n\}|,$$

where $|\{a \in \Upsilon : a \le n\}|$ denotes the number of elements in Υ that do not exceed n.

Let $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ be two non-decreasing sequences of positive real numbers such that $\alpha_n \le \beta_n$ for all $n \in \mathbb{N}$ and $\beta_n - \alpha_n \to \infty$ as $n \to \infty$. Let $\Upsilon \subset \mathbb{N}$. Then, the number $\delta_{\alpha\beta}(\Upsilon)$, called the $\alpha\beta$-density of the set Υ, is defined by

$$\delta_{\alpha\beta}(\Upsilon) = \lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)} |\{a : a \in [\alpha_n, \beta_n] \cap \Upsilon\}|,$$

whenever the limit exists (see [13]). For more extensive treatments of the concept of statistical convergence, see [14–17].

In [18], the concept of a modulus function was introduced. A function φ is called a modulus function (or simply, a modulus) if it satisfies the following conditions:

1. $\varphi(\mathfrak{t}) = 0 \Leftrightarrow \mathfrak{t} = 0$,
2. $\varphi(\mathfrak{t}_1 + \mathfrak{t}_2) \le \varphi(\mathfrak{t}_1) + \varphi(\mathfrak{t}_2)$ for every $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, \infty)$,
3. φ is increasing,
4. φ is continuous from the right at 0.

These conditions imply that a modulus function must be continuous on the set $[0, \infty)$. Such a function can be either bounded or unbounded. For instance, $\varphi(\mathfrak{t}) = \mathfrak{t}^r$ for $r \in (0, 1]$, defines an unbounded modulus, whereas $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t}+1}$ defines a bounded modulus. Several researchers have employed modulus functions to define and investigate various sequence spaces, one may refer to [19–25].

While statistical convergence and strong summability have been studied in various forms for sequences of fuzzy numbers, the integration of modulus functions into this framework remains underdeveloped. A unified approach that extends these concepts, reveals their interrelationships, and refines them under adaptable conditions is still lacking. Such a framework is essential for broadening the theoretical scope and enabling richer applications of convergence theory in fuzzy analysis and contexts involving uncertainty.

2 Main Results

Definition 1: *Let φ be any unbounded modulus and $\rho \in (0,1]$. Then, a sequence $(\mathfrak{D}_k)$ of fuzzy numbers is called $\varphi - \alpha\beta$-statistically convergent of order ρ (or, $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$-convergent) to a fuzzy number $\mathfrak{D}_0$ if*

$$\lim_{n\to\infty} \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{D}_k, \mathfrak{D}_0\right) \geq \varepsilon\right\}\right|\right) = 0$$

for each $\varepsilon > 0$. In this case, we write $\mathfrak{D}_k \to \mathfrak{D}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$. The notation $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$ stands for the set of all $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$-convergent sequences.

Definition 2: *Let φ be any modulus and $\rho \in (0,1]$. Then, a sequence $(\mathfrak{D}_k)$ of fuzzy numbers is called $\varphi - \alpha\beta$-strongly summable of order ρ (or, strongly $\mathcal{N}^{\rho}_{\alpha\beta}(\varphi)$-summable) to a fuzzy number $\mathfrak{D}_0$ if*

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^{\rho}} \sum_{k\in[\alpha_n,\beta_n]} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right) = 0.$$

In this case, we write $\mathfrak{D}_k \to \mathfrak{D}_0 \left[\mathcal{N}^{\rho}_{\alpha\beta}(\varphi)\right]$.

Note that a modulus function is not required to be unbounded in Definition 2.

Theorem 1: *Let φ be any unbounded modulus and $\rho \in (0,1]$. Also, let $(\mathfrak{D}_k)$ and $(\mathfrak{E}_k)$ be sequences of fuzzy numbers such that $\mathfrak{D}_k \to \mathfrak{D}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$ and $\mathfrak{E}_k \to \mathfrak{E}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$. Then,*

1. $\mathfrak{a}\mathfrak{D}_k \to \mathfrak{a}\mathfrak{D}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$ *for every* $\mathfrak{a} \in \mathbb{R}$.
2. $(\mathfrak{D}_k + \mathfrak{E}_k) \to (\mathfrak{D}_0 + \mathfrak{E}_0) \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$.

Proof: Part (1). In case $\mathfrak{a} = 0$, it is clear. We assume that $\mathfrak{a} \neq 0$. Then, for every $\varepsilon > 0$,

$$\frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{a}\mathfrak{D}_k, \mathfrak{a}\mathfrak{D}_0\right) \geq \varepsilon\right\}\right|\right)$$
$$= \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{D}_k, \mathfrak{D}_0\right) \geq \frac{\varepsilon}{|\mathfrak{a}|}\right\}\right|\right).$$

Since $\mathfrak{D}_k \to \mathfrak{D}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$, we get

$$\lim_{n\to\infty} \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{a}\mathfrak{D}_k, \mathfrak{a}\mathfrak{D}_0\right) \geq \varepsilon\right\}\right|\right) = 0.$$

Therefore, $\mathfrak{a}\mathfrak{D}_k \to \mathfrak{a}\mathfrak{D}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$.

Part (2). For every $\varepsilon > 0$, we have

$$\frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{D}_k + \mathfrak{E}_k, \mathfrak{D}_0 + \mathfrak{E}_0\right) \geq \varepsilon\right\}\right|\right)$$
$$\leq \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{D}_k, \mathfrak{D}_0\right) \geq \frac{\varepsilon}{2}\right\}\right|\right)$$
$$+ \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left(\mathfrak{E}_k, \mathfrak{E}_0\right) \geq \frac{\varepsilon}{2}\right\}\right|\right).$$

Since $\mathfrak{D}_k \to \mathfrak{D}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$ and $\mathfrak{E}_k \to \mathfrak{E}_0 \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$,

$$\lim_{n\to\infty} \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k + \mathfrak{E}_k, \mathfrak{D}_0 + \mathfrak{E}_0) \geq \varepsilon\}|\right).$$

Therefore, $(\mathfrak{D}_k + \mathfrak{E}_k) \to (\mathfrak{D}_0 + \mathfrak{E}_0) \left[\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)\right]$. □

Theorem 2: *Let φ be any unbounded modulus and $\rho \in (0, 1]$. Then,*

$$\mathfrak{c}(\mathcal{F}(\mathbb{R})) \subset \mathcal{S}^{\rho}_{\alpha\beta}(\varphi),$$

and the inclusion is strict.

Proof: The first part is clear. To illustrate that the inclusion is strict, define the sequence $(\mathfrak{D}_k)$ of fuzzy numbers by

$$\mathfrak{D}_k = \begin{cases} \left\{\begin{array}{ll} -4 + \mathfrak{v}, & \text{if } 4 \leq \mathfrak{v} \leq 5 \\ 6 - \mathfrak{v}, & \text{if } 5 \leq \mathfrak{v} \leq 6 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if } k = i^3, \\ \left\{\begin{array}{ll} -1 + \mathfrak{v}, & \text{if } 1 \leq \mathfrak{v} \leq 2 \\ 3 - \mathfrak{v}, & \text{if } 2 \leq \mathfrak{v} \leq 3 \\ 0, & \text{otherwise} \end{array}\right\} = \mathfrak{D}_0, & \text{if } k \neq i^3. \end{cases} \quad i \in \mathbb{N}. \tag{1}$$

The σ-level set of $(\mathfrak{D}_k)$ is

$$[\mathfrak{D}_k]^{\sigma} = \begin{cases} [4 + \sigma, 6 - \sigma], & \text{if } k = i^3, \\ [1 + \sigma, 3 - \sigma], & \text{if } k \neq i^3. \end{cases}$$

If we take $(\alpha_n) = (1)$ and $(\beta_n) = (n)$, for any unbounded modulus φ,

$$\frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left([\mathfrak{D}_k]^{\sigma}, [\mathfrak{D}_0]^{\sigma}\right) \geq \varepsilon\right\}\right|\right)$$

$$\leq \frac{\varphi(\sqrt[3]{(\beta_n - \alpha_n + 1)})}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)} \to 0$$

as $n \to \infty$ if $\frac{1}{3} < \rho \leq 1$. So, $(\mathfrak{D}_k)$ is $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$-convergent to $\mathfrak{D}_0$, where $[\mathfrak{D}_0]^{\sigma} = [1 + \sigma, 3 - \sigma]$. On the other hand, it is obvious that $(\mathfrak{D}_k) \notin \mathfrak{c}(\mathcal{F}(\mathbb{R}))$. □

Theorem 3: *Let φ be any unbounded modulus and $\rho \in (0, 1]$.*

1. $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi) \subset \mathcal{S}^{\rho}_{\alpha\beta}$ *and the inclusion is strict.*
2. *If* $\lim_{\mathfrak{t}\to\infty} \frac{\varphi(\mathfrak{t})}{\mathfrak{t}} > 0$, *then* $\mathcal{S}^{\rho}_{\alpha\beta} \subset \mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$.

Proof: (1) Let $\varepsilon > 0$ be given. Suppose $(\mathfrak{D}_k)$ is $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$-convergent to a fuzzy number $\mathfrak{D}_0$. Then, for any $z \in \mathbb{N}$, there is $k_0 \in \mathbb{N}$ (dependsing on z) such that

$$\begin{aligned}\varphi\left(|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}|\right) &\leq \frac{1}{z}\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)\\ &\leq \frac{1}{z}z\varphi\left(\frac{1}{z}(\beta_n - \alpha_n + 1)^{\rho}\right)\\ &= \varphi\left(\frac{1}{z}(\beta_n - \alpha_n + 1)^{\rho}\right)\end{aligned}$$

for every $n \geq k_0$. Since φ is a modulus,

$$|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}| \leq \frac{1}{z}(\beta_n - \alpha_n + 1)^{\rho}.$$

Therefore, $(\mathfrak{D}_k)$ is $\mathcal{S}^{\rho}_{\alpha\beta}$-convergent $\mathfrak{D}_0$. To establish the strictness, consider the sequence $(\mathfrak{D}_k)$ as defined by (1). Let $\varphi(t) = \log(1 + t)$ with $\rho = 1$. Then,

$$\begin{aligned}&\frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)}\varphi\left(\left|\left\{k \in [\alpha_n, \beta_n] : d\left([\mathfrak{D}_k]^{\sigma}, [\mathfrak{D}_0]^{\sigma}\right) \geq \varepsilon\right\}\right|\right)\\ &\geq \frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)}\varphi(\sqrt[3]{(\beta_n - \alpha_n + 1)} - 1)\\ &= \frac{\log(\sqrt[3]{(\beta_n - \alpha_n + 1)})}{\log\left((\beta_n - \alpha_n + 1)^{\rho} + 1\right)} \to \frac{1}{3}\end{aligned}$$

as $n \to \infty$ for every $\varepsilon > 0$. Therefore, $(\mathfrak{D}_k)$ is not $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$-convergent to $\mathfrak{D}_0$, where $[\mathfrak{D}_0]^{\sigma} = [1 + \sigma, 3 - \sigma]$. However,

$$\begin{aligned}&\frac{1}{(\beta_n - \alpha_n + 1)^{\rho}}\left|\left\{k \in [\alpha_n, \beta_n] : d\left([\mathfrak{D}_k]^{\sigma}, [\mathfrak{D}_0]^{\sigma}\right) \geq \varepsilon\right\}\right|\\ &\geq \frac{1}{(\beta_n - \alpha_n + 1)^{\rho}}\sqrt[3]{(\beta_n - \alpha_n + 1)} \to 0\end{aligned}$$

as $n \to \infty$. Hence, $(\mathfrak{D}_k)$ is $\mathcal{S}^{\rho}_{\alpha\beta}$-convergent to $\mathfrak{D}_0$.

(2) Suppose $(\mathfrak{D}_k)$ is $\mathcal{S}^{\rho}_{\alpha\beta}$-convergent to a fuzzy number $\mathfrak{D}_0$. Then, for any $\varepsilon > 0$,

$$\begin{aligned}&\frac{1}{(\beta_n - \alpha_n + 1)^{\rho}}|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}|\\ &\geq \frac{1}{\varphi(1)}\frac{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)}{(\beta_n - \alpha_n + 1)^{\rho}}\frac{1}{\varphi\left((\beta_n - \alpha_n + 1)^{\rho}\right)}\varphi\left(|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}|\right).\end{aligned}$$

Therefore, $(\mathfrak{D}_k)$ is $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi)$ – convergent to $\mathfrak{D}_0$. □

Theorem 4: *Let φ be any unbounded modulus and $\rho, \rho' \in (0, 1]$ such that $\rho \leq \rho'$. Then,*

1. $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi) \subset \mathcal{S}^{\rho'}_{\alpha\beta}(\varphi)$.
2. $\mathcal{S}^{\rho}_{\alpha\beta}(\varphi) \subset \mathcal{S}_{\alpha\beta}(\varphi)$.

Proof: Omitted. □

Theorem 5: *Let* φ *be any modulus and* $\rho, \rho' \in (0, 1]$ *such that* $\rho \leq \rho'$. *Then,*

$$\mathcal{N}^{\rho}_{\alpha\beta}(\varphi) \subset \mathcal{N}^{\rho'}_{\alpha\beta}(\varphi),$$

and the inclusion is strict.

Proof: Suppose that a sequence $(\mathfrak{D}_k)$ is strongly $\mathcal{N}^{\rho}_{\alpha\beta}(\varphi)$-summable to a fuzzy number $\mathfrak{D}_0$. Since $\rho \leq \rho'$, we have

$$\frac{1}{(\beta_n - \alpha_n + 1)^{\rho'}} \sum_{k\in[\alpha_n,\beta_n]} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) \leq \frac{1}{(\beta_n - \alpha_n + 1)^{\rho}} \sum_{k\in[\alpha_n,\beta_n]} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)).$$

It follows that $(\mathfrak{D}_k)$ is strongly $\mathcal{N}^{\rho'}_{\alpha\beta}(\varphi)$-summable to $\mathfrak{D}_0$. To verify the strictness of the inclusion, let us define the sequence $(\mathfrak{D}_k)$ by

$$\mathfrak{D}_k = \begin{cases} \left\{\begin{array}{cc} -3+\mathfrak{v}, & \text{if } 3 \leq \mathfrak{v} \leq 4 \\ 5-\mathfrak{v}, & \text{if } 4 \leq \mathfrak{v} \leq 5 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if } k = i^4 \\ \left\{\begin{array}{cc} \mathfrak{v}, & \text{if } 0 \leq \mathfrak{v} \leq 1 \\ 2-\mathfrak{v}, & \text{if } 1 \leq \mathfrak{v} \leq 2 \\ 0, & \text{otherwise} \end{array}\right\} = \mathfrak{D}_0, & \text{if } k \neq i^4. \end{cases} \quad i \in \mathbb{N}.$$

If we take $(\alpha_n) = (1)$ and $(\beta_n) = (n)$, with parameters satisfying $0 < \rho \leq \frac{1}{4}$ and $\frac{1}{4} < \rho' \leq 1$, then for a modulus $\varphi(\mathfrak{t}) = \mathfrak{t}$,$(\mathfrak{D}_k)$ is strongly $\mathcal{N}^{\rho'}_{\alpha\beta}(\varphi)$-summable to $\mathfrak{D}_0$. However, $(\mathfrak{D}_k)$ is not strongly $\mathcal{N}^{\rho}_{\alpha\beta}(\varphi)$-summable to $\mathfrak{D}_0$. □

Theorem 6: *Let* φ *be any modulus and* $\rho \in (0, 1]$. *Then,*

$$\mathcal{N}^{\rho}_{\alpha\beta} \subset \mathcal{N}^{\rho}_{\alpha\beta}(\varphi).$$

Proof: Suppose $(\mathfrak{D}_k)$ is strongly $\mathcal{N}^{\rho}_{\alpha\beta}$-summable to a fuzzy number $\mathfrak{D}_0$. Then,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^{\rho}} \sum_{k\in[\alpha_n,\beta_n]} d(\mathfrak{D}_k, \mathfrak{D}_0) = 0. \tag{2}$$

Let $\varepsilon > 0$ be given. Select $\delta \in (0, 1)$ such that $\varphi(\mathfrak{t}) < \varepsilon$ for $\mathfrak{t} \in (0, \delta]$. Then,

$$\sum_{k\in[\alpha_n,\beta_n]} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) = \sum_{\substack{k\in[\alpha_n,\beta_n] \\ d(\mathfrak{D}_k,\mathfrak{D}_0)\leq\delta}} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) + \sum_{\substack{k\in[\alpha_n,\beta_n] \\ d(\mathfrak{D}_k,\mathfrak{D}_0)>\delta}} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)).$$

From the fact that $d(\mathfrak{D}_k, \mathfrak{D}_0) \leq \delta$ implies $\varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) < \varepsilon$, we can write

$$\sum_{k\in[\alpha_n,\beta_n]} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) < \varepsilon(\beta_n - \alpha_n + 1)^{\rho}$$

and for $d\left(\mathfrak{D}_k, \mathfrak{D}_0\right) > \delta$,

$$d\left(\mathfrak{D}_k, \mathfrak{D}_0\right) < \frac{d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)}{\delta} < 1 + \left[\frac{d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)}{\delta}\right].$$

Since φ is a modulus,

$$\begin{aligned}\varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right) &\le \varphi\left(1 + \left[\frac{d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)}{\delta}\right]\right)\\ &\le \varphi(1)\left(1 + \left[\frac{d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)}{\delta}\right]\right)\\ &\le 2\varphi(1)\frac{d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)}{\delta}.\end{aligned}$$

This implies

$$\sum_{k\in[\alpha_n,\beta_n]} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right) \le \frac{2\varphi\left(1\right)}{\delta}\left(\sum_{\substack{k\in[\alpha_n,\beta_n]\\ d\left(\mathfrak{D}_k,\mathfrak{D}_0\right)>\delta}} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right)\right).$$

Thus,

$$\begin{aligned}\frac{1}{\left(\beta_n - \alpha_n + 1\right)^{\rho}} \sum_{k\in[\alpha_n,\beta_n]} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right) &\le \varepsilon + \frac{2\varphi\left(1\right)}{\delta}\frac{1}{\left(\beta_n - \alpha_n + 1\right)^{\rho}}\left(\sum_{\substack{k\in[\alpha_n,\beta_n]\\ d\left(\mathfrak{D}_k,\mathfrak{D}_0\right)>\delta}} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right)\right)\\ &\le \varepsilon + \frac{2\varphi\left(1\right)}{\delta}\frac{1}{\left(\beta_n - \alpha_n + 1\right)^{\rho}}\left(\sum_{k\in[\alpha_n,\beta_n]} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right)\right).\end{aligned} \tag{3}$$

From (2) and (3), we get

$$\lim_{n\to\infty} \frac{1}{\left(\beta_n - \alpha_n + 1\right)^{\rho}} \sum_{k\in[\alpha_n,\beta_n]} \varphi\left(d\left(\mathfrak{D}_k, \mathfrak{D}_0\right)\right) = 0.$$

Therefore, $\left(\mathfrak{D}_k\right)$ is strongly $\mathcal{N}^{\rho}_{\alpha\beta}\left(\varphi\right)$-summable to $\mathfrak{D}_0$. □

Theorem 7: *Let φ be any modulus and $\rho \in (0, 1]$. If $\lim_{t\to\infty} \frac{\varphi(t)}{t} > 0$, then $\mathcal{N}^{\rho}_{\alpha\beta}\left(\varphi\right) \subset \mathcal{N}^{\rho}_{\alpha\beta}$.*

Proof: Suppose $\left(\mathfrak{D}_k\right)$ is strongly $\mathcal{N}^{\rho}_{\alpha\beta}\left(\varphi\right)$ – summable to a fuzzy number $\mathfrak{D}_0$. Since φ is a modulus, there exists $z \in \mathbb{R}^+$ such that $z = \lim_{t\to\infty} \frac{\varphi(t)}{t}$ by Prop. 1 of [26]. This implies $\varphi(t) \ge zt$ for all $t \ge 0$ and so that $t \leq \frac{1}{z}\varphi(t)$ for all $t \geq 0$. Now, we may write

$$\frac{1}{(\beta_n - \alpha_n + 1)^\rho} \sum_{k\in[\alpha_n,\beta_n]} d(\mathfrak{D}_k, \mathfrak{D}_0) \leq \frac{1}{z}\frac{1}{(\beta_n - \alpha_n + 1)^\rho} \sum_{k\in[\alpha_n,\beta_n]} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) = 0.$$

Therefore, $(\mathfrak{D}_k)$ is strongly $\mathcal{N}^\rho_{\alpha\beta}$-summable to $\mathfrak{D}_0$. □

Theorem 8: *Let φ be any unbounded modulus and $\rho \in (0, 1]$. Then,*
$\mathcal{N}^\rho_{\alpha\beta}(\varphi) \subset \mathcal{S}^\rho_{\alpha\beta}(\varphi)$,
and the inclusion is strict.

Proof: Let $\varepsilon > 0$ be given. Then, we have

$$\begin{aligned}
\frac{1}{(\beta_n - \alpha_n + 1)^\rho} \sum_{k\in[\alpha_n,\beta_n]} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0)) &= \frac{1}{(\beta_n - \alpha_n + 1)^\rho}\left(\sum_{\substack{k\in[\alpha_n,\beta_n] \\ d(\mathfrak{D}_k,\mathfrak{D}_0)\geq\varepsilon}} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0))\right) \\
&+ \frac{1}{(\beta_n - \alpha_n + 1)^\rho}\left(\sum_{\substack{k\in[\alpha_n,\beta_n] \\ d(\mathfrak{D}_k,\mathfrak{D}_0)<\varepsilon}} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0))\right) \\
&\geq \frac{1}{(\beta_n - \alpha_n + 1)^\rho}\left(\sum_{\substack{k\in[\alpha_n,\beta_n] \\ d(\mathfrak{D}_k,\mathfrak{D}_0)\geq\varepsilon}} \varphi(d(\mathfrak{D}_k, \mathfrak{D}_0))\right) \\
&\geq \frac{1}{(\beta_n - \alpha_n + 1)^\rho} |\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}| \varphi(\varepsilon) \\
&\geq \frac{1}{(\beta_n - \alpha_n + 1)^\rho}\varphi(|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}|) \frac{\varphi(\varepsilon)}{\varphi(1)} \\
&= \frac{\varphi((\beta_n - \alpha_n + 1)^\rho)}{(\beta_n - \alpha_n + 1)^\rho} \frac{\varphi(\varepsilon)}{\varphi(1)} \frac{\varphi(|\{k \in [\alpha_n, \beta_n] : d(\mathfrak{D}_k, \mathfrak{D}_0) \geq \varepsilon\}|)}{\varphi((\beta_n - \alpha_n + 1)^\rho)}.
\end{aligned}$$

Therefore, we obtain that $(\mathfrak{D}_k)$ is $\mathcal{S}^\rho_{\alpha\beta}(\varphi)$-convergent to a fuzzy number $\mathfrak{D}_0$ if it is strongly $\mathcal{N}^\rho_{\alpha\beta}(\varphi)$-summable to $\mathfrak{D}_0$. To establish the strictness, define the sequence $(\mathfrak{D}_k)$ by

$$\mathfrak{D}_k = \begin{cases} \left.\begin{matrix} 1 + \frac{\mathfrak{v}}{k^2}, & \text{if } -k^2 \leq \mathfrak{v} \leq 0 \\ 1 - \frac{\mathfrak{v}}{k^2}, & \text{if } 0 \leq \mathfrak{v} \leq k^2 \\ 0, & \text{otherwise} \end{matrix}\right\}, & \text{if } k = i^3 \\ \left.\begin{matrix} -3 + \mathfrak{v}, & \text{if } 3 \leq \mathfrak{v} \leq 4 \\ 5 - \mathfrak{v}, & \text{if } 4 \leq \mathfrak{v} \leq 5 \\ 0, & \text{otherwise} \end{matrix}\right\} = \mathfrak{D}_0, & \text{if } k \neq i^3. \end{cases} \quad i \in \mathbb{N}.$$

Now, the σ-level set of $(\mathfrak{D}_k)$ is

$$[\mathfrak{D}_k]^\sigma = \begin{cases} [-k^2(1-\sigma), k^2(1-\sigma)], & \text{if } k = i^3, \\ [3+\sigma, 5-\sigma], & \text{if } k \neq i^3. \end{cases}$$

If we take $\frac{1}{3} < \rho \leq 1$, $(\alpha_n) = (1)$, $(\beta_n) = (n)$, and $\varphi(\mathfrak{t}) = \mathfrak{t}$, then $(\mathfrak{D}_k)$ is $\mathcal{S}_{\alpha\beta}^{\rho}(\varphi)$-convergent to a fuzzy number $\mathfrak{D}_0$ but it is not strongly $\mathcal{N}_{\alpha\beta}^{\rho}(\varphi)$-summable to $\mathfrak{D}_0$, where $[\mathfrak{D}_0]^\sigma = [3+\sigma, 5-\sigma]$. □

3 Conclusion

In this work, we have developed generalized forms of statistical convergence and strong summability for sequences of fuzzy numbers through the framework of modulus functions. By imposing suitable conditions on the modulus functions, we have extended and refined several existing convergence notions in the fuzzy context. Our results not only encompass earlier findings as particular cases, but also provide a more flexible and unifying approach to studying convergence and summability phenomena in fuzzy sequences. Furthermore, the interrelationships established between the proposed concepts contribute to a deeper theoretical understanding of the interplay between summability and convergence in this setting. These contributions open the door to further investigations, including potential applications in approximation theory, fuzzy analysis, and other areas where uncertainty and imprecision play a central role.

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