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Controllability and Observability Conditions for Impulsive DAEs on Time Scales

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ABSTRACT

This paper establishes criteria for controllability and observability in a class of linear time-invariant impulsive differential-algebraic equations (DAEs) defined on a time scale. Controllability is defined as the ability to transfer the system state from any initial condition to any desired state within a finite time interval using an appropriate control input. Observability is characterized as the ability to uniquely determine the initial state from output measurements over a finite time interval. The main contributions are the derivation of necessary and sufficient conditions for both state controllability and state observability. These conditions are formulated in terms of rank-based tests and the properties of Gramian matrices, which are constructed using the Drazin inverse to handle the system's algebraic constraints and impulsive dynamics. The theoretical results are validated by solving a illustrative controllability and observability problem, demonstrating the efficacy of the proposed approach.

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1 | Introduction

A differential algebraic equation is a type of differential equation in which the derivatives (in general) are not expressed explicitly and derivatives of some of the dependent variables may not appear in the equations at all. The general form of a system of differential algebraic equations (DAEs) is given by

$$G(t, z, \dot{z}) = 0,$$

where t is an independent variable, z is state vector, and $\dot{z}$ is derivative of z with respect to t.

DAEs are useful for modeling different practical problems in chemical engineering, electrical networks, mechanical systems, multibody dynamics, and economics, see for example, [1, 2], which often cannot be modeled by standard ordinary DAEs. Especially DAEs have then been discovered to be appropriate for modeling and simulation of the dynamical procedures. In general, these equations are implicit differential equations. While in simple case, these equations are combination of differential equations along with algebraic constraints, from which the name DAEs comes from. A huge solution theory for DAEs has been developed, which can be looked up in [3, 4]. Gantmacher discusses the classical solution of DAEs in his famous book, see [5]. Gantmacher's focus is actually on Kronecker canonical form but he does not consider non smooth homogeneities and inconsistent initial values. It is noticed that the determination of Kronecker canonical form is numerically ill posed in practical problems. Therefore, researchers take interest in other solution formulas.

Hence, an efficient tool to analyze the linear differential algebraic system is with the help Drazin inverse (DI) [6]. Let $X \in \mathbb{C}^{n\times n}$, a matrix $F \in \mathbb{C}^{n\times n}$ such that satisfies $F^{k+1}X = F^k, XFX = X$, and $FX = XF$, is called the DI of F and denoted by F^D. The DI of any square matrix always exist and unique, see in [4]. There are applications of DI to solve linear differential algebraic control system [7, 8].

DAEs found its way into control theory after the work of Rosenbrock [9]. He developed ideas of describing linear systems by polynomial system matrices. Subsequently, Rosenbrock [10], Luenberger [11], and Verghese et al. [12] have contributed a lot in area of control theory. Controllability and observability are two of the most fundamental concepts in modern control theory. They have close connections to observer design, quadratic optimal control, pole assignment, structural decomposition, and so forth, see [13]. The concept of controllability provides helps for the design of control law. On the other words, observability describes an ability to use partial information to figure out the whole system states. Kalman [14] was the first one, who proposed the concept of controllability and observability. Controllability and observability of time varying DAEs have already been discussed, see [15, 16].

Impulsive control systems are used as a mathematical tool for analyzing the behavior of most real-life processes. The mathematical tool for designing impulsive control systems is the theory of these equations, which expresses processes that change their states at certain non fixed or fixed moments in the form of jumps. In recent years, some results on differential algebraic sytems with impulsive control have been reported, see for example, [17–21]. Several important developments have been reported in the literature regarding observer and controller design for fractional-order dynamical systems, particularly in observer-based control, nonlinear Hadamard fractional systems, unknown input observer design, and conformable fractional state estimation problems [22–25], demonstrating the growing interest in generalized dynamical frameworks and motivating further extensions to impulsive differential-algebraic systems on time scales.

However, to the best of our knowledge, there has been no result about the controllability and observability of system of impulsive differential algebraic equations on time scale, where time scale is a nonempty closed subset of $\mathbb{R}$. There was a gap that no theory can deal collectively with both the continuous and discrete cases in one place. Bohner and Peterson added significant work in the time scales calculus, see [26, 27]. While advanced control designs for complex systems are actively developed [28, 29], and existence analyzes for differential equations remain a focus [30], the foundational concepts of controllability and observability for this specific hybrid system class have been overlooked.

Extending the existing results of impulsive descriptor systems to the time-scale setting is not straightforward due to several mathematical difficulties. The singularity of the descriptor matrix produces hidden algebraic constraints, while the unified continuous-discrete structure of time scales complicates the construction of solution trajectories. In particular, establishing the regressivity of the transformed coefficient matrix depends critically on the graininess function, which has no counterpart in the classical continuous-time case. Furthermore, impulsive effects interact with the natural jumps of the time scale, making the derivation of state transition matrices and Gramian-based controllability and observability criteria significantly more delicate. These challenges motivate the present generalization and substantially broaden the previous continuous-time results. In this paper, we investigate state controllability and state observability of impulsive differential algebraic system on time scale. We derive the necessary and sufficient conditions on controllability and observability for the impulsive differential algebraic system on time scale.

The paper is organized as follows: Section 2 presents the elementary facts and preliminary results. In Sections 3 and 4, the main results about controllability and observability for the impulsive differential algebraic systems on time scale are given, respectively. Section 5 contains the concluding remarks.

2 | Preliminaries

Consider $\mathbb{R}^n$ be space of n dimensional column vectors with a norm $\|\cdot\|$. A time scale T is a non-empty closed subset of $\mathbb{R}$. The notations $[a, b] := \{t \in T; a \le t \le b\}$, where $a, b \in$T. The set of all rd-continuous functions $g : T \to \mathbb{R}^n$ will be denoted by $C_{rd}(T, \mathbb{R}^n)$ that are continuous at every right dense point and is finite at every left dense point. The set of rd-continuous (respectively rd-continuous and regressive) functions $A : T \to M_n(\mathbb{R})$ is denoted by $C_{rd}(T, M_n(\mathbb{R}))$ (respectively by $C_{rd}\mathcal{R}(T, M_n(\mathbb{R}))$). Moreover, We recall that a matrix-valued function A is called regressive if $I + \mu(t)A(t)$ is invertible $\forall\, t \in$T, where I is $n \times n$ identity matrix. We refer to [26, 27] for more information on time scales. Consider an impulsive differential algebraic system on time scale as follows:

$$
S_1 := \begin{cases} Fz^{\Delta}(t) = Pz(t) + Qu(t), t \neq t_i \\ z\left(t_i^+\right) = \left(1 + d_i\right) z\left(t_i\right), t = t_i, i = 1, 2, \dots, \\ y(t) = Rz(t), \\ z\left(t_0\right) = z_0, \end{cases}
$$

where T is an unbounded above time scale with bounded graininess, $T_0 := \left[t_0, \infty\right) \cap T$, $\left[t_{k-1}, t_k\right) \in T_0$, and $d_i \in \mathbb{R}$ are constants. Throughtout this paper, we suppose that $F, P \in C_{rd}\mathcal{R}(T_0, M_n(\mathbb{R}))$, $Q \in C_{rd}\mathcal{R}(T_0, M_{n\times m}(\mathbb{R}))$, $R \in C_{rd}\mathcal{R}(T_0, M_{p\times n}(\mathbb{R}))$, $z \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, and z^{Δ} is delta derivative of z. Also, consider that system S_1 holds the following assumptions:

(A1): $0 \le t_0 < t_1 < \cdots < t_k < \dots$, provided that $\lim_{k\to\infty} t_k = \infty$ and $t_k \in$T are right dense.

(A2): $z(t_k^+) := \lim_{h\to 0^+} z(t_k + h)$, $z\left(t_k\right) = z(t_k^-) := \lim_{h\to 0^-} z(t_k - h)$.

(A3): Matrix pencil (F, P) is regular, that is, $\exists\, c \in \mathbb{C}$ such that $Fc - P$ is invertible.

Recall from [31], index of a matrix F, denoted by $Ind(F)$, is the smallest non negative integer r such that rank F^r = rank F^{r+1}. A matrix $F^D \in C_{rd}\mathcal{R}(M_n(\mathbb{R}))$ is called the DI of a matrix F if $F^D F = FF^D$, $F^D F F^D = F^D$ and $F^D F^{r+1} = F^r$, where r is the index of the matrix. Note that, DI of a matrix always exists and unique, see [7]. Steps to compute DI F^D of a matrix F are:

(a) Find W, V (a pair of matrices), with $W \in C_{rd}\mathcal{R}(M_{n\times s}(\mathbb{R}))$, $V \in C_{rd}\mathcal{R}(M_{s\times n}(\mathbb{R}))$ such that rank W = rank V = rank $F = s$ and $F = WV$.

(b) Compute non-singular matrix $VFW \in C_{rd}\mathcal{R}(M_{s\times s}(\mathbb{R}))$.

(c) Finally, $F^D = W(VFW)^{-1}V$.

DI can also be expressed in terms of Jordan canonical form as: $F = S\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}S^{-1}, F^D = S\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}S^{-1}$, where J constitutes the Jordan blocks corresponding to non-zero eigen values and N is nilpotent, that is, $N^k = 0$, while $N^{k-1} \neq 0$. From this representation of F^D. We can see (from [8]) that: $R(F^D) = R(F^r), \mathcal{N}(F^D) = \mathcal{N}(F^r)$, and $\mathbb{R}^n = R(F^D) \oplus \mathcal{N}(F^D)$.

Remark 2.1. When T$= \mathbb{R}$, $z(t_i^+) = (1 + d_i)z(t_i) = 0, Q = I$, we have the following initial value problem

$$
\begin{aligned}
Fz'(t) &= Pz(t) + u(t) \\
z(t_0) &= z_0,
\end{aligned} \tag{1}
$$

where $z(\cdot) \in \mathbb{R}^{n\times 1}, F, P \in \mathbb{R}^{n\times n}$ and $u(\cdot) : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and z' is ordinary derivative of z. If E is non-singular matrix, then differential algebraic equation is equivalent to an ordinary differential equation of the form $z'(t) = E^{-1}Az(t) + g$. When E is a singular matrix, the solution of homogeneous initial value problem may be inconsistent, that is, there may not exist a solution. If there is a solution, it need not be unique. For example, the following system

$$
\begin{aligned}
7z_1^{'}(t) + z_2(t) &= u(t), \\
7z_1^{'}(t) + z_2(t) &= u(t),
\end{aligned}
$$

is inconsistent. And for the following system

$$
\begin{aligned}
z_1'(t) - 5z_1(t) &= g_1 \\
5z_3'(t) - z_2(t) &= g_2 \\
z_3(t) &= g_3,
\end{aligned}
$$

for any given g_1 and z_1, the first equation has unique solution. If g_3 is differentiable, the solutions of $z_2(t) = g_2 - 5g_3^{'}\ z_3(t) = g_3$. Hence, the solutions exists if $z_2(0) = g_2^{'}(0) - g_3(0), z_3(0) = g_3(0)$. Hence, if the solution exists, it is unique.

Now, premultiplying first equation of system S_1 with $(Fc - P)^{-1}$, we have:

$$
S_2 := \begin{cases} \overline{F}z^{\Delta}(t) = \overline{P}z(t) + \overline{Q}u(t), t \neq t_i \\ z(t_i^+) = (1 + d_i)z(t_i), t = t_i, i = 1, 2, \dots, \\ y(t) = Cz(t), \\ z(t_0) = z_0, \end{cases}
$$

where $\overline{F} = (Fc - P)^{-1}F, \overline{P} = (Fc - P)^{-1}P, \overline{Q} = (Fc - P)^{-1}Q$. The matrices $\overline{F}, \overline{P}$, and $\overline{Q}$ satisfy further properties, see [32, Remark 2.5 and Lemma 2.6].

Remark 2.2. Regressivity is not preserved under the DI in general.

The following example justifies the above remark:

Example 2.3. Let $\overline{F} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \overline{F}^D = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$. Let us consider the time scales such that $\mu(t) = 2$, then $I + \mu(t)\overline{F} = \begin{bmatrix} 1 + (2)(-2) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$. det $\left(I + \mu(t)\overline{F}\right) = -3 \neq 0$. So $\overline{F}$ is regressive. Now computing $I + \mu(t)\overline{F}^D = \begin{bmatrix} 1 + (2)\frac{-1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. It is clear that det $\left(I + \mu(t)\overline{F}^D\right) = 0$. Hence, $\overline{F}^D$ is not regressive.

Theorem 2.4. *Let $\overline{F}$ be a square matrix and let $\mu(t)$ be scalar valued function. Assume $\overline{F}$ is regressive, that is, $1 + \mu(t)\overline{F}$ is invertible, $\overline{F}$ have a Drazin decomposition $\overline{F} = F_s + F_n$, where F_s is semisimple, F_n is nilpotent, and $\sigma\left(\overline{F}\right) \cap \left\{-\frac{1}{\mu(t)}, -\mu(t)\right\} = \emptyset$, then DI $\overline{F}^D$ is also regressive.*

Proof. For a matrix $\overline{F}$, its DI satisfies, $\lambda \neq 0$ become λ^{-1}, zero eigenvalue map to zero. So,

$$
\sigma\left(\overline{F}^D\right) = \left\{\lambda^{-1} : \lambda \in \sigma\left(\overline{F}\right), \lambda \neq 0\right\} \cup \{0\}.
$$

Let $\overline{F}$ is regressive, so it satisfies $1 + \mu(t)\lambda \neq 0$, for all $\lambda \in \sigma\left(\overline{F}\right), \forall t$. This implies $\lambda \neq -\frac{1}{\mu(t)}$. If $\lambda \neq 0$, then eigenvalues of $1 + \mu(t)\overline{F}^D$ are $1 + \mu(t)\lambda^{-1}$ and if $\lambda = 0$, then eigenvalue is 1. So singularity occurs only if

$$
1 + \mu(t)\lambda^{-1} = 0
$$

which implies $\lambda = -\mu(t)$. From regressivity of $\overline{F}$, $1 + \mu(t)\lambda \neq 0$ implies that $\lambda \neq -\frac{1}{\mu(t)}$. Now compare for $\overline{F}$, $\lambda = -\frac{1}{\mu(t)}$. For $\overline{F}^D$, $\lambda = -\mu(t)$. This cannot both occur for the same spectral structure unless spectrum intersects a time varying forbidden set but assumption exclude this intersection thus $1 + \mu(t)\lambda^{-1} \neq 0\ \forall t$. All eigenvalues of $1 + \mu(t)\overline{F}^D$ are nonzero. So $1 + \mu(t)\overline{F}^D$ is invertible for all t. Hence, $\overline{F}^D$ is regressive. □

Lemma 2.5. *Let $\mu(t)$ be scalar graininess function and $\overline{F}, \overline{P} \in \mathbb{R}^{n\times n}$. Assume that $\mu_{\max} = \sup|\mu(t)| < \infty$, $\overline{F}^D$ exists and $\mu_{\max}\cdot\rho\left(\overline{F}^D\overline{P}\right) < 1$. Then, $\overline{F}^D\overline{P}$ is regressive.*

Proof. Let $\lambda \in \rho\left(\overline{F}^D\overline{P}\right)$, then eigenvalues of $I + \mu(t)\overline{F}^D\overline{P}$ are $1 + \mu(t)\lambda$. we estimate

$$
|\mu(t)\lambda| \leq \mu_{\max}\rho\left(\overline{F}^D\overline{P}\right) < 1
$$

hence,

$$
|1 + \mu(t)\lambda| \geq 1 - |\mu(t)\lambda| > 0
$$

so

$$
1 + \mu(t)\lambda \neq 0\ \forall t
$$

hence

$$
I + \mu(t)\overline{F}^D\overline{P} \text{ is invertible}
$$

thus, $\overline{F}^D\overline{P}$ is regressive.

Example 2.6. Let us consider the time scales such that $\mu(t) \in [0,1]$ for all t. Let $\overline{F} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \overline{F}^D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $\overline{P} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, then $A = \overline{F}^D\overline{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. $\sigma(A) = \left\{ \frac{1}{2}, 0 \right\}$. Now, computing

$$I + \mu(t)A = \begin{bmatrix} 1 + \frac{\mu(t)}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$\det(I + \mu(t)A) = \left(1 + \frac{\mu(t)}{2}\right) \geq 1 > 0$. Hence, $I + \mu(t)A$ is invertible. This implies that A is regressive. □

Throughout in this paper,

(A4) : Matrix $\overline{F}^D\overline{P}$ satisfies the regressive property.

Define the consistency space $\overline{\mathcal{V}}_i^*$ for discussing controllability and observability of S_2 (or S_1) as:

$$\overline{\mathcal{V}}_i^* := \left\{ \begin{array}{c} z_0 \in \mathbb{R}^n \text{ such that there exists } u \in \mathbb{R}^m, \\ z \text{ is a solution of } S_2 (\text{or } S_1) \text{ with } z(0) = z_0 \end{array} \right\}.$$

Moreover, we can express the solution of S_2 (or S_1) as:

Lemma 2.7. *Suppose that assumptions (A1)–(A4) are satisfied. The solution of impulsive system S_2 (or S_1) is given by*

$$z(t) = \begin{cases} e_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}z_0 + \int_{t_0}^{t} e_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)\left(\overline{FP}^D\right)^q\overline{P}^D\overline{Q}u^q(t), \text{ for } t \in [t_0, t_1] \\ \prod_{i=1}^{k}(1+d_i)e_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}z_0 + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i) \\ \quad \times \int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \int_{t_k}^{t} e_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)\left(\overline{FP}^D\right)^q\overline{P}^D\overline{Q}u^q(t), \quad \begin{array}{l} \text{for } t \in (t_k, t_{k+1}], \\ k = 1, 2, \ldots, \end{array} \end{cases} \tag{2}$$

where $r = Ind(F)$ and I is $n \times n$ identity matrix.

3 | Controllability

In this section, some ideas and equivalent criteria associated with the controllability of impulsive differential algebraic system are proved and discussed. Likewise the concept of controllability discussed in [32], the definition for controllability of impulsive differential algebraic system is given as follows:

Definition 3.1. [32] If given any initial state $z_0 \in \overline{\mathcal{V}}_i^*$, $\exists$ a piecewise rd-continuous input signal $u(\cdot)$ provided that the corresponding solution of the impulsive system S_2 (or S_1) satisfies $z(t_f) = 0$. Then, the impulsive system S_2 (or S_1) is controllable on $[t_0, t_f]$ with $t_f > 0$.

For introducing equivalent criteria of controllability of the impulsive system S_2 (or S_1), the controllability Gramian matrices are defined as follows:

$$\begin{aligned} \mathcal{W}_i := \mathcal{W}(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} &\left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right) \\ &\times \left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right)^{\top}\Delta\tau, \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}_k := \mathcal{W}(t_0, t_k, t_f) = \int_{t_k}^{t_f} &\left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right) \\ &\times \left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right)^{\top}\Delta\tau, \end{aligned}$$

for $i = 1, 2, \ldots k$, and $\top$ represents the conjugate transpose.

Theorem 3.2. *Suppose that assumptions (A1)–(A4) are satisfied:*

(a) *If there exist at least $l \in \{1, \ldots, k+1\}$ such that $\mathcal{W}(t_0, t_{l-1}, t_l)$ is invertible then, the impulsive system S_2 (or S_1) is controllable on $[t_0, t_f], (t_f \in (t_k, t_{k+1}])$.*

(b) *Assume that $d_i \neq -1, (i = 1, 2, \ldots, k)$, if the impulsive system S_2 (or S_1) is controllable on $[t_0, t_f], (t_f \in (t_k, t_{k+1}])$, then rank $\left[\mathcal{W}_0 \ \mathcal{W}_1 \cdots \mathcal{W}_k\right] = n$.*

Proof. First, consider that $\exists$ at least one $l \in \{1, \ldots, k+1\}$ such that the controllability Gramian $\mathcal{W}_{l-1}$ is invertible. Then, for given $z_0 \in \overline{\mathcal{V}}_i^*, \mathcal{W}_0^{-1} = \mathcal{W}^{-1}(t_0, t_0, t_1), \mathcal{W}_{l-1}^{-1} = \mathcal{W}^{-1}(t_0, t_{l-1}, t_l)$, choose $u(t)$ to be:

$$u(t) := \begin{cases} \left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(t))\overline{F}^D\overline{Q}\right)^{\top}\mathcal{W}_0^{-1} \times \left(-e_{\overline{F}^D\overline{P}}(t_f, t_0)\overline{F}^D\overline{F}z_0\right) \\ \quad \times a_l\left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(t))\overline{F}^D\overline{Q}\right)^{\top}\mathcal{W}_{l-1}^{-1}\overline{F}^D\overline{F}z_0, \\ \quad \times \text{ for } t \in (t_{l-1}, t_l] \\ 0, \text{ for } t \in [t_0, t_f]/(t_{l-1}, t_l], \end{cases} \tag{3}$$

where a_l as a constant satisfies $\prod_{j=1}^{k}(1+d_j) + a_l\prod_{j=l}^{k}(1+d_j) = 0$. And solution (2) of S_2 (or S_1) at $t = t_f$ becomes

$$z(t_f) = \begin{cases} e_{\overline{F}^D\overline{P}}(t_f,t_0)\overline{F}^D\overline{F}z_0 + \int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_f,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)\left(\overline{FP}^D\right)^q\overline{P}^D\overline{Q}u^q(t_f), \text{ for } t_f \in [t_0, t_1] \\ \prod_{i=1}^{k}(1+d_i)e_{\overline{F}^D\overline{P}}(t_f,t_0)\overline{F}^D\overline{F}z_0 + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i) \\ \quad \times \int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_f,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)\left(\overline{FP}^D\right)^q\overline{P}^D\overline{Q}u^q(t_f), \quad \begin{array}{l} \text{for } t_f \in (t_k, t_{k+1}], \\ k = 1, 2, \ldots. \end{array} \end{cases} \tag{4}$$

Pre-multiplying Equation (4) by $\overline{F}^D$ and using Equation (3) in Equation (4), we have

$$\overline{F}^D z(t_f) = 0. \tag{5}$$

Pre-multiplying Equation (5) by $\overline{P}^D$, we have

$$\overline{P}^D \overline{F}^D z(t_f) = 0.$$

Next, $\overline{P}^D\overline{F}^D = \overline{F}^D\overline{P}^D$ implies that $\overline{F}^D z(t_f)$ belongs to kernel of $\left(\overline{P}^D\right)$ and $\overline{P}^D z(t_f)$ belongs to kernel of $\left(\overline{F}^D\right)$. Finally, we obtain

$$z(t_f) = 0.$$

Hence, the impulsive system S_2 (or S_1) is controllable.

To show the reverse implication, consider that the impulsive system S_2 (or S_1) is controllable and rank $[\mathcal{W}_0\ \mathcal{W}_1 \cdots \mathcal{W}_k] \neq n$, then $\exists$ a non-zero vector $z_\alpha \in \overline{\mathcal{V}}_i^*$ provided that

$$z_\alpha^\top \mathcal{W}_{j-1} z_\alpha = 0 \text{ and } z_\alpha^\top \mathcal{W}_k z_\alpha = 0, \text{ for } t \in (t_j, t_{j+1}], j = 1, 2, \ldots k \tag{6}$$

from equations in (6), we get

$$\int_{t_{j-1}}^{t_j} z_\alpha^\top \left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right)\left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right)^\top z_\alpha \Delta\tau = 0, \tag{7}$$

and

$$\int_{t_k}^{t_f} z_\alpha^\top \left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right)\left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right)^\top z_\alpha \Delta\tau = 0, \tag{8}$$

since, the integrand in both the Equations (7) and (8) is nonnegative. Thus, we have

$$\left\| z_\alpha^\top e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q} \right\|^2 = 0.$$

It implies that

$$z_\alpha^\top \left(e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}\right) = 0. \tag{9}$$

Furthermore, the controllability of impulsive system S_2 (or S_1) implies that (from Equation (4)):

$$\begin{aligned} 0 &= z(t_f) \\ &= \begin{cases} e_{\overline{F}^D\overline{P}}(t_f, t_0)\overline{F}^D\overline{F}z_0 + \int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_f, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)\left(\overline{F}\overline{P}^D\right)^q \overline{P}^D\overline{Q}u^q(t_f), \text{ for } t_f \in [t_0, t_1] \\ \prod_{i=1}^{k}(1+d_i) e_{\overline{F}^D\overline{P}}(t_f, t_0)\overline{F}^D\overline{F}z_0 + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i) \\ \quad \times \int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_f, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)\left(\overline{F}\overline{P}^D\right)^q \overline{P}^D\overline{Q}u^q(t_f), \quad \begin{array}{l}\text{for } t_f \in (t_k, t_{k+1}], \\ k = 1, 2, \ldots.\end{array} \end{cases} \end{aligned} \tag{10}$$

By substituting $z_0 = z_\alpha$ in Equation (10) and using semi-group property of exponential matrix, we get

$$0 = \begin{cases} e_{\overline{F}^D\overline{P}}(t_f, t_0)\overline{F}^D\overline{F}z_\alpha + e_{\overline{F}^D\overline{P}}(t_f, t_0) \\ \quad \int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)e_{\overline{F}^D\overline{P}}(t_f, t_0)e_{\overline{F}^D\overline{P}}(t_0, t_f)\left(\overline{F}\overline{P}^D\right)^q \\ \quad \overline{P}^D\overline{Q}u^q(t_f), \text{ for } t_f \in [t_0, t_1] \\ \prod_{i=1}^{k}(1+d_i) e_{\overline{F}^D\overline{P}}(t_f, t_0)\overline{F}^D\overline{F}z_\alpha + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i) \\ \quad \times e_{\overline{F}^D\overline{P}}(t_f, t_0)\int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + e_{\overline{F}^D\overline{P}}(t_f, t_0)\int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)e_{\overline{F}^D\overline{P}}(t_f, t_0)e_{\overline{F}^D\overline{P}}(t_0, t_f)\left(\overline{F}\overline{P}^D\right)^q \\ \quad \overline{P}^D\overline{Q}u^q(t_f), \begin{array}{l}\text{for } t_f \in (t_k, t_{k+1}], \\ k = 1, 2, \ldots,\end{array} \end{cases}$$

then, we have

$$0 = \begin{cases} \overline{F}^D\overline{F}z_\alpha + \int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)e_{\overline{F}^D\overline{P}}(t_0, t_f)\left(\overline{F}\overline{P}^D\right)^q \\ \quad \times \overline{P}^D\overline{Q}u^q(t_f), \text{ for } t_f \in [t_0, t_1] \\ \prod_{i=1}^{k}(1+d_i)\overline{F}^D\overline{F}z_\alpha + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i) \\ \quad \times \int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ \quad + \sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F} - I\right)e_{\overline{F}^D\overline{P}}(t_0, t_f)\left(\overline{F}\overline{P}^D\right)^q \\ \quad \times \overline{P}^D\overline{Q}u^q(t_f), \begin{array}{l}\text{for } t_f \in (t_k, t_{k+1}], \\ k = 1, 2, \ldots.\end{array} \end{cases} \tag{11}$$

Pre-multiplying Equation (11) by $\overline{F}^D$ and simplifying, we get

$$0 = \begin{cases} \overline{F}^D\left(z_\alpha + \int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\right) \text{ for } t_f \in [t_0, t_1] \\ \overline{F}^D\begin{pmatrix} \prod_{i=1}^{k}(1+d_i)z_\alpha + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i) \\ \times \int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \\ + \int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \end{pmatrix} \begin{array}{l}\text{for } t_f \in (t_k, t_{k+1}], \\ k = 1, 2, \ldots.\end{array} \end{cases} \tag{12}$$

Since $\overline{F}^D\overline{P}^D = \overline{P}^D\overline{F}^D$, so Equation (12) implies that

$$\overline{P}^D\left(z_\alpha + \int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_0, \sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\right) \in \ker\left(\overline{F}^D\right)$$

and

$$\overline{F}^D\left(z_\alpha+\int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\right)\in \ker\left(\overline{P}^D\right), \tag{13}$$

for $t_f\in[t_0,t_1]$. Also,

$$\overline{P}^D\left(\begin{array}{l}\prod_{i=1}^{k}(1+d_i)z_\alpha\\ +\sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i)\int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\\ +\int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\end{array}\right)$$
$$\in\ker\left(\overline{F}^D\right)$$

and

$$\overline{F}^D\left(\begin{array}{l}\prod_{i=1}^{k}(1+d_i)z_\alpha\\ +\sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i)\int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\\ +\int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\end{array}\right)$$
$$\in\ker\left(\overline{P}^D\right), \tag{14}$$

for $t_f\in(t_k,t_{k+1}]$, $k=1,2,\ldots$.

By using [32, Remark 2.7], from (13) and (14), we have

$$0=\begin{cases} z_\alpha+\int_{t_0}^{t_f} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau \text{ for } t_f\in[t_0,t_1]\\ \prod_{i=1}^{k}(1+d_i)z_\alpha\\ \quad+\sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i)\int_{t_{j-1}}^{t_j} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau)) \text{ for } t_f\in(t_k,t_{k+1}],\\ \overline{F}^D\overline{Q}u(\tau)\Delta\tau \qquad\qquad k=1,2,\ldots.\\ \quad+\int_{t_k}^{t_f} e_{\overline{F}^D\overline{P}}(t_0,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\end{cases}$$

Pre-multiplying Equation (12) by $z_\alpha^\top$ and using Equation (9), we have

$$\|z\|^2=0;$$

which is contradiction to that $z\neq 0$. Thus, this completes our proof. □

Theorem 3.3. *Suppose that assumptions A1–A4 are satisfied, the necessary and sufficient condition for impulsive system S_2 (or S_1) to be controllable on $[t_0,t_f]$, $(t_f\in(t_k,t_{k+1}])$ is*

$$rank\ \mathrm{Q_c}=n,$$

where $\mathrm{Q_c}=\left[\overline{F}^D\overline{Q}|\left(\overline{F}^D\overline{P}\right)\overline{F}^D\overline{Q}|\cdots|\left(\overline{F}^D\overline{P}\right)^{n-1}\overline{F}^D\overline{Q}\right]$.

Proof. The proof of this theorem is on the similar pattern as in [32, Theorem 3.4]. □

Example 3.4. Consider the following impulsive differential algebraic system on arbitrary time scales such that $\mu_{\max}.1.354328<1$.

$$\begin{aligned} Fz^\Delta(t)&=Pz(t)+Qu(t),\ t\neq t_i,\ i=1,2,\ldots\\ z(t^+)&=\frac{1}{3}z(t),\ t=t_i \text{ where } t_i=\frac{(6i-5)}{2}\\ z\left(\begin{bmatrix}0.5\\0.5\\0.5\end{bmatrix}\right)&=\begin{bmatrix}1\\0\\0\end{bmatrix},\ t\in[0.5,27.5],\end{aligned} \tag{15}$$

and

$$F=\begin{bmatrix}5&0&0\\0&4&0\\0&6&0\end{bmatrix}, P=\begin{bmatrix}5&7&1\\2&4&3\\8&9&6\end{bmatrix},$$

$$Q=\begin{bmatrix}5\\9\\7\end{bmatrix}, C=\begin{bmatrix}4&2&1\end{bmatrix}.$$

The impulsive points satisfy (A1) and (A2). It is clear that matrix F is not invertible and rank $(F)=1$. Moreover, for $c\in\mathbb{C}$, det $(Fc-P)\neq 0$, that is matrix pencil is regular. Hence (A3) holds. Taking $c=1+3i$, we have

$$Fc-P=\begin{bmatrix}15i&-7&-1\\-2&12i&-3\\-8&-3+18i&-6\end{bmatrix},$$

and

$$(Fc-P)^{-1}$$
$$=\begin{bmatrix}0.0268-0.0709i&-0.0498-0.1540i&0.0204+0.0888i\\0.0307+0.0332i&0.2288-0.2522i&-0.1195+0.1206i\\-0.1508+0.1700i&0.7086+1.0177i&-0.4958-0.5372i.\end{bmatrix}$$

Pre-multiplying first equation of impulsive differential algebraic system (15) with $(Fc-P)^{-1}$, we obtain

$$\overline{F}z^\Delta(t)=\overline{P}z(t)+\overline{Q}u(t),\ t\neq\frac{(6i-5)}{2},\ i=1,2,\ldots \tag{16}$$

where

$$\overline{F}=(Fc-P)^{-1}F=\begin{bmatrix}0.1342-0.3546i&-0.0767-0.0831i&0\\0.1534+0.1661i&0.1981-0.2854i&0\\-0.7540+0.8498i&-0.1406+0.8477i&0\end{bmatrix},$$

$$\overline{P}=(Fc-P)^{-1}P=\begin{bmatrix}0.1981+0.0479i&0.1725-0.3131i&0\\-0.3450+0.6262i&0.0543+0.3088i&0\\-3.3035-1.4121i&-2.6837+0.4260i&-1\end{bmatrix},$$

and

$$\overline{Q}=(Fc-P)^{-1}Q=\begin{bmatrix}-0.1712-1.1188i\\1.3757-1.2596i\\2.1527+6.2488i\end{bmatrix}.$$

Following the steps to find DI of $\overline{F}$, we get

$$W=\begin{bmatrix}0.1342-0.3546i & 0\\ 0.1534+0.1661i & 0.1333-0.2666i\\ -0.7540+0.8498i & -0.2667+0.5332i\end{bmatrix},$$

$$V=\begin{bmatrix}1 & 0.1334-0.2668i & 0\\ 0 & 1 & 0\end{bmatrix}.$$

Now, computing

$$V\overline{F}W=\begin{bmatrix}-0.0962-0.1684i & -0.0632+0.0105i\\ 0.1573-0.0430i & -0.0497-0.0908i\end{bmatrix},$$

which is a non-singular matrix. The desired DI is given by

$$\overline{F}^D=\begin{bmatrix}0.6668+3.0006i & -1.0008 & 0\\ 2.0007-0.0006i & 1.5004+3.0008i & 0\\ -6.2255-10.0015i & 0.3358-6.0027i & 0\end{bmatrix}.$$

To check the regressivity of $\overline{F}^D\overline{P}$, the eigenvalues of $\overline{F}^D\overline{P}$ are $0, -0.083477-1.351753i, -0.083153+1.351303i$. The spectral radius is $\rho\left(\overline{F}^D\overline{P}\right)=1.354328$. It means $\mu_{\max}.1.354328<1$.

So Lemma 2.5 holds and $\overline{F}^D\overline{P}$ is regressive. Hence,

$$Q_c=\begin{bmatrix}1.8662+0.0008i & 6.1242-0.0015i & 5.7104-0.0026i\\ 5.5006+0.0000i & -6.4847-0.0031i & 8.8438+0.0047i\\ -17.2229-0.0036i & -7.4443+0.0149i & 45.5257-0.0185i\end{bmatrix},$$

since $rank\ Q_c=3$, it implies that impulsive differential algebraic system (15) (correspondingly (16)) is controllable on $[0.5, 27.5]$.

4 | Observability

In this section, some ideas and equivalent criteria associated with the observability of impulsive differential algebraic system are proved and discussed. Likewise the concept of observability discussed in [32], the definition for observability of impulsive differential algebraic system is given as follows:

Definition 4.1 ([32]). If any initial state $z(t_0)=z_0\in\overline{\mathcal{V}}_i^*$ is uniquely determined by the corresponding input signal $u(t)$ and output signal $y(t), t\in[t_0,t_f]$, $(t_f\in(t_k,t_{k+1}])$. Then, the impulsive differential algebraic system S_2 (or S_1) is said to be state observable on $[t_0,t_f]$, $(t_f>t_0)$.

Next, to discuss the equivalent criteria for observability of the impulsive differential algebraic system S_2 (or S_1), we define the following matrices

$$M_i^{i+1}=M(t_0,t_i,t_{i+1})=\int_{t_i}^{t_{i+1}}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top \times R^\top R\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)\Delta\tau, \tag{17}$$

for $i=1,2,\ldots,k$,

$$M_k^f=M(t_0,t_k,t_f)=\int_{t_k}^{t_f}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top \times R\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)\Delta\tau, \tag{18}$$

and

$$M_0^1=M(t_0,t_0,t_1)=\int_{t_0}^{t_1}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top \times R\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)\Delta\tau, \tag{19}$$

Theorem 4.2. *Suppose that assumptions (A1)–(A4) are satisfied, the impulsive differential algebraic system S_2 (or S_1) is observable on $[t_0,t_f]$, $(t_f\in(t_k,t_{k+1}])$, if and only if observability Gramian matrix*

$$M(t_0;t_f)=M_0^1+\sum_{j=1}^{k-1}\prod_{j=1}^{i}(1+d_j)M_i^{i+1} +\prod_{j=1}^{k}(1+d_j)M_k^f,$$

is invertible, where $(1+d_i)\geq 0$, $i=1,2,\ldots k$, M_i^{i+1}, M_k^f, and M_0^1 are defined in Equations (17–19), respectively.

Proof. Suppose that the matrix $M(t_0;t_f)$ is invertible, the output of the impulsive differential algebraic system S_2 (or S_1) is:

$$y(t)=\begin{cases}Re_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}z_0+\int_{t_0}^{t}Re_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\\ \quad +R\sum_{q=0}^{r-1}\left(\overline{F}^D\overline{F}-I\right)\left(\overline{F}\overline{P}^D\right)^q\overline{P}^D\overline{Q}u^q(t),\ \text{for } t\in[t_0,t_1]\\ \prod_{i=1}^{k}(1+d_i)Re_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}z_0+\sum_{j=1}^{k}\prod_{i=j}^{k}(1+d_i)\\ \quad\times\int_{t_{j-1}}^{t_j}Re_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\\ \quad+\int_{t_k}^{t}Re_{\overline{F}^D\overline{P}}(t,\sigma(\tau))\overline{F}^D\overline{Q}u(\tau)\Delta\tau\\ \quad+\sum_{q=0}^{r-1}R\left(\overline{F}^D\overline{F}-I\right)\left(\overline{F}\overline{P}^D\right)^q\overline{P}^D\overline{Q}u^q(t),\quad \text{for } t\in(t_k,t_{k+1}],\ k=1,2,\ldots,\end{cases}$$

from Definition 4.1, observability of S_2 (or S_1) is equivalent to the observability of $y(t)$, given as follows:

$$y(t)=\begin{cases}Re_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}z_0,\ \text{for } t\in[t_0,t_1]\\ \prod_{i=1}^{k}(1+d_i)Re_{\overline{F}^D\overline{P}}(t,t_0)\\ \quad\times\overline{F}^D\overline{F}z_0,\ \text{for } t\in(t_k,t_{k+1}], k=1,2,\ldots,\end{cases} \tag{20}$$

as $u(t)=0$. Now pre-multiplying both sides of (20) by $\left(e_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}\right)^\top R^\top$ and integrating from t_0 to t_f with respect to t, we have

$$\int_{t_0}^{t_f}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top y(\tau)\Delta\tau$$

$$=\begin{pmatrix}\int_{t_0}^{t_1}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top Re_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\Delta\tau\\ +\sum_{i=1}^{k-1}\prod_{j=1}^{i}(1+d_j)\int_{t_i}^{t_{i+1}}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top\\ \times Re_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\Delta\tau\\ +\prod_{j=1}^{k}(1+d_j)\int_{t_k}^{t_f}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top\\ \times Re_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\Delta\tau\end{pmatrix}z_0 \quad (21)$$

Keeping in view Equations (17–19), an Equation (21) yields that

$$\int_{t_0}^{t_f}\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top y(\tau)\Delta\tau \quad (22)$$

$$=\left(M_0^1+\sum_{j=1}^{k-1}\prod_{i=j}^{k}(1+d_i)M_j^{j+1}+\prod_{i=j}^{k}(1+d_j)M_k^f\right)z_0. \quad (23)$$

It is clear that left-hand side of Equation (22) depends on system output $y(\cdot)$ and an algebraic expression in z_0. Since the matrix $M(t_0,t_f)$ is invertible, the initial state can be uniquely determined by the corresponding output of the system $y(\cdot)$.

Conversely, consider that the matrix $M(t_0,t_f)$ is not invertible, then $\exists\ x\neq 0$ and $x\in\overline{\mathcal{V}}_i^*$ such that

$$x^\top M(t_0,t_f)x=0.$$

Since $1+d_i\geq 0$ and M_i^{i+1}, M_k^f are positive semi-definite matrices. Therefore, for $i=0,1,\ldots k-1$,

$$x^\top M_i^{i+1}x=0 \quad (24)$$

and

$$x^\top M_k^f x=0. \quad (25)$$

Put $x=z_0$, then from Equations (20) and (24), (25), we get

$$\int_{t_0}^{t_f}y^\top(\tau)y(\tau)\Delta\tau$$

$$=\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}y^\top(\tau)y(\tau)\Delta\tau+\int_{t_k}^{t_f}y^\top(\tau)y(\tau)\Delta\tau$$

$$=\int_{t_0}^{t_1}z_0^\top\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top R^\top Re_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}z_0\Delta\tau$$

$$+\sum_{i=1}^{k-1}\left(\prod_{j=1}^{k}(1+d_j)\right)^2\int_{t_i}^{t_{i+1}}z_0^\top\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top$$

$$\times R^\top Re_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}z_0\Delta\tau$$

$$+\left(\prod_{j=1}^{k}(1+d_j)\right)^2\int_{t_k}^{t_f}z_0^\top\left(e_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}\right)^\top$$

$$\times R^\top Re_{\overline{F}^D\overline{P}}(\tau,t_0)\overline{F}^D\overline{F}z_0\Delta\tau$$

which returns into the following equation

$$\int_{t_0}^{t_f}y^\top(\tau)y(\tau)\Delta\tau=z_0^\top M_0^1z_0+\sum_{i=1}^{k-1}\left(\prod_{j=1}^{k}(1+d_j)\right)^2z_0^\top M_i^{i+1}z_0$$

$$+\left(\prod_{j=1}^{k}(1+d_j)\right)^2z_0^\top M_k^f z_0$$

$$=0.$$

Therefore, we have

$$\int_{t_0}^{t_f}\|y(\tau)\|^2\Delta\tau=0. \quad (26)$$

This follows that

$$0=y(t)$$

$$=\begin{cases}R\left(e_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}\right)z_0,\ t\in[t_0,t_1]\\ \prod_{j=1}^{l}(1+d_i)R\left(e_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}\right)z_0, & \text{for } t\in(t_l,t_{l+1}],\\ & l=1,2,\ldots,k-1,\\ \prod_{j=1}^{k}(1+d_i)R\left(e_{\overline{F}^D\overline{P}}(t,t_0)\overline{F}^D\overline{F}\right)z_0, & \text{for } t\in(t_k,t_f]\end{cases} \quad (27)$$

from (27), it implies that system is unobservable on $[t_0,t_f],t_f\in(t_k,t_{k+1}]$, which is contradiction. □

Theorem 4.3. *Suppose that assumptions A1–A4 are satisfied. The necessary and sufficient criteria for impulsive differential algebraic system S_2 (or S_1) is observable on $[t_0,t_f],t_f\in(t_k,t_{k+1}]$, is*

$$\text{rank }(Q)=n,$$

where $Q=\begin{bmatrix}C\\ C\left(\overline{FP}^D\right)\\ C\left(\overline{FP}^D\right)^2\\ \vdots\\ C\left(\overline{FP}^D\right)^{n-1}\end{bmatrix}.$

Proof. The proof of this theorem is on the similar pattern as in [32, Theorem 4.3]. □

Example 4.4. Consider the following impulsive differential algebraic system on arbitrary time scale such that $\mu_{\max}<1.0204$, where the matrices are defined as

$$Fz'(t)=Pz(t)+Qu(t),\ t\neq t_i,\ i=1,2,\ldots \quad (28)$$

$$z(t^+)=\frac{1}{3}z(t),\ t=t_i \text{ where } t_i=\frac{(6i-5)}{2}$$

$$y(t)=Cx(t) \quad (29)$$

$$z\left(\begin{bmatrix}0.5\\0.5\\0.5\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\ t\in[0.5,27.5]$$

$$F = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}, P = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 56 & 8 \\ 1 & 14 & 2 \end{bmatrix},$$

$$Q = \begin{bmatrix} 5 \\ 9 \\ 7 \end{bmatrix}, C = \begin{bmatrix} 11 & 17 & 13 \end{bmatrix}.$$

The impulsive points and conditions (A1) and (A2) satisfy. It is clear that $\det F = 0$, so F is not invertible and rank $F = 2$. Moreover, there exists $c \in \mathbb{C}$, such that $\det(Fc - P) \neq 0$, that is matrix pencil is regular. Taking $c = 1$, we have

$$F - P = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -52 & -8 \\ 0 & -8 & -2 \end{bmatrix}$$

and

$$(F - P)^{-1} = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.05 & 0.2 \\ 0 & 0.2 & -1.3 \end{bmatrix}$$

Pre-multiplying first equation of impulsive differential algebraic system (28) with $(Fc - P)^{-1}$, we obtain

$$\overline{F}z'(t) = \overline{P}z(t) + \overline{Q}u(t), \; t \neq \frac{(6i-5)}{2}, \; i = 1, 2, \ldots \tag{30}$$

where

$$\overline{F} = (F - P)^{-1}F = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 0 \end{bmatrix},$$

$$\overline{P} = (F - P)^{-1}P = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -7 & -1 \end{bmatrix}$$

and

$$\overline{Q} = (F - P)^{-1}Q = \begin{bmatrix} -0.5 \\ 0.95 \\ -7.3 \end{bmatrix}$$

Following the steps to find DI of $\overline{P}$, we get

$$\overline{P}^D = \begin{bmatrix} -0.6667 & 0 & 0 \\ 0 & 0 & -0.14 \\ 0 & -0 & -0.02 \end{bmatrix}.$$

and

$$\overline{FP}^D = \begin{bmatrix} 0.333 & 0 & 0 \\ 0 & 0 & -0.14 \\ 0 & -0 & 0.98 \end{bmatrix}$$

Eigenvalues are 0.333, 0, 0.98. Thus, $\rho\left(\overline{FP}^D\right) = 0.98$. $\mu_{\max}(0.98) < 1$ which implies that $\mu_{\max} < \frac{1}{0.98}$ and $\mu_{\max} < 1.0204$. Hence, $\overline{FP}^D$ is regressive. Hence,

$$Q = \begin{bmatrix} 11 & 17 & 13 \\ 1.2223 & 539 & 11 \\ 3.6668 & -77 & -11 \end{bmatrix}.$$

Since, rank$(Q) = 3$. This implies that the considered impulsive differential algebraic system on time scale is observable.

5 | Conclusion

This work substantially extends the existing continuous-time results to impulsive differential-algebraic systems on time scales, where several mathematical difficulties arise naturally. The singularity of the descriptor matrix leads to hidden algebraic constraints, while the hybrid continuous-discrete structure of time scales complicates the solution construction. Overcoming these difficulties, the obtained results broaden and generalize the classical theory and provide a unified framework for further developments in descriptor dynamic systems on time scales. We have developed necessary and sufficient criteria in terms of Gramian matrices for the controllability and observability of linear time invariant impulsive differential algebraic control systems on time scale. We have obtained these criteria using DI approach. Also, we have presented necessary and sufficient criteria with rank conditions for the controllability and observability of linear time invariant impulsive differential algebraic control systems. Moreover, we illustrate our results with examples. In future, we may develop necessary and sufficient criteria for the controllability and observability of linear time variant impulsive differential algebraic control systems on time scale.

Author Contributions

Awais Younus: conceptualization, investigation, writing – original draft, supervision, methodology. **Zoubia Dastgeer:** investigation, conceptualization, formal analysis, writing – original draft. **Thabet Abdeljawad:** conceptualization, formal analysis, validation, writing – review and editing, investigation, methodology. **Meriem Louafi:** writing – review and editing, validation, investigation.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Appendix

The following nomenclature is used in this paper:

$\mathbb{C}^{n\times n}$	n by n matrix with complex entries
$M_n(\mathbb{R})$	n by n matrix with real entries
$M_{n\times m}(\mathbb{R})$	n by m matrix with real entries
$\mathrm{R}(F^D)$	range of F^D
$\mathcal{N}(F^D)$	null of F^D

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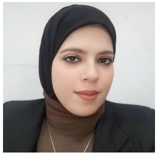

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