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Qualitative analysis of non-local fractional Sturm–Liouville–Langevin differential equation and inclusion with time delays

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ABSTRACT

We introduce time delays into the fractional Sturm–Liouville–Langevin framework to model hereditary effects and reaction lags in both single-valued and multivalued formulations. The problems are formulated using the generalized κ-Caputo fractional derivative, and we establish existence and uniqueness results. For the single-valued case, uniqueness is first obtained by applying the Banach fixed-point theorem with a κ-Bielecki norm. The effectiveness of this norm lies in its capability to relax the strong sufficient hypothesis commonly imposed in the application of Banach's fixed point theorem under the classical supremum norm — specifically, the contraction constant condition — by treating it in a more flexible and efficient manner through the use of this norm. In a special case, by subdividing the time interval and applying Burton's progressive contraction method, we obtain uniqueness under the relaxed Lipschitz assumption, thus extending beyond the strict hypothesis required by Banach's theorem — specifically, the contraction Lipschitz condition —. For the multivalued case, the Leray–Schauder nonlinear alternative is employed, which significantly broadens the existence theory by accommodating noncompact operators under L^1-Carathéodory conditions. Finally, illustrative examples validate the theoretical results.

1. Introduction

Statement of the problem. The following fractional time-delayed Sturm–Liouville–Langevin (FDSLL) problem has been investigated:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha_1;\kappa}\left(\rho(\mathfrak{z})\,{}^{C}D_{a^+}^{\alpha_2;\kappa}\varpi(\mathfrak{z}) + r(\mathfrak{z})\varpi(\mathfrak{z})\right) \\ \quad = \Psi(\mathfrak{z}, \varpi(\mathfrak{z}), \varpi(h(\mathfrak{z}))), \quad \mathfrak{z} \in [a,T], \quad 0 < a < T, \\ \varpi(\mathfrak{z}) = g(\mathfrak{z}), \qquad \mathfrak{z} \in [-q,a], \quad q > 0 \\ {}^{C}D_{a^+}^{\alpha_2;\kappa}\varpi(a) = V_{\alpha_2}, \end{cases} \tag{1.1}$$

where $\rho \in C([a,T], \mathbb{R} \setminus \{0\})$, $r \in C([a,T], \mathbb{R})$, $g \in C([-q,a], \mathbb{R})$, and $h \in C([a,T], [-q,T])$, with $h(\mathfrak{z}) \le \mathfrak{z}$ for all $\mathfrak{z} \in [a,T]$. Moreover, ${}^{C}D_{a^+}^{\alpha_i;\kappa}$ represents the κ-Caputo fractional derivative of order $0 < \alpha_i < 1$, defined with respect to an increasing function κ, where $i \in \{1,2\}$.

As a second problem, the multivalued version of the FDSLL problem (1.1) is addressed. This work significantly extends previous research by considering the following FDSLL inclusion problem:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha_1;\kappa}\left(\rho(\mathfrak{z})\,{}^{C}D_{a^+}^{\alpha_2;\kappa}\varpi(\mathfrak{z}) + r(\mathfrak{z})\varpi(\mathfrak{z})\right) \in \Pi(\mathfrak{z}, \varpi(\mathfrak{z}), \varpi(\delta\mathfrak{z})), \quad \mathfrak{z} \in [a,T], \\ \varpi(\mathfrak{z}) = g(\mathfrak{z}), \qquad \mathfrak{z} \in [-q,a], \quad q > 0, \\ {}^{C}D_{a^+}^{\alpha_2;\kappa}\varpi(a) = V_{\alpha_2}, \end{cases} \tag{1.2}$$

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where $\Pi : [-q,T] \times \overline{\mathcal{B}_r} \to \mathcal{P}(\mathbb{R})$ is a multivalued operator, $\mathcal{P}(\mathbb{R})$ represents the family of all non-empty subsets of $\mathbb{R}$, $\mathcal{B}_r = \{x \in \mathbb{R}^2 : \|x\| < r\}$ is the open ball of radius $r > 0$ centered at the origin, and $0 < \alpha_i < 1$, with $i \in \{1,2\}$.

Despite recent advances in the analysis of fractional Sturm–Liouville and Langevin systems, no previous study has undertaken a qualitative investigation of nonlinear models that simultaneously incorporate time delays and multivalued mappings using the κ-Caputo derivative. Addressing this gap is crucial for understanding the dynamics of systems with complex memory effects and their engineering applications, such as delayed-feedback control and viscoelastic materials. In our approach, a κ-Bielecki-type norm is first employed to establish uniqueness of solutions. In specific cases, Burton's progressive contraction method further extends uniqueness under a relaxed Lipschitz condition, while the Leray–Schauder nonlinear alternative is applied to prove existence for the multivalued case.

Relevant works and motivations. Modeling many real-world phenomena requires mathematical tools that capture both memory effects and the influence of past states. Fractional calculus, through its intrinsic nonlocal operators, naturally incorporates memory, while explicit time-delay terms more accurately represent a system's historical dependence [1,2]. Incorporating these elements into a Sturm–Liouville–Langevin framework yields a flexible model capable of reproducing both dissipative and oscillatory behavior [3]. This combined approach is particularly useful in applications such as viscoelasticity, anomalous diffusion, and delayed-feedback control systems, where classical integer-order models often give inadequate descriptions.

It is well established that non-integer-order calculus, commonly referred to as fractional calculus, constitutes a fundamental and versatile branch of mathematics that offers several advantages over classical calculus. One of its main strengths lies in its ability to model hereditary and memory effects inherent in many materials, as well as in the flexibility it provides by allowing the order of differentiation to be variable [4,5], distributed [6,7], or even complex [8,9]. Unlike conventional integer-order derivatives, most fractional derivatives are nonlocal operators. Consequently, improving analytical and numerical techniques for fractional differential models, including both equations and inclusions, is essential for advancing our understanding and accurate interpretation of complex dynamical phenomena.

Recent research has shown a growing interest in developing new fractional operators due to their superior ability to characterize memory and hereditary behaviors. In particular, various extensions of the Caputo fractional derivative have been proposed to enhance both analytical flexibility and computational tractability. Since some definitions of fractional differentiation generalize others, it is useful to study fractional differential equations and inclusions involving such generalized operator factors called the κ-Caputo fractional derivative, ${}^{C}D_{a^+}^{\alpha_1;\kappa}$ (Definition 3, below).

These advancements, along with recent investigations on the existence and approximate controllability of fractional delayed stochastic differential inclusions models [10], neutral differential systems with delay [11], and the application of topological degree methods to Ψ-Caputo fractional Langevin equations [12], provide a solid theoretical foundation that motivates and supports the formulation and analysis of the κ-Caputo-based Sturm–Liouville–Langevin models with time delays presented in this work. Furthermore, fixed point theorems play a major role as effective tools in the study of ordinary, partial, and fractional differential problems; we refer the reader to [13–17].

The historical context of our study traces back to Scottish scientist Robert Brown, who first observed Brownian motion—the random movement of microscopic particles suspended in a fluid. Because of its inherent randomness, mathematicians and physicists have extensively investigated stochastic processes to model this phenomenon. In 1908, Paul Langevin proposed a phenomenological model that describes the stochastic motion of a heavy Brownian particle of mass m immersed in a liquid. The system is governed by Newton's equation of motion, which forms the basis of the model. The resulting equation, known as the Langevin differential equation, is expressed as follows:

$$m\ddot{\mathfrak{y}}(\mathfrak{z}) = -\nu\dot{\mathfrak{y}}(\mathfrak{z}) + \mathfrak{g}(\mathfrak{z}).$$

The fractional nonlinear Langevin equation was first studied in [18–20], and is given by:

$$D_{a^+}^{\alpha_1}\left(D_{a^+}^{\alpha_2}\varpi(\mathfrak{z}) + \nu\varpi(\mathfrak{z})\right) = g(\mathfrak{z}, \varpi(\mathfrak{z})), \qquad \mathfrak{z} \in [0,1].$$

Recent studies have concentrated on the existence and uniqueness of solutions for initial boundary value problems related to nonlinear fractional Langevin problems. Significant contributions comprise [21–23], as well as the sources mentioned therein.

Complementing this, Sturm and Liouville theory, advanced in the mid-19th century, revolutionized mathematical analysis. Many real-world applications demonstrate its importance, especially in mathematics and engineering. The classical Sturm–Liouville problem is a BVP for a second-order linear differential equation and is given by:

$$\begin{cases} -\dfrac{d}{d\mathfrak{z}}\left[\rho(\mathfrak{z})\dfrac{d\varpi(\mathfrak{z})}{d\mathfrak{z}}\right] + \nu(\mathfrak{z})\varpi(\mathfrak{z}) = \lambda r(\mathfrak{z})\varpi(\mathfrak{z}), \quad \mathfrak{z} \in [a,b], \\ a_1\varpi(a) + a_2\varpi'(a) = 0, \\ b_1\varpi(b) + b_2\varpi'(b) = 0. \end{cases}$$

For further details on this topic, the reader is referred to [24–26].

Furthermore, the fractional expansion of the Sturm–Liouville system has recently been addressed by several authors [27,28]. The equation is written as follows:

$$-D_{b-}^{\alpha}\left(\rho(\mathfrak{z})D_{a+}^{\alpha}\varpi(\mathfrak{z})\right) + \nu(\mathfrak{z})\varpi(\mathfrak{z}) = \lambda r(\mathfrak{z})\varpi(\mathfrak{z}), \qquad \mathfrak{z} \in \,]a,b[\,,$$

where $\alpha \in \,]0,1]$, D_{b-}^{α} and D_{a+}^{α} denote the Riemann–Liouville fractional derivatives, while $\mathcal{I}_{b-}^{\alpha}$ represents the Riemann–Liouville fractional integral.

A significant development in this field is the generalized Sturm–Liouville–Langevin (GSLL) equation, which was proposed by [25], and is given by:

$$\begin{cases} D^{\beta}\left(p(\mathfrak{z})D^{\alpha}\varpi(\mathfrak{z}) + r(\mathfrak{z})\varpi(\mathfrak{z})\right) = h(\mathfrak{z}, \varpi(\mathfrak{z})), \quad 1 < \mathfrak{z} < T, \\ \varpi(1) = -\varpi(T), \quad D^{\alpha}\varpi(1) = -D^{\alpha}\varpi(T), \end{cases}$$

and has been extensively studied for its existence and uniqueness properties [29–31].

Fractional differential inclusions, which generalize fractional differential equations, have been effectively used to model various scientific, control, and engineering phenomena [32–34]. Recent advances include the study of multi-term semilinear fractional differential inclusions under Riemann–Liouville fractional operators [35] and the analysis of hybrid κ-Caputo fractional Langevin equations [36].

Despite these advancements, a critical gap remains in the literature. To the best of our knowledge, no previous study has undertaken a qualitative analysis of a nonlinear fractional Sturm–Liouville–Langevin (FSLL) problem that simultaneously incorporates time delays and is extended to the multivalued case. This paper fills these gaps by presenting a qualitative analysis of a novel class of nonlinear problems. Our work is based on the generalized κ-Caputo fractional derivative and the associated κ-Riemann–Liouville fractional integral [37].

The main contributions of this study are summarized as follows:

1. **A novel FSLL model with delays and inclusions.** We introduce, for the first time, a unified FSLL model that incorporates time delays and is extended to the multivalued setting. This model significantly broadens the range of phenomena that can be described, allowing for systems with memory effects, past-state dependence, and intrinsic uncertainties.
2. **Methodological advances in uniqueness theory.** We develop two complementary approaches to establish uniqueness:

- We introduce a κ-**Bielecki-type weighted norm** and apply the Banach fixed-point theorem within this framework. This approach mitigates the strong condition typically imposed when using the classical supremum norm, as encountered in related works such as [23]. Consequently, we obtain less restrictive and more widely applicable existence and uniqueness conditions.
- In a special case, we employ **Burton's progressive contraction method** to establish uniqueness under a condition weaker than the standard Lipschitz assumption, thereby extending the applicability of the theory to a broader class of nonlinear functions.

3. **A robust existence theory for multivalued problems.** For the multivalued FSLL inclusion, we prove the existence of solutions under general L^1-Carathéodory conditions by applying the **Leray–Schauder nonlinear alternative**. This approach is particularly suitable for multivalued mappings and does not require compactness assumptions, which significantly broadens the existence theory for such complex problems.
4. **This κ-Caputo fractional operator** provides a unified framework that includes, as particular cases, several classical–fractional derivatives, such as the Caputo, Hadamard, and Erdélyi–Kober derivatives. The introduction of the scaling function κ enhances the modeling capability by allowing the memory kernel to adapt to the underlying hereditary structure of the system, thus extending beyond the fixed power-law behavior inherent in standard fractional operators. From an analytical standpoint, the use of this generalized derivative requires the development of novel tools, notably a κ-Bielecki-type weighted norm, which effectively manages potential singularities introduced by κ (for instance, when $\kappa(\mathfrak{z}) = \ln \mathfrak{z}$ and the domain contains nonpositive points) and results in less restrictive conditions for the existence and uniqueness of solutions.
5. **A broad unification and recovery of classical models.** The existence, uniqueness and qualitative results obtained for both the FDSLL problem (1.1) and the FDSLL inclusion (1.2) encompass and extend a number of previously studied fractional models. In particular, our framework recovers the following important special cases:

    - If $r(\mathfrak{z}) = 0$ for every $\mathfrak{z} \in [a, T]$, the FDSLL problem (1.1) and the FDSLL inclusion (1.2) reduce, respectively, to a nonlinear fractional time-delayed Sturm–Liouville problem and to the corresponding nonlinear fractional time-delayed Sturm–Liouville inclusion. Thus all existence, uniqueness and qualitative statements derived here apply a fortiori to that class of Sturm–Liouville models.
    - If $\rho(\mathfrak{z}) = 1$ and $r(\mathfrak{z}) = \lambda$ with $\lambda \in \mathbb{R}$ for every $\mathfrak{z} \in [a, T]$, then (1.1) and (1.2) reduce, respectively, to the nonlinear fractional time-delayed Langevin problem and its inclusion analogue. Hence the present results also recover and extend the corresponding well-posedness and qualitative conclusions available for these classical Langevin-type models.

    Collectively, these reductions demonstrate that the proposed FSLL framework both unifies a range of models studied in the literature and strictly extends them by allowing spatially varying coefficients ρ and r, the generalized κ-operators, and multivalued nonlinearities.
6. One of the main contributions of this work is to demonstrate that both the fractional Sturm–Liouville–Langevin models, namely the FDSLL Eq. (1.1) and the FDSLL inclusion (1.2), provide a unified framework that encompasses and generalizes four well-known problems with concrete applications. More precisely, by appropriate choices of the involved coefficients, the proposed models reduce to:

    - a nonlinear fractional time-delayed Sturm–Liouville problem and its corresponding inclusion when $r(\mathfrak{z}) = 0$ for $\mathfrak{z} \in [a, T]$; and
    - the ordinary form of a nonlinear fractional time-delayed Langevin problem and its inclusion counterpart when $\rho(\mathfrak{z}) = 1$ and $r(\mathfrak{z}) = \lambda$, with $\lambda \in \mathbb{R}$, for $\mathfrak{z} \in [a, T]$. Hence, the theoretical results obtained in this paper simultaneously extend and unify these four classes of problems as particular cases.

Organization of the paper. The paper is organized as follows. In Section 2 we present the definitions, lemmas, and multivalued analysis used throughout the paper, which form the foundation for the subsequent theorems. In Section 3.1 we obtain the uniqueness of the solution to the FDSLL problem (1.1) by employing κ-Bielecki-type norms. Section 3.2 shows that, in certain cases, the uniqueness condition for problem (3.6) can be relaxed using Burton's method. In Section 3.3 we establish existence results for the inclusion version of problem (1.2) via the Leray–Schauder alternative. Section 4 illustrates three examples. Finally, Section 5 offers additional remarks, and Section 6 presents the study's conclusions.

2. Essential concepts and auxiliary results

The generalized fractional operators concerning strictly increasing function κ, known as the κ-Riemann–Liouville fractional operators and the κ-Caputo fractional derivative, are introduced. These operators generalize the classical notions of non-integer-order derivatives and are presented as a powerful framework for modeling a wide range of phenomena involving memory and heredity. To support subsequent results and facilitate analysis, a key lemma characterizing the relationship between these operators is provided.

Definition 1 (*[37]*). Let $\mu > 0$, $n \in \mathbb{N}$, $[a, T] \subseteq \mathbb{R}$ with $0 < a < T \leq \infty$. Consider a function $\Psi : [a, T] \to \mathbb{R}$ that is integrable in $[a, T]$, and a function $\kappa \in C^1_+$. The κ-Riemann–Liouville fractional integral of fractional order μ for a function Ψ is defined by

$$\mathcal{I}^{\mu;\kappa}_{a^+}\Psi(\mathfrak{z}) = \frac{1}{\Gamma(\mu)}\int_a^{\mathfrak{z}} \kappa'(s)(\kappa(\mathfrak{z}) - \kappa(s))^{\mu-1}\Psi(s)ds.$$

Definition 2 (*[37]*). Let $\mu \in (n-1, n)$, such that $n = \lfloor\mu\rfloor + 1$ and $\lfloor\mu\rfloor$ represents the integer piece of μ. Let a function $\kappa \in C^1_+$. The κ-Riemann–Liouville fractional derivative of fractional order μ for a function Ψ is expressed as

$$\mathcal{D}^{\mu;\kappa}_{a^+}\Psi(\mathfrak{z}) = \left(\frac{1}{\kappa'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}\right)^n \mathcal{I}^{n-\mu;\kappa}_{a^+}\Psi(\mathfrak{z})$$

or equivalently,

$$\mathcal{D}^{\mu;\kappa}_{a^+}\Psi(\mathfrak{z}) = \frac{1}{\Gamma(n-\mu)}\left(\frac{1}{\kappa'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}\right)^n \int_a^{\mathfrak{z}} \kappa'(s)(\kappa(\mathfrak{z}) - \kappa(s))^{n-\mu-1}\Psi(s)ds,$$

Definition 3 (*[37]*). Let $\mu > 0$, $n \in \mathbb{R}$, $[a, T] \subseteq \mathbb{R}$ with $0 < a < T \leq \infty$, and $\kappa, \Psi \in C^n([a, T], \mathbb{R})$, and $\kappa \in C^1_+$ The left-sided κ-Caputo fractional derivative of the function Ψ of order μ is defined as

$${}^c\mathcal{D}^{\mu;\kappa}_{a^+}\Psi(\mathfrak{z}) = \mathcal{I}^{n-\mu;\kappa}_{a^+}\left(\frac{1}{\kappa'(\mathfrak{z})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{z}}\right)^n\Psi(\mathfrak{z}),$$

such that $n = [\mu] + 1$ if $\mu \notin \mathbb{N}$ and $n = \mu$ if $\mu \in \mathbb{N}$.

For simplicity notation, let $\Psi^{[n]}_\kappa(\mathfrak{z}) = \left(\frac{1}{\kappa'(\mathfrak{z})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{z}}\right)^n\Psi(\mathfrak{z})$, so that the κ-Caputo fractional derivative can be rewritten as

$${}^c\mathcal{D}^{\mu;\kappa}_{a^+}\Psi(\mathfrak{z}) = \begin{cases} \displaystyle\int_a^{\mathfrak{z}} \frac{\kappa'(s)(\kappa(\mathfrak{z}) - \kappa(s))^{n-\mu-1}}{\Gamma(n-\mu)}\Psi^{[n]}_\kappa(s)\mathrm{d}s & \text{if } \mu \notin \mathbb{N} \\ \Psi^{[n]}_\kappa(\mathfrak{z}) & \text{if } \mu \in \mathbb{N} \end{cases}$$

Note that the generalized κ-Caputo fractional derivative considered in this study extends the classical Caputo framework by introducing a kernel defined through a strictly increasing function κ. The operator preserves linearity and, under suitable smoothness and norm conditions — particularly in the Banach space $(C([-q,T],\mathbb{R}),\|\cdot\|_{\theta,\alpha})$ equipped with the κ-Bielecki norm — is bounded. In general, an exact integration-by-parts formula is not available in the κ-setting without additional assumptions. However, the construction reduces to the classical Caputo fractional derivative when $\kappa(\mathfrak{z})=\mathfrak{z}$, and recovers several other well-known cases (e.g., Hadamard, Erdélyi–Kober). Importantly, while the κ-Caputo fractional operator does not satisfy a semigroup property in general, we have established conditions under which the semigroup laws hold. These structural features underscore the flexibility and mathematical richness of the κ-Caputo fractional approach for modeling memory-dependent phenomena..

Lemma 1 ([37]). *Let $\mu>0$ and $\Psi:[a,T]\to\mathbb{R}$. Then the next results hold:*

- *If* $\Psi\in C([a,T])$, *then* ${}^{C}\mathcal{D}^{\mu,\kappa}_{a^+}\mathcal{I}^{\mu,\kappa}_{a^+}\Psi(\mathfrak{z})=\Psi(\mathfrak{z})$.
- *If* $\Psi\in C^{n-1}([a,T])$, *then* $\mathcal{I}^{\mu,\kappa\,C}_{a^+}\mathcal{D}^{\alpha,\kappa}_{a^+}\Psi(\mathfrak{z})=\Psi(\mathfrak{z})-\sum_{k=0}^{n-1}c_k\,(\kappa(\mathfrak{z})-\kappa(a))^k$.

where $c_k=\dfrac{\Psi^{[k]}_\kappa(a)}{k!}$, $\Psi^{[k]}_\kappa(a)=\left[\dfrac{1}{\kappa'(\mathfrak{z})}\dfrac{d}{d\mathfrak{z}}\right]^k\Psi(a)$ *and* $n-1<\mu\le n$.

Particularly, for $\mu\in(0,1)$, it is found that $\mathcal{I}^{\mu,\kappa\,C}_{a^+}\mathcal{D}^{\mu,\kappa}_{a^+}\Psi(\mathfrak{z})=\Psi(\mathfrak{z})-\Psi(a)$.

To broaden the scope of analysis, the focus is now shifted to the study of multivalued analysis. This transition enables a wider range of mathematical problems to be addressed. For more on the topic, see [29].

For a normed space $(E,\|\cdot\|)$, the following is defined:
$\mathcal{P}(E)=\{F\subset E:\quad F\neq\emptyset\}$ *and* $\mathcal{P}_{c,cp}(X)=\{F\subset E:$
Y is convex and compact$\}$

For the foundational concepts of multivalued analysis, those seeking deeper understanding can consult [38,39]

Definition 4. A multivalued operator $\Pi:[a,b]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is called *Carathéodory* if:

1. The map $\mathfrak{z}\mapsto\Pi(\mathfrak{z},\varpi)$ is measurable for each $\mathfrak{z}\in\mathbb{R}$, and
2. The map $\varpi\mapsto\Pi(\mathfrak{z},\varpi)$ is upper semicontinuous for a.e. $\varpi\in[a,b]$.

Additionally, a Carathéodory map Π is said to be $\mathbf{L}^1$-*Carathéodory* if, for every $\rho>0$, there exists a function $\varphi_\rho\in\mathbf{L}^1([a,b];\mathbb{R})$ such that

$\|\Pi(\mathfrak{z},\varpi)\|=\sup\{|\mathfrak{z}|:\mathfrak{z}\in\Pi(\mathfrak{z},\varpi)\}\le\varphi_\rho(\mathfrak{z})$,

for all $\varpi\in\mathbb{R}$ with $\|\varpi\|\le\rho$, and for almost every $\mathfrak{z}\in[a,b]$.

To establish a uniqueness result for the problem (1.1), and an existence result for the inclusion version problem (1.2), the following three fixed-point theorems will be utilized:

The following result, known as the Banach fixed point theorem, establishes the existence and uniqueness of fixed points for contraction mappings.

Theorem 1 ([40]). *Let (F,d) be a complete metric space, and let $\Psi: F\to F$ be a contraction mapping. Then Ψ admits one fixed point $\varpi^*\in F$, satisfying $\Psi\varpi^*=\varpi^*$.*

The following theorem, known as the Arzelà–Ascoli Theorem, provides a characterization of relatively compact subsets in spaces of continuous functions.

Theorem 2 ([41]). *Consider (F,d) be a compact metric space. A subset Ψ of the space of continuous functions $C(\varpi)$ is relatively compact iff Ψ is uniformly bounded and equicontinuous.*

The Leray–Schauder nonlinear alternative theorem for multivalued maps provides conditions ensuring the existence of fixed points in Banach spaces (as stated below).

Theorem 3. *[Nonlinear Alternative for Kakutani maps [42,43]]*

Let E be a Banach space and let $\mathcal{N}_1$ a closed convex subset of E, U an open subset of $\mathcal{N}_1$, with $0\in U$. Suppose $\Pi:\overline{U}\to\mathcal{P}_{c,cp}(\mathcal{N}_1)$ be an upper semicontinuous and compact multivalued map.

Then, at least one of the following statements is true:

(A) Π has a fixed point in $\overline{U}$ (i.e., there exists $\varpi^*\in\overline{U}$ such that $\varpi^*\in\Pi(\varpi^*)$).
(B) There exist $\varpi\in\partial U$ and $\varepsilon\in(0,1)$ such that $\varpi\in\varepsilon\Pi(\varpi)$.

3. Main results

Differential equations and differential inclusions and their solutions are fundamental to mathematical analysis and offer valuable insights into a range of scientific and engineering problems. By examining the existence and uniqueness results within various mathematical frameworks, the FDSLL problem (1.1) and the multivalued version (1.2) are the main emphasis, to add to the body of current knowledge.

Initially, the uniqueness of solutions to the proposed FDSLL problem (1.1) is explored employed the Banach contraction technique endowed with a κ-Bielecki-type norm. This approach establishes conditions under which a unique solution exists, thereby providing a robust mathematical foundation. To further expand the uniqueness results, Burton progressive contraction approach is then used for the FDSLL problem (3.6), in which a more generic condition **(H4)** is used in place of the original condition **(H2)**. This modification improves our comprehension of the behavior of the solution while also expanding the theorem application.

The following lemma, which proves the equivalence between the FDSLL problem (1.1) and the corresponding integral Eq. (3.1), forms a fundamental part of the analysis. This finding establishes a fundamental connection between the integral equation and the differential equation. This equivalence is essential because it enables the analysis to make use of the potent instruments of integral equation theory.

Lemma 2. *Let $0<\alpha_1,\alpha_2\le 1$. Assume that $\Psi\in\mathbf{L}^1([a,T]\times\mathbb{R}^2,\mathbb{R})$, $\rho\in C([a,T],\mathbb{R}^\star)$, and $r\in C([a,T],\mathbb{R})$. Then, the function y is a solution to the problem (1.1) if and only if y satisfies the equivalent integral equation:*

$$\begin{aligned}\varpi(\mathfrak{z})=&\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\Psi\big(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(h(\mathfrak{z}))\big)}{\rho(\mathfrak{z})}\right)-\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\right)\\&+\big(r(a)g(a)+\rho(a)V_{\alpha_2}\big)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{1}{\rho(\mathfrak{z})}\right)+g(a),\end{aligned}\tag{3.1}$$

for $\mathfrak{z}\in[a,T]$, where $\varpi(\mathfrak{z})=g(\mathfrak{z})$ for $\mathfrak{z}\in[-q,a]$.

Proof. By applying the operator $\mathcal{I}^{\alpha_1,\kappa}_{a^+}$ to both sides of Eq. (1.1) and invoking Lemma 1, the following result is obtained.

$$\begin{aligned}&\rho(\mathfrak{z})^{C}\mathfrak{D}^{\alpha_2;\kappa}_{a^+}\varpi(\mathfrak{z})+r(\mathfrak{z})\varpi(\mathfrak{z})-\rho(a)^{C}\mathfrak{D}^{\alpha_2;\kappa}_{a^+}\varpi(a)-r(a)\varpi(a)\\&=\mathcal{I}^{\alpha_1,\kappa}_{a^+}\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(h(\mathfrak{z}))),\end{aligned}$$

and hence

$${}^{C}\mathfrak{D}^{\alpha_2;\kappa}_{a^+}\varpi(\mathfrak{z})=\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(h(\mathfrak{z})))}{\rho(\mathfrak{z})}-\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}+\frac{\rho(a)V_{\alpha_2}}{\rho(\mathfrak{z})}+\frac{r(a)g(a)}{\rho(\mathfrak{z})}.\tag{3.2}$$

By employing Lemma 1 and applying the operator $\mathcal{I}^{\alpha_2,\kappa}_{a^+}$ to both sides of Eq. (3.2), the following result is obtained.

$$\begin{aligned}\varpi(\mathfrak{z})=&\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(h(\mathfrak{z})))}{\rho(\mathfrak{z})}\Big)-\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\Big)\\&+\Big(r(a)g(a)+\rho(a)V_{\alpha_2}\Big)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big)+g(a).\end{aligned}\tag{3.3}$$

On the other hand, it is simple to see that the solution (3.3) verifies the fractional differential problem and the boundary conditions in (1.1) with the help of Lemma 1. □

Throughout the paper, the following notations will be utilized for readability and clarity:

$$\begin{aligned}\Phi_1 &= \frac{\big(\kappa(T)-\kappa(a)\big)^{\alpha_1+\alpha_2}}{\underline{\rho}\Gamma(\alpha_1+\alpha_2+1)}, \qquad \Phi_2 = \frac{\bar{r}\big(\kappa(T)-\kappa(a)\big)^{\alpha_2}}{\underline{\rho}\Gamma(\alpha_2+1)},\\ \Phi_3 &= \frac{\big(|r(a)|\,|g(a)|+|\rho(a)|\,\big|V_{\alpha_2}\big|\big)\big(\kappa(T)-\kappa(a)\big)^{\alpha_2}}{\underline{\rho}\Gamma(\alpha_2+1)},\end{aligned} \tag{3.4}$$

$$\underline{\rho} = \inf_{\mathfrak{z}\in[a,T]} |\rho(\mathfrak{z})|, \quad \text{and} \quad \bar{r} = \sup_{\mathfrak{z}\in[a,T]} |r(\mathfrak{z})|. \tag{3.5}$$

To enhance the clarity, readability, and overall flow, we present the underlying assumptions (**(H1)–(H8)**) in a structured manner.

1. **(H1)** $\Psi \in C([a,T]\times\mathbb{R}^2,\mathbb{R})$ and $h \in C([a,T],[-q,T])$, with $h(\mathfrak{z}) \le \mathfrak{z}$ for all $\mathfrak{z}\in[a,T]$.
2. **(H2)** There exists a constant $L_\Psi \in \mathbb{R}_+$ such that
$$\left|\Psi(\mathfrak{z},\varpi_1,\varpi_2)-\Psi(\mathfrak{z},\widetilde{\varpi}_1,\widetilde{\varpi}_2)\right| \le L_\Psi\left(|\varpi_1-\widetilde{\varpi}_1|+|\varpi_2-\widetilde{\varpi}_2|\right),$$
for all $\varpi_1,\widetilde{\varpi}_1,\varpi_2,\widetilde{\varpi}_2\in\mathbb{R}$ and $\mathfrak{z}\in[a,T]$.
3. **(H3)** The function $\Psi : [a,T]\times\mathbb{R}^2\to\mathbb{R}$ is continuous.
4. **(H4)** There exists a constant $L_\Psi \in \mathbb{R}_+$ such that
$$\left|\Psi(\mathfrak{z},\varpi_1,\varpi_2)-\Psi(\mathfrak{z},\widetilde{\varpi}_1,\varpi_2)\right| \le L_\Psi\left|\varpi_1-\widetilde{\varpi}_1\right|,$$
for all $\varpi_1,\widetilde{\varpi}_1,\varpi_2\in\mathbb{R}$ and $\mathfrak{z}\in[a,T]$.
5. **(H5)** $\Phi_2 < 1$, where Φ_2 is defined in (3.4).
6. **(H6)** The map $\Pi : [a,T]\times\mathbb{R}^2 \to \mathcal{P}_{c,cp}(\mathbb{R})$ is an $\mathbf{L}^1$-Carathéodory multivalued map with non-empty, compact, and convex values. Furthermore, for any fixed $\varpi\in C([a,T],\mathbb{R})$, the set
$$S_{\Pi,\varpi} = \left\{v\in\mathbf{L}^1([a,T],\mathbb{R}) : v(\mathfrak{z})\in\Pi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\theta t)) \quad \text{for a.e. } \mathfrak{z}\in[a,T]\right\}$$
is non-empty and convex.
7. **(H7)** The map Π satisfies
$$|\Pi(\mathfrak{z},x,\varpi)| := \sup\{|v| : v\in\Pi(\mathfrak{z},x,\varpi)\} \le p(\mathfrak{z})\chi(|x|+|\varpi|)$$
for all $\mathfrak{z}\in[a,T]$ and all $x,\varpi\in C([a,T],\mathbb{R})$, where $p\in\mathbf{L}^1([a,T],\mathbb{R}^+)$ and $\chi : \mathbb{R}^+\to[0,\infty)$ is a continuous, non-decreasing function.
8. **(H8)** There exists a constant $M>0$ and $M>M_g$ such that
$$\frac{(1-\Phi_2)M}{\|p\|\chi(2M)\Phi_1+\Phi_3} > 1,$$
where Φ_1,Φ_2,Φ_3 are as defined in (3.4).

The space $C([a,T],\mathbb{R})$ is equipped with the κ-Bielecki-type norm $\|\cdot\|_{\theta,\alpha}$ is defined by

$$\Theta_\alpha(\theta,\mathfrak{z}) := \begin{cases}\mathbb{E}_\alpha\big(\theta(\kappa(\mathfrak{z})-\kappa(a))^\alpha\big), & \mathfrak{z}\in(a,T],\\ \mathbb{E}_\alpha\big(\theta(\kappa(a)-\kappa(a))^\alpha\big)=\mathbb{E}_\alpha(0)=1, & \mathfrak{z}\in[-q,a].\end{cases}$$

Then set

$$\|\varpi\|_{\theta,\alpha} = \sup_{\mathfrak{z}\in[-q,T]}\frac{|\varpi(\mathfrak{z})|}{\Theta_\alpha(\theta,\mathfrak{z})}, \quad \theta,\alpha>0,$$

where $\mathbb{E}_\alpha$ denotes the Mittag-Leffler function with a single parameter α and $\kappa\in C^1_+ = \{g : g\in C^1([a,T],\mathbb{R})$ and $g'(\mathfrak{z})>0$ on $\mathfrak{z}\in[a,T]\}$. If $\alpha\to1$, the norm $\|\varpi\|_{\theta,\alpha}$ reduces to

$$\Theta(\theta,\mathfrak{z}) := \begin{cases} e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}, & \mathfrak{z}\in(a,T],\\ e^{\theta(\kappa(a)-\kappa(a))}=1, & \mathfrak{z}\in[-q,a].\end{cases}$$

Then set

$$\|\varpi\|_\theta = \sup_{\mathfrak{z}\in[-q,T]}\frac{|\varpi(\mathfrak{z})|}{\Theta(\theta,\mathfrak{z})}, \quad \theta>0.$$

Under this norm, the pair $\big(C([-q,T],\mathbb{R}),\|\cdot\|_\theta\big)$ forms a Banach space.

Remark 1. Note that the function κ need not be defined at every point of the interval $[-q,a]$ (where $q,a>0$). For instance, if $\kappa(\mathfrak{z})=\ln(\mathfrak{z})$ (${}^C D^{\alpha_1;\kappa}_{a^+} := {}^H D^{\alpha_1}_{a^+}$ Hadamard fractional derivative) and $-q<0<a$, then κ is not defined at $\mathfrak{z}\le0$. Such possible singularities motivate the introduction of the weighted norm associated with the function $\Theta_\alpha(\theta,\mathfrak{z})$ defined above. The weighted norm is designed to control the behavior of functions (and of the operator) near points where κ may fail to defined, and hence to guarantee finiteness of the relevant norms and applicability of the subsequent estimates.

3.1. Banach contraction and uniqueness of solution: A theoretical analysis

In this subsection, we use two κ-Caputo fractional derivatives to formulate the proposed FDSLL problem (1.1) and study the uniqueness of its solution.

Theorem 4. *Assume that the hypotheses* **(H1)** *and* **(H2)** *are valid. Then, the FDSLL problem (1.1) has a unique solution in $C([-q,T],\mathbb{R})$.*

To prove Theorem 4, heavy reliance will be placed on the following partial result:

Lemma 3 ([29]). *Let $\mu,\theta>0$ and $\kappa\in C^n([a,T],\mathbb{R})$, where $\kappa\in C^1_+$. Then, the next inequality verifies:*

$$\mathcal{I}^{\mu;\kappa}_{a^+} e^{\theta(\kappa(\mathfrak{z})-\kappa(a))} \le \frac{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}}{\theta^\mu}, \qquad a\le\mathfrak{z}\le T.$$

Proof ([29]). By applying the κ-Riemann–Liouville operator $\mathcal{I}^{\mu;\kappa}_{a^+}$ to the function $\mathfrak{z}\mapsto e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}$, and using the substitution $\varpi=\kappa(\mathfrak{z})-\kappa(s)$ and $z=\theta\varpi$, the following result is obtained:

$$\begin{aligned}\mathcal{I}^{\mu;\kappa}_{a^+} e^{\theta(\kappa(\mathfrak{z})-\kappa(a))} &= \frac{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}}{\Gamma(\mu)}\int_a^{\kappa(\mathfrak{z})-\kappa(a)} \varpi^{\mu-1}e^{-\theta\varpi}\,\mathrm{d}\varpi\\ &= \frac{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}}{\Gamma(\mu)\theta^\mu}\int_a^{\theta(\kappa(\mathfrak{z})-\kappa(a))} z^{\mu-1}e^{-z}\,\mathrm{d}z\\ &\le \frac{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}}{\Gamma(\mu)\theta^\mu}\int_a^{\infty} z^{\mu-1}e^{-z}\,\mathrm{d}z\\ &= \frac{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}}{\theta^\mu}.\end{aligned}$$ □

Proof. Define the operator $\mathcal{F} : C([-q,T],\mathbb{R})\to C([-q,T],\mathbb{R})$ by

$$\mathcal{F}\varpi(\mathfrak{z}) = \begin{cases} g(\mathfrak{z}), & -q\le\mathfrak{z}\le a,\\ \text{the right hand side of Eq. (3.1)}, & a<\mathfrak{z}\le T.\end{cases}$$

By Lemma 2, the fixed points of $\mathcal{F}$ coincide exactly with the solutions of Eq. (1.1). We now show that $\mathcal{F}$ is a contraction for a sufficiently large choice of $\theta>0$.

Fix $\varpi,\widetilde{\varpi}\in C([-q,T],\mathbb{R})$. Since $\mathcal{F}\varpi=\mathcal{F}\widetilde{\varpi}=g$ on $[-q,a]$, it remains to estimate the difference for $\mathfrak{z}\in[a,T]$. Using the Lipschitz condition **(H2)**, we have

$$\begin{aligned}\left|\mathcal{F}\varpi(\mathfrak{z})-\mathcal{F}\widetilde{\varpi}(\mathfrak{z})\right| &\le \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Bigg(\frac{1}{|\rho(\mathfrak{z})|}\,\mathcal{I}^{\alpha_1,\kappa}_{a^+}\Big|\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(h(\mathfrak{z})))\\ &\quad -\Psi(\mathfrak{z},\widetilde{\varpi}(\mathfrak{z}),\widetilde{\varpi}(h(\mathfrak{z})))\Big|\Bigg)\\ &\quad + \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{|r(\mathfrak{z})|}{|\rho(\mathfrak{z})|}\,|\varpi(\mathfrak{z})-\widetilde{\varpi}(\mathfrak{z})|\Big)\\ &\le \frac{L_\Psi}{\underline{\rho}}\,\mathcal{I}^{\alpha_2,\kappa}_{a^+}\big(\mathcal{I}^{\alpha_1,\kappa}_{a^+}\big(|\varpi(\mathfrak{z})-\widetilde{\varpi}(\mathfrak{z})|+|\varpi\circ h(\mathfrak{z})-\widetilde{\varpi}\circ h(\mathfrak{z})|\big)\big)\\ &\quad + \frac{\bar{r}}{\underline{\rho}}\,\mathcal{I}^{\alpha_2,\kappa}_{a^+}\big(|\varpi(\mathfrak{z})-\widetilde{\varpi}(\mathfrak{z})|\big).\end{aligned}$$

where $\underline{\rho}$ and $\bar{r}$ are as defined in (3.5).

Now use the Bielecki-type norm: for any $\mathfrak{z}\in[a,T]$,

$$|\varpi(s)-\widetilde{\varpi}(\mathfrak{z})| \le \|\varpi-\widetilde{\varpi}\|_\theta\,\Theta(\theta,s) = \|\varpi-\widetilde{\varpi}\|_\theta\,e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}.$$

Also, since $h(\mathfrak{z})\le s$ and κ is increasing,

$$|\varpi(h(\mathfrak{z}))-\widetilde{\varpi}(h(\mathfrak{z}))| \le \|\varpi-\widetilde{\varpi}\|_\theta\,e^{\theta(\kappa(h(\mathfrak{z}))-\kappa(a))} \le \|\varpi-\widetilde{\varpi}\|_\theta\,e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}.$$

Therefore the sum $|\varpi(\mathfrak{z})-\widetilde{\varpi}(\mathfrak{z})|+|\varpi(h(\mathfrak{z}))-\widetilde{\varpi}(h(\mathfrak{z}))|$ is bounded by $2\|\varpi-\widetilde{\varpi}\|_\theta\, e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}$.

Applying these bounds inside the fractional integrals gives

$$\left|\mathcal{F}\varpi(\mathfrak{z})-\mathcal{F}\widetilde{\varpi}(\mathfrak{z})\right| \le \frac{2L_\Psi}{\underline{\rho}}\|\varpi-\widetilde{\varpi}\|_\theta\, \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\mathcal{I}^{\alpha_1,\kappa}_{a^+} e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}\right) + \frac{\bar{r}}{\underline{\rho}}\|\varpi-\widetilde{\varpi}\|_\theta\, \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}\right).$$

Combine the two the κ-Riemann–Liouville fractional integrals: $\mathcal{I}^{\alpha_2,\kappa}_{a^+}\mathcal{I}^{\alpha_1,\kappa}_{a^+}=\mathcal{I}^{\alpha_1+\alpha_2,\kappa}_{a^+}$. Thus

$$\left|\mathcal{F}\varpi(\mathfrak{z})-\mathcal{F}\widetilde{\varpi}(\mathfrak{z})\right| \le \left(\frac{2L_\Psi}{\underline{\rho}}\, \mathcal{I}^{\alpha_1+\alpha_2,\kappa}_{a^+}\left(e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}\right) + \frac{\bar{r}}{\underline{\rho}}\, \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}\right)\right)\|\varpi-\widetilde{\varpi}\|_\theta.$$

Now apply Lemma 3: for any $\mu>0$,

$$\mathcal{I}^{\mu,\kappa}_{a^+}\left(e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}\right) \le \frac{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}}{\theta^\mu}.$$

Using this with $\mu=\alpha_1+\alpha_2$ and $\mu=\alpha_2$ yields

$$\left|\mathcal{F}\varpi(\mathfrak{z})-\mathcal{F}\widetilde{\varpi}(\mathfrak{z})\right| \le e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}\left(\frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}}+\frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\right)\|\varpi-\widetilde{\varpi}\|_\theta.$$

Divide both sides by $\Theta(\theta,z)=e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}$ and take supremum over $\mathfrak{z}\in[-q,T]$ to obtain the norm estimate

$$\|\mathcal{F}\varpi-\mathcal{F}\widetilde{\varpi}\|_\theta \le \left(\frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}}+\frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\right)\|\varpi-\widetilde{\varpi}\|_\theta.$$

Set

$$\frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}}+\frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}} \le \frac{2L_\Psi+\bar{r}}{\underline{\rho}\,\min\{\theta^{\alpha_1+\alpha_2},\theta^{\alpha_2}\}}.$$

Since $\alpha_1,\alpha_2>0$, $\left(\frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}}+\frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\right)\le \frac{2L_\Psi+\bar{r}}{\underline{\rho}\,\min\{\theta^{\alpha_1+\alpha_2},\theta^{\alpha_2}\}} \to 0$ as $\theta\to\infty$. Hence we can choose $\theta>0$ large enough such that $\frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}}+\frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\le \frac{2L_\Psi+\bar{r}}{\underline{\rho}\,\min\{\theta^{\alpha_1+\alpha_2},\theta^{\alpha_2}\}}<1$. For such θ the operator $\mathcal{F}$ is a contraction on $(C([-q,T],\mathbb{R}),\|\cdot\|_\theta)$. By the Banach fixed-point theorem $\mathcal{F}$ admits a unique fixed point in $C([-q,T],\mathbb{R})$, which is the unique solution of (1.1). □

3.2. Burton fixed-point approach to establish uniqueness of solution under weaker conditions

In this subsection, the uniqueness result for the integral Eq. (3.7) is established using the Burton approach, also known as the progressive contraction method [44]. This is achieved by replacing condition **(H2)** with the subsequent condition **(H4)**, which is more general. Consequently, in a special case, under a weaker assumption and by subdividing the time interval, we obtain a unique solution to the FDSLL problem (1.1), thereby contributing to a deeper understanding of the behavior of solutions within this class of equations.

Theorem 5. *Assume that the hypotheses **(H3)** and **(H4)** are valid. Then, the FDSLL problem (1.1) has a unique solution in $C([-q,T],\mathbb{R})$.*

Remark 2.

1. It is observed that if we consider a specific form of the delay function $h(\mathfrak{z})=\mathfrak{z}-q$ in problem (1.1), where $q>0$ is a constant delay, then the Lipschitz condition (**H2**) with respect to the third variable can be relaxed, allowing us to replace (**H2**) with a weaker condition (**H4**) that does not require Lipschitz continuity in the third argument. In problem (1.1), $h(\mathfrak{z})=\mathfrak{z}-q$ is considered, where $q>0$ represents a constant delay.

$$\begin{cases} {}^C\mathfrak{D}^{\alpha_1;\kappa}_{a^+}\left(\rho(\mathfrak{z})\,{}^C\mathfrak{D}^{\alpha_2;\kappa}_{a^+}\varpi(\mathfrak{z})+r(\mathfrak{z})\varpi(\mathfrak{z})\right)=\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\mathfrak{z}-q)), & \mathfrak{z}\in[a,T]\\ \varpi(\mathfrak{z})=g(\mathfrak{z}) & \mathfrak{z}\in[-q,a]\\ {}^C\mathfrak{D}^{\alpha_2;\kappa}_{a^+}\varpi(a)=V_{\alpha_2} \end{cases} \tag{3.6}$$

where $0<\alpha_i<1 \quad i\in\{1,2\}$.

Solving the above problem is equivalent to solving the following integral equation in $[a,T]$ (see Lemma 2), where the solution coincides with the initial function g on the interval $[-q,a]$.

$$\begin{aligned}\varpi(\mathfrak{z}) = &\,\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\mathfrak{z}-q))}{\rho(\mathfrak{z})}\right)-\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\right)\\ &+\left(r(a)g(a)+\rho(a)V_{\alpha_2}\right)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{1}{\rho(\mathfrak{z})}\right)+g(a).\end{aligned} \tag{3.7}$$

where $\underline{\rho}$ and $\bar{r}$ are as defined in (3.5).

2. **Choice of Partition.** To make the Burton (progressive-contraction) partitioning fully explicit we choose a step size S with $0<S<q$. A convenient specific choice is $S:=q/2$. Let $n:=\left\lceil\frac{T-a}{S}\right\rceil$ and set $T_0=a$, $T_i:=a+iS$ for $i=1,\dots,n-1$, and $T_n:=T$. Then each subinterval satisfies $T_i-T_{i-1}\le S<q$. For any $\mathfrak{z}\in[T_{i-1},T_i]$ we have $\mathfrak{z}-q\le T_i-q\le T_{i-1}$, so the delayed argument $\mathfrak{z}-q$ lies in the previously determined region $[-q,T_{i-1}]$. This choice guarantees the delayed value is known when solving on $[T_{i-1},T_i]$, and it avoids the (misleading) formula $nS=T$ which should instead read $nS\ge T-a$ (with equality only in the idealized equal-length case).

   Illustration of a Misleading Formulation. For concreteness take $a=0$, $T=5$ and $q=2$, and choose $S:=q/2=1$. The partition with step S is $[0,1],[1,2],[2,3],[3,4],[4,5]$, so $n=5$ and $nS=5=T-a$. If we take $\mathfrak{z}=3.7\in[3,4]$ then $\mathfrak{z}-q=3.7-2=1.7\in[1,2]$, i.e. $\mathfrak{z}-q$ lies in the previous subinterval and is therefore known when solving on $[3,4]$.
   If instead $T=4.3$ then $n=\lceil (T-a)/S\rceil=\lceil 4.3\rceil=5$, so $nS=5>4.3$ and the last subinterval becomes $[4,4.3]$ (length $0.3<S$). The same argument still holds: for any $\mathfrak{z}\in[T_{i-1},T_i]$ we have $\mathfrak{z}-q\le T_{i-1}$, hence $\mathfrak{z}-q$ belongs to the already processed region $[-q,T_{i-1}]$.

Proof. Partition and choice of S. Choose $S\in(0,q)$; for concreteness set $S:=q/2$. Define $n:=\lceil (T-a)/S\rceil$, set $T_0:=a$, $T_i:=a+iS$ for $i=1,\dots,n-1$, and $T_n:=T$. Then each subinterval $[T_{i-1},T_i]$ satisfies $T_i-T_{i-1}\le S<q$.

The key inequality. Take any $i=1,\dots,n$ and any $\mathfrak{z}\in[T_{i-1},T_i]$. Since $\mathfrak{z}\le T_i$ and $T_i-T_{i-1}\le S$ we get

$$\mathfrak{z}-q\le T_i-q=T_{i-1}+(T_i-T_{i-1})-q\le T_{i-1}+S-q.$$

Because $S<q$ we have $S-q<0$, hence

$$\mathfrak{z}-q\le T_{i-1}+S-q\le T_{i-1}.$$

Therefore for every $\mathfrak{z}\in[T_{i-1},T_i]$ the delayed argument $\mathfrak{z}-q$ lies in $(-\infty,T_{i-1}]$, and in particular in the previously processed region $[-q,T_{i-1}]$. In the first subinterval $i=1$ this yields $\mathfrak{z}-q\le a$, so $\mathfrak{z}-q\in[-q,a]$ (the initial data region).

By employing these partitions, it will be demonstrated in each step that a unique continuous solution exists. In the subsequent step, the solution obtained in the previous phase is utilized as the initial function. Additionally, $\theta>0$ is selected to be sufficiently large such that

$$\frac{L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}}+\frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}} \le \frac{L_\Psi+\bar{r}}{\underline{\rho}\,\min\{\theta^{\alpha_1+\alpha_2},\theta^{\alpha_2}\}}<1.$$

Step 1: existence and uniqueness on $[-q,T_1]$**.** Let $C([-q,T_1],\mathbb{R})$ with the Bielecki-type norm

$$\|\varpi\|_1 := \sup_{\mathfrak{z}\in[-q,T_1]} \frac{|\varpi(\mathfrak{z})|}{\Theta(\mathfrak{z})}, \qquad \Theta(\mathfrak{z}) := \begin{cases} e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}, & \mathfrak{z}\in(a,T_1],\\ 1, & \mathfrak{z}\in[-q,a]. \end{cases}$$

Define $\mathcal{F}_1 : C([-q,T_1],\mathbb{R}) \to C([-q,T_1],\mathbb{R})$ by the right-hand side of integral Eq. (3.7) for $\mathfrak{z}\in[a,T_1]$ and set $\mathcal{F}_1\varpi(\mathfrak{z}) = g(\mathfrak{z})$ for $\mathfrak{z}\in[-q,a]$. For $\mathfrak{z}\in[a,T_1]$ and $\varpi,\tilde{\varpi}\in C([-q,T_1],\mathbb{R})$, the key observation above gives $\varpi(\mathfrak{z}-q) = \tilde{\varpi}(\mathfrak{z}-q) = g(\mathfrak{z}-q)$. Hence by hypothesis **(H4)** (Lipschitz in the non-delayed argument only),

$$\left|\Psi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\mathfrak{z}-q)) - \Psi(\mathfrak{z},\tilde{\varpi}(\mathfrak{z}),\varpi(\mathfrak{z}-q))\right| \le L_\Psi\,|\varpi(\mathfrak{z})-\tilde{\varpi}(\mathfrak{z})|.$$

Proceeding as in Theorem 4 and using $I^{\alpha_2}I^{\alpha_1} = I^{\alpha_1+\alpha_2}$ and Lemma 3 (fractional integral of exponential weight), we obtain for $\mathfrak{z}\in[a,T_1]$

$$\frac{|\mathcal{F}_1\varpi(\mathfrak{z}) - \mathcal{F}_1\tilde{\varpi}(\mathfrak{z})|}{e^{\theta(\kappa(\mathfrak{z})-\kappa(a))}} \le \Big(\frac{L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}} + \frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\Big)\|\varpi-\tilde{\varpi}\|_1.$$

Taking the supremum over $\mathfrak{z}\in[-q,T_1]$ yields

$$\|\mathcal{F}_1\varpi - \mathcal{F}_1\tilde{\varpi}\|_1 \le \Big(\frac{L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}} + \frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\Big)\|\varpi-\tilde{\varpi}\|_1. \tag{3.8}$$

Since $\dfrac{L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}} + \dfrac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}} \le \dfrac{L_\Psi+\bar{r}}{\underline{\rho}\,\min\{\theta^{\alpha_1+\alpha_2},\theta^{\alpha_2}\}} < 1$,

$\mathcal{F}_1$ is a contraction on $C([-q,T_1],\mathbb{R})$ and admits a unique fixed point $\varpi_1\in C([-q,T_1],\mathbb{R})$.

Step $k\ge 2$**: inductive extension to** $[-q,T_k]$**.** Assume a unique $\varpi_{k-1}\in C([-q,T_{k-1}],\mathbb{R})$ solving the problem on $[-q,T_{k-1}]$ has been constructed. Define $C([-q,T_k],\mathbb{R})$ with the analogous Bielecki norm $\|\cdot\|_k$, and define $\mathcal{F}_k : C([-q,T_k],\mathbb{R}) \to C([-q,T_k],\mathbb{R})$ to equal ϖ_{k-1} on $[-q,T_{k-1}]$ and to be given by the integral equation on $[T_{k-1},T_k]$. For $\mathfrak{z}\in[T_{k-1},T_k]$ the partition inequality implies $\mathfrak{z}-q\le T_{k-1}$, so $\varpi(\mathfrak{z}-q) = \tilde{\varpi}(\mathfrak{z}-q) = \varpi_{k-1}(\mathfrak{z}-q)$ for any $\varpi,\tilde{\varpi}\in C([-q,T_k],\mathbb{R})$. Thus the same **(H4)**-based estimate as in Step 1 holds, yielding

$$\begin{aligned}\|\mathcal{F}_k\varpi - \mathcal{F}_k\tilde{\varpi}\|_k &\le \Big(\frac{L_\Psi}{\underline{\rho}\,\theta^{\alpha_1+\alpha_2}} + \frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}}\Big)\|\varpi-\tilde{\varpi}\|_k\\ &\le \frac{L_\Psi+\bar{r}}{\underline{\rho}\,\min\{\theta^{\alpha_1+\alpha_2},\theta^{\alpha_2}\}}\,\|\varpi-\tilde{\varpi}\|_k.\end{aligned}$$

Hence $\mathcal{F}_k$ is a contraction and admits a unique fixed point $\varpi_k\in C([-q,T_k],\mathbb{R})$ which extends ϖ_{k-1}.

Step: conclusion. After n steps we obtain a unique $\varpi = \varpi_n \in C([-q,T])$ solving the integral equation (and thus the original FD-SLL problem) on $[-q,T]$. Uniqueness over the whole interval follows from uniqueness on each subinterval and compatibility at partition points. $\square$

3.3. Existence results for the multivalued FDSLL problem

In this part, existence theorems are established for a multivalued version of the FDSLL equation, referred to as problem (1.2). The main result is derived using the Leray–Schauder Theorem 3 in its nonlinear alternative form for multivalued maps. By applying this theorem, the existence of solutions is demonstrated under $\mathbf{L}^1$-Carathéodory-type conditions imposed on the nonlinearity. Additionally, the findings extend previous results by addressing a broader class of multivalued operators.

The inclusion version problem (1.2) requires an understanding of multivalued analysis. The following definition and lemma lay the groundwork for the subsequent analysis.

Definition 5. A continuous function ϖ is defined as a solution to Eq. (1.2) if it verifies the following conditions:

- $\varpi(\mathfrak{z}) = g(\mathfrak{z}) \quad$ for $\mathfrak{z}\in[-q,a]$,
- ${}^{C}\mathfrak{D}^{\alpha_2;\kappa}_{a^+}\varpi(a) = V_{\alpha_2}$,
- There is a function $\upsilon\in\mathbf{L}^1([a,T],\mathbb{R})$, with $\upsilon\in\Pi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta_\mathfrak{z}))$ almost everywhere on $[a,T]$, such that

$$\begin{aligned}\varpi(\mathfrak{z}) = {}&\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon(\mathfrak{z})}{\rho(\mathfrak{z})}\right) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\right)\\ &+\left(r(a)g(a)+\rho(a)V_{\alpha_2}\right)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{1}{\rho(\mathfrak{z})}\right) + g(a)\end{aligned} \tag{3.9}$$

for each ϖ, the set of selections of Π is defined as

$$\mathcal{S}_{\Pi,\varpi} := \left\{\upsilon\in\mathbf{L}^1([a,T],\mathbb{R}) : \upsilon\in\Pi(\mathfrak{z},\varpi(\mathfrak{z}),\varpi(\delta_\mathfrak{z}))\text{ a.e. on }[a,T]\right\}.$$

Lemma 4 *([40]). Let X be a Banach space, and let $\Pi : [a,T]\times\mathbb{R}^2 \to \mathcal{P}_{c,cp}$ be a $\mathbf{L}^1$-Carathéodory multivalued map. Let Υ be a continuous linear operator mapping from $\mathbf{L}^1([a,T],X)$ to $C([-q,T],X)$. Then, the mapping expressed as:*

$$\Upsilon\circ\mathcal{S}_\Pi : \begin{array}{ccc} C([-q,T],X) & \longrightarrow & \mathcal{P}_{c,cp}(C([-q,T],X))\\ \varpi & \longrightarrow & (\Upsilon\circ\mathcal{S}_\Pi)(\varpi) = \Upsilon\left(\mathcal{S}_{\Pi,\varpi}\right).\end{array}$$

In the space $C([-q,T],X)\times C([-q,T],X)$, is a closed graph mapping.

Theorem 6. *Assume that the hypotheses **(H5)–(H8)** are satisfied. Then, under these conditions, the inclusion problem (1.2) admits at least one solution on the interval $[-q,T]$.*

Proof. To convert problem (1.2) to a fixed point problem, let us add the operator:

$$\mathbb{K} : \begin{array}{ccc} C([-q,T],\mathbb{R}) & \longrightarrow & \mathcal{P}_{c,cp}(C([-q,T],\mathbb{R}))\\ \varpi & \longrightarrow & \mathbb{K}(\varpi)\end{array}$$

as follows:

$$\mathbb{K}(\varpi) = \left\{ k\in C([-q,T],\mathbb{R}) : k(\mathfrak{z}) \right.$$

$$= \left\{\begin{array}{ll} g(\mathfrak{z}), & \mathfrak{z}\in[-q,a]\\ \text{the right-hand side of integral equation (3.9)}, & \mathfrak{z}\in(a,T]\end{array}\right\}$$

where $\upsilon\in\mathcal{S}_{\Pi,\varpi}\subset L^1([a,T],\mathbb{R})$.

It will be demonstrated that the operator $\mathbb{K}$ fulfills the requirements stipulated in Theorem 3. The detailed proof is provided in a step-by-step format.

Step 1: The set $\mathbb{K}(\varpi)$ is convex for every $\varpi\in C([-q,T],\mathbb{R})$.

Moreover, if k_1 and k_2 are elements of $\mathbb{K}(\varpi)$, then υ_1 and υ_2 exist in $\mathcal{S}_{\Pi,\varpi}$ such that for each $\mathfrak{z}$ in the interval $[a,T]$, the following holds:

$$\begin{aligned}k_j(\mathfrak{z}) = {}&\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon_j(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\Big)\\ &+\left(r(a)g(a)+\rho(a)V_{\alpha_2}\right)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big) + g(a)\end{aligned}$$

For $j=1,2$, let $0\le\lambda\le 1$; then, for each $t\in[a,T]$, it is given that:

$$\begin{aligned}\lambda k_1(\mathfrak{z}) + (1-\lambda)k_2(\mathfrak{z}) = {}&\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}(\lambda\upsilon_1(\mathfrak{z})+(1-\lambda)\upsilon_2(\mathfrak{z}))}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\Big)\\ &+\left(r(a)g(a)+\rho(a)V_{\alpha_2}\right)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big) + g(a)\end{aligned}$$

Since $\mathcal{S}_{\Pi,\varpi}$ is convex, it can be concluded that $(\lambda k_1(\mathfrak{z}) + (1-\lambda)k_2(\mathfrak{z}))\in\mathbb{K}(\varpi)$ also belongs to $\mathbb{K}(\varpi)$. Since $K(\varpi)$ equals the singleton $\{g\}$ on $[-q,a]$ and a convex set on $[-q,a]$, it follows that $K(\varpi)\subset C([-q,a])$. Consequently, $\mathbb{K}(\varpi)$ is convex for every $\varpi\in C([-q,T],\mathbb{R})$. **Step 2:** We show that bounded sets are mapped to bounded sets in $C([-q,T],\mathbb{R})$ via a mapping $\mathbb{K}$.

To establish the existence of a positive constant η such that for any $k\in\mathbb{K}(\varpi)$ and

$\varpi \in \mathcal{B}_{\varrho} = \{\varpi \in C([-q,T],\mathbb{R}) : \|\varpi\| \leq \varrho\}$, it is given that $\|k\| \leq \eta$, it is sufficient to show that such a condition is satisfied.

If $k \in \mathbb{K}(\varpi)$, then there exists $\upsilon \in \mathcal{S}_{\Pi,\varpi}$, then for every $\mathfrak{z} \in [a,T]$ it is given that:

$$\begin{aligned}
|k(\varpi)(\mathfrak{z})| =& sup_{\mathfrak{z}\in[0,T]} \Big\{ \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}|\upsilon(\mathfrak{z})|}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{|r(\mathfrak{z})|\,|\varpi(\mathfrak{z})|}{|\rho(\mathfrak{z})|}\Big) \\
&+ \Big(|r(a)|\,|g(a)| + |\rho(a)|\,\Big|V_{\alpha_2}\Big|\Big)\mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{|\rho(\mathfrak{z})|}\Big) + |g(a)| \Big\} \\
\leq& \frac{\|p\|\,\chi(2\,\|\varpi\|)}{\underline{\rho}\Gamma(\alpha_1+\alpha_2+1)}\big(\kappa(T)-\kappa(a)\big)^{\alpha_1+\alpha_2} + \frac{\bar{r}\|\varpi\|}{\underline{\rho}\Gamma(\alpha_2+1)}\big(\kappa(T)-\kappa(a)\big)^{\alpha_2} \\
&+ \frac{|r(a)|\,|g(a)| + |\rho(a)|\,\Big|V_{\alpha_2}\Big|}{\underline{\rho}\Gamma(\alpha_2+1)}\big(\kappa(T)-\kappa(a)\big)^{\alpha_2} \\
\leq& \|p\|\,\chi(2\,\|\varpi\|)\Phi_1 + + \|\varpi\|\,\Phi_2 + \Phi_3,
\end{aligned}$$

then

$$\|k\| \leq \|p\|\,\chi(2\varrho)\Phi_1 + +\varrho\Phi_2 + \Phi_3 := \eta.$$

On $[-q,a]$ each $\mathbb{K}(\varpi)$ is equal to the fixed continuous function g, hence bounded. Therefore $\mathbb{K}$ maps bounded subsets of $C([-q,T],\mathbb{R})$ into bounded subsets of $\mathcal{P}_{c,cp}(\mathbb{R})$.

Therefore, $\mathbb{K}(\varpi)$ maps bounded sets to bounded and equicontinuous sets in $C([-q,T],\mathbb{R})$.

Step 3: We prove that bounded sets are mapped to equicontinuous sets in $C([-q,T],\mathbb{R})$ via a mapping $\mathbb{K}$.

Assuming $\mathfrak{z}_1, \mathfrak{z}_2 \in [a,T]$; $\mathfrak{z}_1 < \mathfrak{z}_2$, and $\varpi \in \mathcal{B}_{\varrho}$, where $\mathcal{B}_{\varrho}$ is a bounded set of $C([a,T],\mathbb{R})$.

For every $\varpi \in \mathcal{B}_{\varrho}$ and $k \in \mathbb{K}(\varpi)$, there exists $\upsilon \in \mathcal{S}_{\Pi,\varpi}$. Then, it is obtained that:

$$\begin{aligned}
\Big|k(\mathfrak{z}_2) - k(\mathfrak{z}_1)\Big| \leq& \left|\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon(\mathfrak{z}_2)}{\rho(\mathfrak{z}_2)}\right) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon(\mathfrak{z}_1)}{\rho(\mathfrak{z}_1)}\right)\right| \\
&+ \left|\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{r(\mathfrak{z}_1)\varpi(\mathfrak{z}_1)}{\rho(\mathfrak{z}_1)}\right) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{r(\mathfrak{z}_2)\varpi(\mathfrak{z}_2)}{\rho(\mathfrak{z}_2)}\right)\right| \\
&+ \Big|r(a)g(a)+\rho(a)V_{\alpha_2}\Big|\left|\mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{1}{\rho(\mathfrak{z}_2)}\right) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\left(\frac{1}{\rho(\mathfrak{z}_1)}\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\Big|k(\mathfrak{z}_2) - k(\mathfrak{z}_1)\Big| \\
&\leq \frac{1}{\underline{\rho}\Gamma(\alpha_2)\,\Gamma(\alpha_1)}\Big[\int_a^{\mathfrak{z}_1}\kappa'(s)\Big|\big(\kappa(\mathfrak{z}_2)-\kappa(s)\big)^{\alpha_2-1} - \big(\kappa(\mathfrak{z}_1)-\kappa(s)\big)^{\alpha_2-1}\Big| \times \\
&\quad \int_a^s \kappa'(x)\,(\kappa(s)-\kappa(x))^{\alpha_1-1}\,|\upsilon(x)|\,dxds \\
&\quad + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\kappa'(s)\big(\kappa(\mathfrak{z}_2)-\kappa(s)\big)^{\alpha_2-1}\int_a^s \kappa'(x)\,(\kappa(s)-\kappa(x))^{\alpha_1-1}\,|\upsilon(x)|\,dxds\Big] \\
&\quad + \frac{\bar{r}\,\|\varpi\|}{\underline{\rho}\Gamma(\alpha_2)}\Big[\int_a^{\mathfrak{z}_1}\kappa'(s)\Big|\big(\kappa(\mathfrak{z}_2)-\kappa(s)\big)^{\alpha_2-1} - \big(\kappa(\mathfrak{z}_1)-\kappa(s)\big)^{\alpha_2-1}\Big|ds \\
&\quad + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\kappa'(s)\big(\kappa(\mathfrak{z}_2)-\kappa(s)\big)^{\alpha_2-1}ds\Big] \\
&\quad + \frac{\Big|r(a)g(a)+\rho(a)V_{\alpha_2}\Big|}{\underline{\rho}\Gamma(\alpha_2)}\Big[\int_a^{\mathfrak{z}_1}\kappa'(s)\Big|\big(\kappa(\mathfrak{z}_2)-\kappa(s)\big)^{\alpha_2-1} - \big(\kappa(\mathfrak{z}_1)-\kappa(s)\big)^{\alpha_2-1}\Big|ds \\
&\quad + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\kappa'(s)\big(\kappa(\mathfrak{z}_2)-\kappa(s)\big)^{\alpha_2-1}ds\Big]
\end{aligned}$$

$$\begin{aligned}
&\Big|k(\mathfrak{z}_2) - k(\mathfrak{z}_1)\Big| \\
&\leq \frac{\|p\|\,\kappa(2x)\,(\kappa(T)-\kappa(a))^{\alpha_1}\Big(\big(\kappa(\mathfrak{z}_1)-\kappa(a)\big)^{\alpha_2} - \big(\kappa(\mathfrak{z}_2)-\kappa(a)\big)^{\alpha_2}\Big)}{\underline{\rho}\Gamma(\alpha_1+1)\,\Gamma(\alpha_2+1)} \\
&+ \frac{\bar{r}\,\|\varpi\|\,(\big(\kappa(\mathfrak{z}_1)-\kappa(a)\big)^{\alpha_2} - \big(\kappa(\mathfrak{z}_2)-\kappa(a)\big)^{\alpha_2})}{\underline{\rho}\Gamma(\alpha_2+1)} \\
&+ \frac{\bar{r}\,\|\varpi\|\,\big(\kappa(\mathfrak{z}_2)-\kappa(\mathfrak{z}_1)\big)^{\alpha_2}}{\underline{\rho}\Gamma(\alpha_2+1)} \\
&+ \frac{(\big(\kappa(\mathfrak{z}_1)-\kappa(a)\big)^{\alpha_2} - \big(\kappa(\mathfrak{z}_2)-\kappa(a)\big)^{\alpha_2})}{\underline{\rho}\Gamma(\alpha_2+1)} + \frac{\big(\kappa(\mathfrak{z}_2)-\kappa(\mathfrak{z}_1)\big)^{\alpha_2}}{\underline{\rho}\Gamma(\alpha_2+1)}.
\end{aligned} \tag{3.10}$$

As $\mathfrak{z}_2 \to \mathfrak{z}_1$, it is evident that the right-hand side of (3.10) remains independent of ϖ, and $\Big|k(\mathfrak{z}_2)-k(\mathfrak{z}_1)\Big| \to 0$. The proof for $\mathbb{K}(\varpi)$ establishes uniform equicontinuous of its values on $[a,T]$. On $[-q,a]$ the image is the single continuous function g, so equicontinuous is immediate there. This suggests that $\mathbb{K}(\varpi)$ is equicontinuous from this. The Arzelà–Ascoli Theorem 2 is used to determine that $\mathbb{K}$ is completely continuous and that $\mathbb{K}$ is relatively compact.

To establish the upper semicontinuity of the operator $\mathbb{K}$, it suffices to demonstrate that $\mathbb{K}$ possesses a closed graph.

Step 4: The operator $\mathbb{K}$ possesses a closed graph property.

Suppose:

- $\varpi_n \to \varpi_*$ in $C([-q,T],\mathbb{R})$.
- $k_n \in \mathbb{K}(\varpi_n)$ with $k_n \to k_*$ in $C([-q,T],\mathbb{R})$, Our goal is to establish that $k_* \in \mathbb{K}(\varpi_*)$.
  Since $k_n \in \mathbb{K}(\varpi_n)$, there exists $\upsilon_n \in \mathcal{S}_{\Pi,\varpi_n}$,xn such that, for every $\mathfrak{z} \in [a,T]$,

$$k_n(\mathfrak{z}) = \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon_n(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi_n(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) + \Big(r(a)g(a)+\rho(a)V_{\alpha_2}\Big) \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big) + g(a).$$

We prove that there exists a function $\upsilon_* \in \mathcal{S}_{\Pi,\varpi_*}$ such that, for every $\mathfrak{z} \in [a,T]$, one has

$$k_*(\mathfrak{z}) = \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon_*(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi_*(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) + \Big(r(a)g(a)+\rho(a)V_{\alpha_2}\Big) \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big) + g(a)$$

Consider the linear operator :

$$\Upsilon : \begin{array}{ccc} \mathbf{L}^1([-q,T],\mathbb{R}) & \longrightarrow & C([-q,T],\mathbb{R}) \\ \upsilon & \longrightarrow & \Upsilon(\upsilon)(\mathfrak{z}) \end{array}$$

with

$$\Upsilon(\upsilon)(\mathfrak{z}) = \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) + \Big(r(a)g(a)+\rho(a)V_{\alpha_2}\Big) \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big) + g(a)$$

Observe that

$$\|k_n - k_*\| \to 0 \quad \text{as } n \to \infty.$$

Then, by Lemma 4, the operator $\Upsilon \circ \mathcal{S}_{\Pi}$ has a closed graph. Moreover, we have

$$k_n(\mathfrak{z}) \in \Upsilon(\mathcal{S}_{\Pi,\varpi_n}).$$

Since $\varpi_n \to \varpi_*$, it follows that

$$k_*(\mathfrak{z}) = \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{\mathcal{I}^{\alpha_1,\kappa}_{a^+}\upsilon_*(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) - \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{r(\mathfrak{z})\varpi_*(\mathfrak{z})}{\rho(\mathfrak{z})}\Big) + \Big(r(a)g(a)+\rho(a)V_{\alpha_2}\Big) \mathcal{I}^{\alpha_2,\kappa}_{a^+}\Big(\frac{1}{\rho(\mathfrak{z})}\Big) + g(a). \text{ for some } \upsilon_* \in \mathcal{S}_{\Pi,\varpi_*}.$$

For $\mathfrak{z} \in [-q,a]$ we have $k_n(\mathfrak{z}) = g(\mathfrak{z})$ for all n, hence $k_*(\mathfrak{z}) = g(\mathfrak{z})$. Thus $k_* \in \mathbb{K}(\varpi_*)$. This proves that $\mathbb{K}$ has a closed graph (equivalently it is upper semicontinuous with compact convex values).

Step 5: The operator $\mathbb{K}$ admits a fixed point in $\mathcal{B}_r$.

It is shown that condition (ii) of Theorem 3 cannot occur. Hence, assume that there exists $\varpi \in \varepsilon\mathbb{K}(\varpi)$ for some $\varepsilon \in (0,1)$. Then there exists $\upsilon \in \mathcal{S}_{\Pi,\varpi}$ such that

$$\varpi(\mathfrak{z}) = \varepsilon\,\mathbb{K}(\varpi)(\mathfrak{z}) = \varepsilon\,k(\mathfrak{z}), \quad \forall\,\mathfrak{z} \in [-q,T], \tag{3.11}$$

which implies that

$$|\varpi(\mathfrak{z})| \leq |k(\mathfrak{z})|, \quad \forall\,\mathfrak{z} \in [-q,T].$$

We now analyze the fixed-point equation (3.11) separately on the subintervals $[-q,a]$ and $(a,T]$.

Case 1: $\mathfrak{z} \in [-q,a]$ On this interval, we have $\mathbb{K}(\varpi)(\mathfrak{z}) = g(\mathfrak{z})$, and therefore

$$\varpi(\mathfrak{z}) = \varepsilon\,g(\mathfrak{z}), \quad \mathfrak{z} \in [-q,a].$$

Since $g \in C([-q,a],\mathbb{R})$, it is bounded. Define $M_g = \max_{\mathfrak{z}\in[-q,a]} |g(\mathfrak{z})|$. Then it follows that

$$|\varpi(\mathfrak{z})| \le M_g, \quad \forall \mathfrak{z} \in [-q,a].$$

In particular, at $\mathfrak{z} = a$,

$$\varpi(a) = \varepsilon\, g(a) \quad \Rightarrow \quad |\varpi(a)| \le |g(a)|.$$

This observation is consistent with the initial condition and yields no contradiction on the interval $[-q,a]$.

Case 2: $\mathfrak{z} \in (a,T]$ On this subinterval, the following estimate holds:

$$\|\varpi\| \le \|p\|\, \chi(2\|\varpi\|)\, \Phi_1 + \|\varpi\|\, \Phi_2 + \Phi_3,$$

which leads to

$$(1-\Phi_2)\|\varpi\| \le \|p\|\, \chi(2\|\varpi\|)\, \Phi_1 + \Phi_3, \tag{3.12}$$

where $\|\varpi\| = \sup_{\mathfrak{z}\in[-q,T]} |\varpi(\mathfrak{z})|$. The supremum norm $\|\varpi\|$ is taken over the entire interval $[-q,T]$. Combining the two subinterval analyses yields:

- On $[-q,a]$: $|\varpi(\mathfrak{z})| \le M_g$;
- On $(a,T]$: the inequality (3.12) implies that if $\|\varpi\| = M$, then
  $$\frac{(1-\Phi_2)M}{\|p\|\, \chi(2M)\, \Phi_1 + \Phi_3} \le 1,$$
  which contradicts hypothesis **(H8)**.

To ensure that the contradiction is global, we observe that if $M > M_g$ as required by **(H8)**, then the supremum $\|\varpi\| = M$ is attained on $(a,T]$, and the bound on $[-q,a]$ does not affect the contradiction.

Therefore, the assumption that $\varpi = \varepsilon\mathbb{K}(\varpi)$ for some $\varepsilon \in (0,1)$ leads to a violation of hypothesis **(H8)**. Consequently, by the Leray–Schauder nonlinear alternative type [43] or Theorem 3, the operator $\mathbb{K}$ admits a fixed point in $C([-q,T],\mathbb{R})$. This fixed point constitutes a solution to the inclusion problem (1.2). □

4. Illustrative checks

This example is designed to verify the theoretical assumptions **(H1)-(H2)** and demonstrate the application of the Banach fixed-point theorem (Theorem 4) in a concrete setting, rather than to model a specific engineering system. The parameters are chosen for mathematical clarity and to illustrate the contraction mapping principle.

- The fractional orders are chosen as $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$, satisfying $0 < \alpha_1, \alpha_2 < 1$. This choice guarantees that the problem remains within the class of FDSLL problems with memory effects, while allowing a clear distinction between the outer and inner fractional dynamics.
- Limiting behavior with respect to α_1 and α_2: The qualitative behavior of the system is sensitive to the fractional orders α_1 and α_2. As $\alpha_1 \to 0^+$, the outer fractional operator loses its memory effect and the model reduces to a lower-order FDSLL equation dominated by the inner derivative of order α_2. In contrast, when $\alpha_2 \to 1^-$, the inner fractional derivative approaches the classical first-order derivative, leading to a mixed classical–fractional FDSLL model with weakened hereditary effects.

Example 1. Consider the following FDSLL problem:

$$\begin{cases} {}^{C}D_{a^+}^{0.4;\mathfrak{z}}\left(2\,{}^{C}D_{a^+}^{0.6;\mathfrak{z}}\varpi(\mathfrak{z}) + (0.01\mathfrak{z}+1)\varpi(\mathfrak{z})\right) \\ \quad = \dfrac{1}{1000}\left(\sin(\varpi(\mathfrak{z})) + \dfrac{\varpi(\mathfrak{z}/4)}{1+\mathfrak{z}^2}\right), \quad \mathfrak{z} \in]\tfrac{1}{2},1] \\ \varpi(\mathfrak{z}) = \mathfrak{z}+2, \quad \mathfrak{z} \in [-1,\tfrac{1}{2}], \\ {}^{C}D_{a^+}^{0.6;\mathfrak{z}}\varpi(\tfrac{1}{2}) = 2. \end{cases} \tag{4.1}$$

Model Parameters and Setup:

- The function $\kappa(\mathfrak{z}) = \mathfrak{z}$ reduces the κ-Caputo fractional derivative to the classical Caputo fractional derivative. This operator models processes with power-law memory, characterized by the kernel $(\mathfrak{z}-s)^{\alpha-1}$, and is widely used in viscoelasticity and anomalous diffusion.
- $\rho(\mathfrak{z}) = 2$, hence $\underline{\rho} = \inf_{\mathfrak{z}\in[\frac{1}{2},1]} |\rho(\mathfrak{z})| = 2$.
- $r(\mathfrak{z}) = 0.01\mathfrak{z}+1$, hence $\bar{r} = \sup_{\mathfrak{z}\in[\frac{1}{2},1]} |r(\mathfrak{z})| = 1.01$.
- The delay function is $h(\mathfrak{z}) = \mathfrak{z}/4$, satisfying $h(\mathfrak{z}) \le \mathfrak{z}$ for all $\mathfrak{z} \in [\frac{1}{2},1]$.
- The nonlinearity is given by $\Psi(\mathfrak{z},\varpi_1,\varpi_2) = \dfrac{1}{1000}\left(\sin(\varpi_1) + \dfrac{\varpi_2}{1+\mathfrak{z}^2}\right)$.

Verification of Hypotheses:

(H1) The function Ψ is continuous on $[\frac{1}{2},1]\times\mathbb{R}^2$, and $h(\mathfrak{z}) = \mathfrak{z}/4$ is continuous on $[\frac{1}{2},1]$ with $h(\mathfrak{z}) \le \mathfrak{z}$. Thus, **(H1)** is satisfied.

(H2) We verify the Lipschitz condition for Ψ. For any $\varpi_1,\varpi_2,\tilde{\varpi}_1,\tilde{\varpi}_2 \in \mathbb{R}$ and $\mathfrak{z} \in [\frac{1}{2},1]$, we have:

$$\begin{aligned} |\Psi(\mathfrak{z},\varpi_1,\varpi_2) - \Psi(\mathfrak{z},\tilde{\varpi}_1,\tilde{\varpi}_2)| &= \frac{1}{1000}\left|\sin(\varpi_1) - \sin(\tilde{\varpi}_1) + \frac{\varpi_2-\tilde{\varpi}_2}{1+\mathfrak{z}^2}\right| \\ &\le \frac{1}{1000}\left(|\sin(\varpi_1)-\sin(\tilde{\varpi}_1)| + \frac{|\varpi_2-\tilde{\varpi}_2|}{1+\mathfrak{z}^2}\right) \\ &\le \frac{1}{1000}\left(|\varpi_1-\tilde{\varpi}_1| + |\varpi_2-\tilde{\varpi}_2|\right), \end{aligned}$$

using the inequalities $|\sin(x)-\sin(y)| \le |x-y|$ and $\frac{1}{1+\mathfrak{z}^2} \le 1$. Thus, **(H2)** is satisfied with $L_\Psi = \frac{1}{1000}$.

Contraction Constant Verification To apply the Banach fixed-point theorem (Theorem 4), we must ensure that the operator $\mathcal{F}$ is a contraction. This requires finding a $\theta > 0$ such that the contraction constant

$$\phi(\theta) = \frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_2+\alpha_1}} + \frac{\bar{r}}{\underline{\rho}\,\theta^{\alpha_2}} = \frac{2\cdot\frac{1}{1000}}{2\theta^{1.0}} + \frac{1.01}{2\theta^{0.6}} < 1.$$

We compute $\phi(\theta)$ for a range of θ values. The results, presented in Table 1, show that for $\theta \ge 0.5$, the condition $\phi(\theta) < 1$ is satisfied. This table demonstrates that for $\theta \ge 0.5$, the value of $\phi(\theta)$ drops below 1, satisfying the sufficient condition for the Banach contraction mapping principle to guarantee a unique solution for the FDSLL problem in Example 1.

Table 1
Evaluation of the contraction constant $\phi(\theta)$ for various values of θ.

θ	$\phi(\theta)$
0.003	16.815664 (> 1)
0.03	4.173508 (> 1)
0.5	0.767437 (< 1)
1	0.506000 (< 1)
2	0.333676 (< 1)
5	0.192469 (< 1)
100	0.031873 (< 1)
1000	0.008005 (< 1)

The behavior of $\phi(\theta)$ is visualized in Fig. 1. The graph clearly shows the monotonic decrease of the contraction constant as θ increases, crossing the critical threshold of 1 between $\theta = 0.03$ and $\theta = 0.5$. This confirms that a sufficiently large θ can always be chosen to ensure the existence of a unique solution.

Since all assumptions of Theorem 4 are satisfied, we conclude that the FDSLL Eq. (4.1) possesses a unique solution on the interval $[-1,1]$.

This example illustrates the utility of Burton's progressive contraction method (Theorem 5) under the weaker Lipschitz condition **(H4)**, which is not satisfied by the stronger condition **(H2)**. The focus is on the mathematical technique for handling specific nonlinearities with delayed arguments.

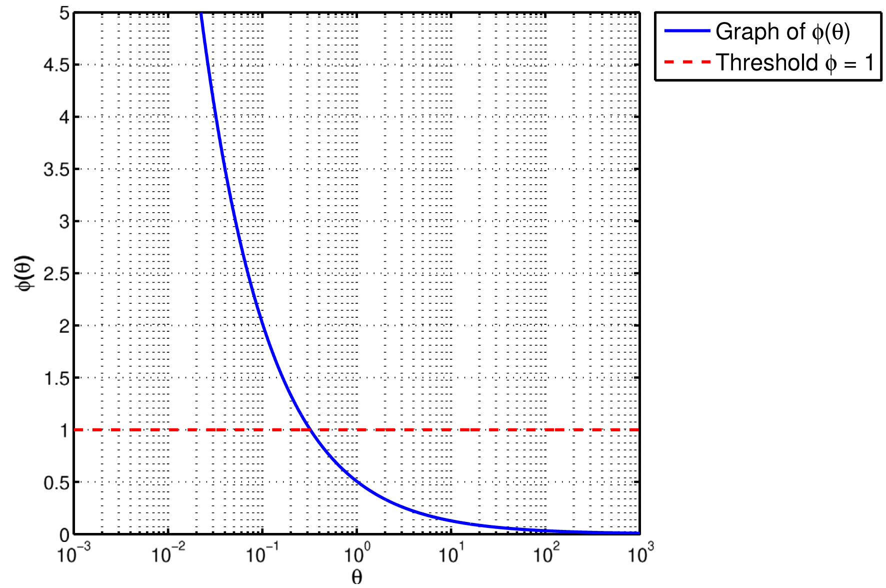

Fig. 1. Graph of the contraction constant $\phi(\theta)$ versus θ for Example 1.

Example 2. Consider the following FDSLL problem with a constant delay $q > 0$:

$$\begin{cases} {}^{C}D^{0.4;\mathfrak{z}}_{a^+}\left(2\,{}^{C}D^{0.6;\mathfrak{z}}_{a^+}\varpi(\mathfrak{z}) + (0.01\mathfrak{z}+1)\varpi(\mathfrak{z})\right) = \varpi(\mathfrak{z}) + \varpi^2(\mathfrak{z}-q), \quad \mathfrak{z}\in\left[\frac{1}{2},1\right], \\ \varpi(\mathfrak{z}) = \mathfrak{z}+2, \quad \mathfrak{z}\in\left[-1,\frac{1}{2}\right], \\ {}^{C}D^{0.6;\mathfrak{z}}_{a^+}\varpi\left(\frac{1}{2}\right) = 2. \end{cases} \tag{4.2}$$

Model Parameters and Setup

- The orders of the derivatives are as follows: $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$, respectively.
- The function $\kappa(\mathfrak{z}) = \mathfrak{z}$ reduces the κ-Caputo fractional derivative to the classical Caputo fractional derivative.
- $\rho(\mathfrak{z}) = 2$, hence $\underline{\rho} = \inf_{\mathfrak{z}\in[\frac{1}{2},1]}|\rho(\mathfrak{z})| = 2$.
- $r(\mathfrak{z}) = 0.01\mathfrak{z}+1$, hence $\overline{r} = \sup_{\mathfrak{z}\in[\frac{1}{2},1]}|r(\mathfrak{z})| = 1.01$.
- The delay function is $h(\mathfrak{z}) = \mathfrak{z} - q$, satisfying $h(\mathfrak{z}) \le \mathfrak{z}$ for all $\mathfrak{z}\in[\frac{1}{2},1]$.
- The nonlinearity is given by $\Psi(\mathfrak{z},\varpi_1,\varpi_2) = \varpi_1 + \varpi_2^2$.

Verification of Hypotheses

- **(H1)** The function Ψ is continuous on $[\frac{1}{2},1]\times\mathbb{R}^2$, and $h(\mathfrak{z}) = \mathfrak{z} - q$ is continuous on $[\frac{1}{2},1]$ with $h(\mathfrak{z}) \le \mathfrak{z}$. Thus, **(H1)** is satisfied.
- **(H4)** The function Ψ satisfies condition **(H4)** but does not satisfy the stronger condition **(H2)**:

  1. **Checking Condition (H4):** By fixing ϖ_2, we have

     $\left|\Psi(\mathfrak{z},\varpi_1,\varpi_2) - \Psi(\mathfrak{z},\widetilde{\varpi}_1,\varpi_2)\right| = \left|\varpi_1 - \widetilde{\varpi}_1\right|.$

     Thus, Ψ is Lipschitz in ϖ_1 with $L_\Psi = 1$, satisfying **(H4)**.
  2. **Why (H2) Fails:** For varying ϖ_1,ϖ_2, we find

     $$\left|\Psi(\mathfrak{z},\varpi_1,\varpi_2) - \Psi(\mathfrak{z},\widetilde{\varpi}_1,\widetilde{\varpi}_2)\right| \le |\varpi_1 - \widetilde{\varpi}_1| + |\varpi_2 - \widetilde{\varpi}_2|\cdot|\varpi_2 + \widetilde{\varpi}_2|.$$

     For **(H2)** to hold, $|\varpi_2 + \widetilde{\varpi}_2|$ must be bounded by a global constant L_Ψ, which is impossible due to the quadratic term ϖ_2^2. Hence, **(H2)** is not satisfied.

Application of Burton's Method (Theorem 5). Since **(H2)** fails but **(H4)** holds, we apply Theorem 5 (Burton's method). The sufficient condition for uniqueness becomes:

$$\frac{2L_\Psi}{\underline{\rho}\,\theta^{\alpha_2+\alpha_1}} + \frac{\overline{r}}{\underline{\rho}\,\theta^{\alpha_2}} = \left(\frac{2}{2\theta^{1.0}} + \frac{1.01}{2\theta^{0.6}}\right) < 1.$$

We now verify that a parameter $\theta > 0$ exists such that this inequality holds.

Numerical Verification of the Contraction Condition. Table 2 presents the value of the contraction constant for a range of θ values. The results demonstrate that for $\theta \ge 2$, the constant falls below 1, satisfying the condition required by Theorem 5. The condition $\phi(\theta) < 1$ is satisfied for $\theta \ge 2$, confirming the applicability of Burton's fixed-point theorem.

Graphical Interpretation. The behavior of the contraction constant $\phi(\theta)$ is visualized in Fig. 2. The plot clearly shows a monotonic decrease in $\phi(\theta)$ as θ increases. The critical threshold of 1 is crossed between $\theta = 1$ and $\theta = 2$, illustrating that a sufficiently large θ can always be chosen to guarantee a unique solution.

Fig. 2 shows the graph of the contraction constant $\phi(\theta)$ versus θ for Example 2. The dashed line indicates the threshold $\phi(\theta) = 1$. The plot provides an intuitive confirmation of the numerical results in Table 2, and confirms that the operator becomes a contraction for $\theta \gtrsim 2$, validating the use of Burton's progressive contraction method.

Conclusion of the Example. Since all assumptions of Theorem 5 are satisfied, we conclude that the FDSLL Eq. (4.1) possesses a unique solution on the interval $[-1,1]$. This example successfully demonstrates the utility of Burton's method in establishing uniqueness under the weaker Lipschitz condition **(H4)**.

This example transitions from a single-valued equation to a differential inclusion, validating the existence theory based on the Leray–Schauder alternative (Theorem 6). It demonstrates how to handle problems where the system's dynamics are not precisely known but can be bounded within a set.

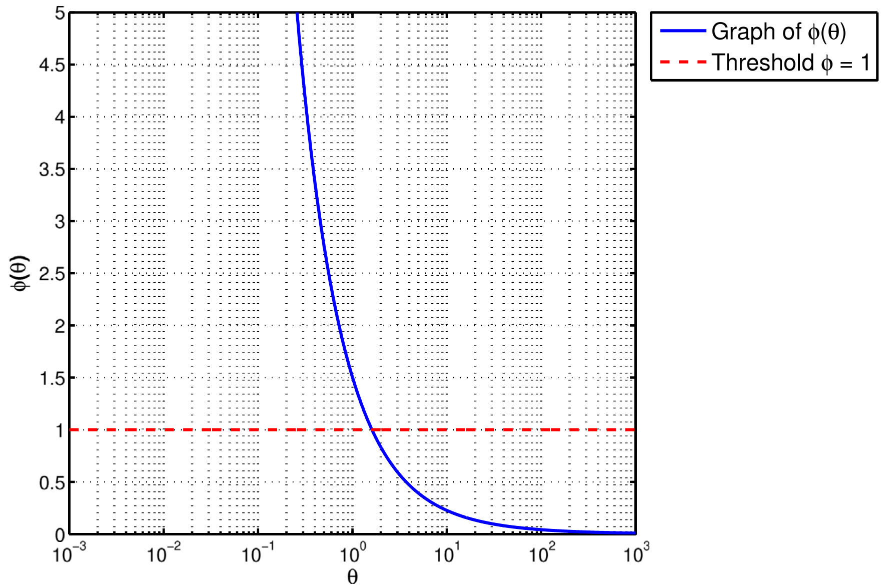

Fig. 2. Graph of the contraction constant $\phi(\theta)$ versus θ for Example 2.

Table 2
Numerical evaluation of the contraction.

θ	$\left(\frac{2L_{\Psi}}{\underline{\rho}\theta^{\alpha_2+\alpha_1}}+\frac{\bar{r}}{\underline{\rho}\theta^{\alpha_2}}\right)$
0.5	2.77 >1
1	1.51 >1
2	0.83 <1
5	0.39 <1
10	0.22 <1

Example 3. Consider the following FDSLL inclusion problem:

$$\begin{cases} {}^{C}D^{0.5;\mathfrak{z}}_{a^+}\left((2+\mathfrak{z})\,{}^{C}D^{0.6;\mathfrak{z}}_{a^+}\varpi(\mathfrak{z})+(0.5\mathfrak{z})\varpi(\mathfrak{z})\right)\in[0,\mathfrak{z}\cdot(|\varpi(\mathfrak{z})|+|\varpi(\delta\mathfrak{z})|)], & \mathfrak{z}\in\left[\frac{1}{2},1\right], \\ \varpi(\mathfrak{z})=\mathfrak{z}^2, & \mathfrak{z}\in\left[-1,\frac{1}{2}\right], \\ {}^{C}D^{0.6;\mathfrak{z}}_{a^+}\varpi\left(\frac{1}{2}\right)=1, \end{cases} \tag{4.3}$$

where $\delta\in(0,1)$ is a constant scaling the delay.

Model Setup and multivalued Map. We analyze this problem within the framework of Theorem 6 (the Leray–Schauder alternative for multivalued maps).

- The orders of the derivatives are as follows: $\alpha_1=0.5$ and $\alpha_2=0.6$, respectively.
- The function $\kappa(\mathfrak{z})=\mathfrak{z}$ corresponds to the classical Caputo fractional derivative, ensuring the model captures power-law memory effects, a standard assumption in viscoelasticity and anomalous diffusion.
- The coefficient functions are: $\rho(\mathfrak{z})=2+\mathfrak{z}$, hence $\underline{\rho}=\inf_{\mathfrak{z}\in[1/2,1]}|\rho(\mathfrak{z})|=2.5$, and $r(\mathfrak{z})=0.5\mathfrak{z}$, hence $\bar{r}=\sup_{\mathfrak{z}\in[1/2,1]}|r(\mathfrak{z})|=0.5$.
- The multivalued right-hand side is defined by the mapping:

  $$\Pi(\mathfrak{z},x,\varpi)=[0,\ \mathfrak{z}\cdot(|x|+|\varpi|)].$$

Verification of Hypotheses for the multivalued Map. We now verify that the multivalued map Π satisfies the necessary conditions for Theorem 6.

- Verification of (H6) and (H7) (L^1-Carathéodory and Closed-Graph Properties): Let $\phi(\mathfrak{z},x,\varpi):=\mathfrak{z}(|x|+|\varpi|)$ be the function defining the upper endpoint of the interval $\Pi(\mathfrak{z},x,\varpi)$.

(i) **Measurability and Upper Semicontinuity:** The function ϕ is continuous on its domain as a composition of continuous functions (absolute value and multiplication). Consequently:

  - For each fixed (x,ϖ), the map $\mathfrak{z}\mapsto\Pi(\mathfrak{z},x,\varpi)$ is a continuous, hence measurable, interval-valued function.
  - For each fixed $\mathfrak{z}$, the map $(x,\varpi)\mapsto\Pi(\mathfrak{z},x,\varpi)$ is continuous with respect to the Hausdorff metric, as small changes in (x,ϖ) lead to small changes in the endpoints of the interval. This implies Π is upper semicontinuous in (x,ϖ).

  Therefore, Π is a Carathéodory multivalued map. Furthermore, its values are non-empty, compact, and convex intervals in $\mathbb{R}$.

(ii) **L^1-Growth Condition (H7):** For any $\upsilon\in\Pi(\mathfrak{z},x,\varpi)$, we have the pointwise bound:

  $$|\upsilon|\leq\mathfrak{z}(|x|+|\varpi|).$$

  This can be written as

  $$\|\Pi(\mathfrak{z},x,\varpi)\|:=\sup\{|\upsilon|\,:\,\upsilon\in\Pi(\mathfrak{z},x,\varpi)\}\leq p(\mathfrak{z})\,\chi(|x|+|\varpi|),$$

  where we define $p(\mathfrak{z})=\mathfrak{z}$ and $\chi(u)=u$. The function $p(\mathfrak{z})$ is continuous on $[1/2,1]$ and thus integrable ($p\in L^1([1/2,1])$), and χ is continuous and non-decreasing. This verifies hypothesis **(H7)**.

(iii) **Closed-Graph Property:** Let $(\mathfrak{z}_n,x_n,\varpi_n,\upsilon_n)\to(\mathfrak{z},x,\varpi,\upsilon)$ be a convergent sequence with $\upsilon_n\in\Pi(\mathfrak{z}_n,x_n,\varpi_n)$ for all n. By definition, $0\leq\upsilon_n\leq\mathfrak{z}_n(|x_n|+|\varpi_n|)$. Taking the limit and using the continuity of ϕ, we obtain:

  $$0\leq\upsilon\leq\mathfrak{z}(|x|+|\varpi|).$$

  This implies $\upsilon\in[0,\mathfrak{z}(|x|+|\varpi|)]=\Pi(\mathfrak{z},x,\varpi)$, proving that Π has a closed graph.

(iv) **Existence of Selections:** Since Π is upper semicontinuous with non-empty, compact, convex values, the existence of continuous ε-approximate selections is guaranteed by Cellina's theorem [45]. This ensures the set of Lebesgue integrable selections $S_{\Pi,\varpi}$ is non-empty for any continuous ϖ, satisfying the remaining part of **(H6)**.

Verification of (H8), (The A Priori Bound). We now verify the existence of a constant $M > 0$ satisfying the inequality in **(H8)**:

$$\frac{(1-\Phi_2)M}{\|p\|_{L^1}\,\chi(2M)\,\Phi_1+\Phi_3} > 1.$$

First, we compute the constants Φ_1, Φ_2, Φ_3 from Eq. (3.4). Using the parameters $a = 1/2$, $T = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, and $\kappa(\mathfrak{z}) = \mathfrak{z}$ (so that $\kappa(T) - \kappa(a) = 1 - 0.5 = 0.5$), we calculate:

$$\Phi_1 = \frac{(\kappa(T)-\kappa(a))^{\alpha_1+\alpha_2}}{\underline{\rho}\,\Gamma(\alpha_1+\alpha_2+1)} = \frac{0.5^{1.1}}{2.5\cdot\Gamma(2.1)} \approx \frac{0.466}{2.5\cdot 1.1} \approx 0.169,$$

$$\Phi_2 = \frac{\overline{r}\,(\kappa(T)-\kappa(a))^{\alpha_2}}{\underline{\rho}\,\Gamma(\alpha_2+1)} = \frac{0.5\cdot 0.5^{0.6}}{2.5\cdot\Gamma(1.6)} \approx \frac{0.5\cdot 0.66}{2.5\cdot 0.893} \approx \frac{0.33}{2.2325} \approx 0.148,$$

$$\Phi_3 = \frac{\left(|r(a)g(a)| + |\rho(a)V_{\alpha_2}|\right)(\kappa(T)-\kappa(a))^{\alpha_2}}{\underline{\rho}\,\Gamma(\alpha_2+1)}.$$

At $\mathfrak{z} = a = 1/2$, we have $g(a) = a^2 = 0.25$, $r(a) = 0.5\cdot 0.5 = 0.25$, and $\rho(a) = 2.5$. With $V_{\alpha_2} = 1$, this yields:

$$\Phi_3 \approx \frac{(0.25\cdot 0.25 + 2.5\cdot 1)\cdot 0.5^{0.6}}{2.5\cdot 0.893} = \frac{(0.0625+2.5)\cdot 0.66}{2.2325} \approx \frac{1.69}{2.2325} \approx 0.757.$$

We also have $\|p\|_{L^1} = \int_{1/2}^{1} \mathfrak{z}\, d\mathfrak{z} = \left[\frac{\mathfrak{z}^2}{2}\right]_{1/2}^{1} = \frac{1}{2} - \frac{1}{8} = 0.375$, and $\chi(u) = u$.

To satisfy **(H8)**, we must find an $M > 0$ such that:

$$(1-\Phi_2)M > \|p\|_{L^1}\,\chi(2M)\,\Phi_1 + \Phi_3.$$

Substituting the known values and $\chi(2M) = 2M$, this inequality becomes:

$$(1-0.148)M > (0.375)\cdot(2M)\cdot(0.169) + 0.757$$

$$0.852M > (0.375\cdot 0.338)M + 0.757$$

$$0.852M > 0.12675M + 0.757$$

$$0.852M - 0.12675M > 0.757$$

$$0.72525M > 0.757$$

$$M > \frac{0.757}{0.72525} \approx 1.044.$$

Justification for the Choice $M = 5$: The calculation above shows that any $M > 1.044$ theoretically satisfies the inequality. However, the Leray–Schauder alternative requires the existence of a priori bounds on potential solutions. The value M represents a bound on the solution's norm, $\|\varpi\|$. Choosing $M = 1.044$ would be the minimal mathematical bound, but it provides no practical margin.

To ensure a clear and robust validation of hypothesis **(H8)**, and to account for the supremum norm over the entire interval $[-1, 1]$ (noting that the initial function $g(\mathfrak{z}) = \mathfrak{z}^2$ on $[-1, 1/2]$ has a maximum of 1 at $\mathfrak{z} = -1$), we select a larger, round value for M. Let us verify the original inequality in **(H8)** for $M = 5$:

$$\begin{aligned}\frac{(1-\Phi_2)M}{\|p\|_{L^1}\,\chi(2M)\,\Phi_1+\Phi_3} &= \frac{(1-0.148)\cdot 5}{0.375\cdot(2\cdot 5)\cdot 0.169+0.757}\\ &= \frac{0.852\cdot 5}{0.375\cdot 10\cdot 0.169+0.757}\\ &= \frac{4.26}{0.634+0.757}\\ &= \frac{4.26}{1.391} \approx 3.06 > 1.\end{aligned}$$

This inequality satisfies **(H8)**.

Conclusion of the Example. Since all hypotheses **(H5)–(H8)** of Theorem 6 are satisfied, we conclude that the FDSLL inclusion problem (4.3) admits at least one solution on the interval $[-1, 1]$. This example demonstrates the effective application of the Leray–Schauder alternative to a non-trivial multivalued FDSLL problem.

5. Remark

The primary differences and benefits of using fractional differential equations and fractional differential inclusions, compared to integer-order models, are noteworthy. Integer-order models are unable to represent the memory and hereditary properties of various processes, whereas fractional models provide such capabilities. Complex dynamical systems are, therefore, represented with greater accuracy and realism, particularly in domains such as anomalous diffusion. A more thorough explanation of system dynamics is facilitated by the non-local character of fractional derivatives. Additionally, fixed-point theorems such as Burton progressive contractions, the Banach contraction principle, and the nonlinear alternative of Leray–Schauder have been employed to establish uniqueness and existence results for both differential equations and differential inclusions. This approach produces solutions that are more reliable and offers insights that surpass those obtained through conventional integer-order techniques. These advantages demonstrate that fractional models are more applicable and efficient across various real-world scenarios. An analysis of the FDSLL problem is provided, with an emphasis on the presence and uniqueness of the solution. Several mathematical concepts, including Burton progressive contractions and the Banach contraction principle, are utilized to derive significant results. The conditions under which these results hold are extended, thereby contributing to a deeper understanding of the behavior of solutions to this class of equations.

6. Conclusion

This paper has established a unified analytical framework for the FDSLL problem (1.1) and its multivalued inclusion (1.2). Existence and uniqueness of solutions were investigated under general assumptions, yielding results that relax several classical restrictions found in the literature.

The main contributions include the introduction of a novel delayed and multivalued FDSLL model, the development of complementary uniqueness approaches based on a κ-Bielecki-type weighted norm and Burton's progressive contraction method, and the derivation of existence results for multivalued inclusions under $\mathbf{L}^1$-Carathéodory conditions via the Leray–Schauder nonlinear alternative. The use of the κ-Caputo fractional operator provides a unifying setting that recovers several classical–fractional Sturm–Liouville and Langevin-type models as particular cases, while the illustrative examples confirm the applicability of the theoretical results.

Despite these advances, the analysis is limited to specific contraction-type assumptions and to $\mathbf{L}^1$-Carathéodory nonlinearities in the multivalued case, and no numerical or stability analysis is addressed. Future research will focus on extending the present framework to nonlinear FDSLL problems involving κ-Hilfer or variable-order fractional derivatives [46], employing common fixed-point results for α-κ-contractions [47], and developing numerical schemes and stability analyses to support applications of the proposed models.

CRediT authorship contribution statement

Hacen Serrai: Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Brahim Tellab:** Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Sabri T.M. Thabet:** Writing – review & editing, Validation, Methodology, Investigation, Formal analysis. **Thabet Abdeljawad:** Writing – review & editing, Validation, Methodology, Investigation, Formal analysis. **Aiman Mukheimer:** Writing – review & editing, Methodology, Investigation, Formal analysis. **Bahaaeldin Abdalla:** Writing – review & editing, Methodology, Investigation, Formal analysis. **Imed Kedim:** Validation, Methodology, Investigation, Formal analysis.

Ethics approval and consent to participate

Not applicable.

Declaration of competing interest

The authors declare that they have no competing interests.

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Data availability

No data were used to support the findings of this study.

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