symmetry

MDPI

Article

Parallel Darboux Equidistant Ruled Surfaces in E^3

Ceyda Cevahir Yıldız [1,*], Süleyman Şenyurt [2] and Luca Grilli [3]

[1] Department of Computer Programming, Istanbul Gelisim Vocational School, Istanbul Gelisim University, 34310 Istanbul, Türkiye
[2] Department of Mathematics, Ordu University, 52200 Ordu, Türkiye; ssenyurt@odu.edu.tr
[3] Department of Economics, Management and Territory, University of Foggia, 71121 Foggia, Italy; luca.grilli@unifg.it
* Correspondence: ccevahir@gelisim.edu.tr

Abstract

In this study, equidistant ruled surfaces generated by the Darboux vector, which has significant kinematic importance and characterizes the instantaneous rotation of a moving frame, are investigated specifically for the Frenet frame. By establishing a structural relationship between a surface and its equidistant ruled surface, transition formulas are provided for shape operators, Gaussian and mean curvatures, and fundamental forms, revealing that the equidistant surface is a scaled transformation of the original one. The obtained results demonstrate that both surfaces are developable and that the geometric properties of the equidistant ruled surfaces can be expressed dependently on each other. Furthermore, it is shown that the geometric character of the equidistant surface, including the invariance of asymptotic lines and the preservation of umbilical points under constant angle conditions, is determined by the rotational dynamics of the base curve. These findings constitute a theoretical foundation for cases involving the use of Darboux axes of different frames in higher dimensions or the investigation of similar structures in different geometric spaces. The geometric interpretation of this theoretical framework is elucidated through the fundamental properties of the surfaces. Finally, a concrete example is presented, where the symmetry of the central planes of the equidistant ruled surfaces at appropriate points is visualized using Maple 2017 software.

Keywords: equidistant ruled surface; Darboux vector; shape operator

MSC: 53A04; 53A05

1. Introduction

In three-dimensional Euclidean space E^3, a ruled surface is defined as the surface generated by the motion of a straight line along a specific curve [1–5], while broader studies on the properties of constant angle and constant curvature ruled surfaces in Minkowski space have also been conducted [6–8]. These surfaces find extensive applications in diverse fields such as architectural design, computer-aided geometric design, and kinematics [9]. The geometric properties of ruled surfaces have also been studied practically in engineering, specifically in spatial axis prediction for spiral tubes [10] and 3D surface reconstruction [11]. Furthermore, detailed geometric analyses have been conducted on surfaces with specific structural properties, including surfaces foliated by curves, quadric surfaces, and translation surfaces [12–14].

The concept of parallel p-equidistant ruled surfaces was first introduced by Valeontis in 1986. He defined these surfaces such that the generator vectors of two ruled surfaces are

Academic Editor: Zhibin Du

Received: 28 November 2025
Revised: 25 December 2025
Accepted: 4 January 2026
Published: 7 January 2026

parallel along their striction curves, and the distance between their polar planes remains constant [15]. Following this conceptualization, the algebraic and integral invariants of p-equidistant surfaces were examined extensively [16–19]. Researchers also extended this definition by considering different frames; for instance, parallel p-equidistant surfaces were investigated using modified orthogonal frames [20]. As interest in these structures grew, the subject of parallel equidistant ruled surfaces gained depth through the selection of different generator vectors and investigations in various spaces. Researchers analyzed parallel z-equidistant ruled surfaces by selecting the normal vector of the striction curve's Frenet frame as the generator [21] and parallel q-equidistant ruled surfaces by selecting the binormal vector [22]. In terms of dimensional generalization, parallel p-equidistant ruled surfaces have been expanded to higher-dimensional spaces [23]. Parallel to these developments, the geometric properties of equidistant ruled surfaces have been studied in various ambient spaces. In the Dual space, parallel equidistant ruled surfaces have been characterized [24]. In Minkowski space, specific attention has been given to parallel p-equidistant ruled surfaces [25–27].

Previous studies have extensively investigated parallel equidistant surfaces generated by various frame vectors; however, these works primarily focus on positional parallel ruled surfaces defined by standard static vector fields. The choice of the Darboux vector as the generator introduces a unique geometric significance. Unlike standard basis vectors, the Darboux vector represents the instantaneous axis of rotation of the frame moving along the curve, and ruled surfaces generated by this vector have been the subject of specific investigation [28]. Consequently, constructing a ruled surface with the Darboux vector transforms the kinematic properties of the curve, specifically its rotational dynamics, into the static geometric invariants of the surface. Therefore, this study distinguishes itself from previous works by utilizing the Darboux vector to bridge the gap between the rotational dynamics of curves and the theory of ruled surfaces. Furthermore, the Darboux vector is significant as it exists in higher-dimensional spaces only in odd dimensions of the form $2n+1$ [29].

In this study, the selection of the Darboux vector as the generator signifies that the ruled surface is generated by a rotation axis carried kinematically along the curve. The characteristic polynomials, first and second fundamental forms, and shape operators of the derived parallel d-equidistant ruled surfaces are analyzed in detail. Based on the obtained findings, the geometric interpretations and algebraic invariant properties of two surfaces $\mathcal{S}$ and $\mathcal{S}^*$ are presented.

2. Preliminaries

Let $r : I \to E^3$ be a unit speed curve and $\{V_1(t), V_2(t), V_3(t)\}$ denote the Frenet frame of α. The Frenet formulas are given by

$$V_1' = \kappa V_2,\ V_2' = -\kappa V_1 + \tau V_3,\ V_3' = -\tau V_2. \tag{1}$$

where $\kappa = \langle V_1, V_2' \rangle$ and $\tau = \langle V_2, V_3' \rangle$ denote the curvature and torsion of α, respectively.

The Darboux vector is defined as $W(t) = \tau(t)V_1(t) + \kappa(t)V_3(t)$. Its normalized form, the unit Darboux vector, is given by $C = \frac{W}{\|W\|}$. If $\theta = \angle(W, V_3)$ denotes the angle between $W(t)$ and $V_3(t)$, then $\sin\theta = \frac{\tau}{\|W\|}$ and $\cos\theta = \frac{\kappa}{\|W\|}$. Consequently, the unit Darboux vector can be expressed as [30]

$$C(t) = \sin\theta V_1(t) + cos\theta V_3(t)\,. \tag{2}$$

https://doi.org/10.3390/sym18010111

Let M be a surface in E^3 and let N be the unit normal vector field of M. Let D denote the Riemannian connection on E^3. For all $X \in \chi(M)$, the definition is given by

$$S(X) = D_X N. \tag{3}$$

The transformation defined by $S : \chi(M) \to \chi(M)$ is called the shape operator or the Weingarten transform on M. The shape operator is a linear transformation [3]. Let M be a surface in E^3. For $1 \leq q \leq 3$, let $I^q : \chi(M) \times \chi(M) \to C^\infty(M, R)$

$$I(X,Y) = \langle X, Y \rangle, \quad II(X,Y) = \langle S(X), Y \rangle, \quad III(X,Y) = \langle S^2(X), Y \rangle \tag{4}$$

the function I^q defined in this form is called the q-th fundamental form on the surface M [3]. These functions are called the first, second, and third fundamental forms on the surface M. Let M be a surface in E^3. If the principal curvatures of M at point P are $k_1(P), \ k_2(P)$, the Gaussian curvature at point P is [3]

$$K(P) = k_1(P)k_2(P). \tag{5}$$

Let M be a surface in E^3. If the principal curvatures of M at the point P are $k_1(P), k_2(P)$, the mean curvature at P is [3]

$$H(P) = \frac{k_1(P) + k_2(P)}{2}. \tag{6}$$

Let M be a surface in E^3. The fundamental forms on M are denoted by I, II, III, respectively. The Gaussian curvature K and the mean curvature H satisfy the following relation [3]:

$$III - HII + KI = 0. \tag{7}$$

Let M be a surface in E^3 and α be a curve in M. Let T be the tangent vector field of α and S be the shape operator of M. The curve α is said to be a curvature line on M if the tangent vector field T corresponds to the principal directions of S along α.

Let M be a surface in E^3 and let S be the shape operator at the point $P \in M$. Two tangent vectors $X_p, Y_p \in T_M(P)$ are called conjugate if

$$\langle S(X_p), Y_p \rangle = 0. \tag{8}$$

For a tangent vector $X_p \neq 0$, if

$$\langle S(X_p), X_p \rangle = 0 \tag{9}$$

then the direction X_p is said to be an asymptotic direction at point P on M.

The curve α, which admits X_p M as its tangent vector at point $\forall P \in \alpha$, is called an asymptotic line on M.

Let $\bar{D}$ be the standard Riemann connection of Euclidean space E^3 and let D be the Riemann connection of manifold M. For every pair of X and Y vector fields on M, the following equation holds:

$$\bar{D}_X Y = D_X Y + V(X,Y) \tag{10}$$

where V is a vector-valued tensor field representing the second fundamental form of the manifold M. This equation is called the Gauss equation [31]. Here, $D_X Y$ and $V(X,Y)$ are the tangent and normal components of $\bar{D}_X Y$, respectively. Let ζ be a normal vector field of M. Then $\bar{D}_X \zeta$ can be expressed in terms of its tangent and normal components as follows:

$$\bar{D}_X \zeta = -A_\zeta(X) + D_X^\perp \zeta. \tag{11}$$

https://doi.org/10.3390/sym18010111

This equation is known as the Weingarten equation. Here, A_ζ is a self-adjoint linear transformation from $T_M(P) \to T_M(P)$ at each point $P \in M$ and $D^\perp$ is the metric connection on the unit normal bundle of M.

From this point onward, we will use the notation A_ζ to refer to both linear transformations and its matrix representation.

A ruled surface in E^3 is a surface that contains a one-parameter family of straight lines. Thus, a ruled surface can be parametrized as

$$\varphi : I \times \mathbb{R} \to E^3, \quad \varphi(t,v) = r(t) + vX(t) \tag{12}$$

where $r(t)$ is called the base curve and $X(t)$ is the generator vector of ruled surface [3].

Let the unit tangent vectors of two curves called base curves $r : I \to E^3$ and $r^* : I \to E^3$ be denoted by V_1 and ${V_1}^*$, respectively. These are also referred to as the generator vectors. The corresponding ruled surfaces are given by the parametric equations $\varphi_{V_1}(t,v) = r(t) + vV_1(t)$ and $\varphi_{V_1^*}(t,v) = r(t) + vV_1^*(t)$ in E^3.

If the generator vectors V_1 and ${V_1}^*$ are parallel, and the distance p between the corresponding polar planes ($Sp\{V_2, V_3\}$) at suitable points is constant, then the pair of ruled surfaces is called a p-equidistant ruled surface pair [15].

Let r and r^* be two curves with associated Frenet frames V_1, V_2, V_3 and ${V_1}^*, {V_1}^*, {V_1}^*$, respectively, at the points $r(t)$ and $r^*(t)$. Let C and C^* be the unit Darboux vectors of the Frenet frames of the curves $r(t)$ and $r^*(t)$, respectively. Then the ruled surfaces generated by these Darboux vectors are given by

$$\varphi_C(t,v) = r(t) + vC(t) \quad \textit{and} \quad \varphi_{C^*}(t,v) = r^*(t) + vC^*(t).$$

If the vectors C and C^* are parallel and the distance d between the corresponding central planes $Sp\{V_1, V_3\}$ at suitable points is constant, then the pair of ruled surfaces is called a d-equidistant ruled surfaces pair. For d-equidistant ruled surfaces, let θ denote the angle between the vectors C and V_3, and let θ^* denote the angle between C^* and V_3^*. Furthermore, let α denote the angle between the vectors V_1 and V_1^* (as well as between V_3 and V_3^*). Then, the corresponding moving frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ are related by the transformation

$$\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \tag{13}$$

where $\alpha = \theta^* - \theta$ [18].

Let the curvature and torsion of the base curves of the d-equidistant ruled surfaces be denoted by κ and τ, respectively. The following relation holds between these curvatures:

$$\kappa^* = (\kappa\cos\alpha - \tau\sin\alpha)\frac{dt}{dt^*}, \text{ and } \tau^* = (\kappa\sin\alpha + \tau\cos\alpha)\frac{dt}{dt^*} \tag{14}$$

The distance between the central planes at the corresponding points of the d-equidistant ruled surfaces is

$$d = \frac{\kappa\left(1 + p' - \frac{\kappa^2}{\kappa^2+\tau^2}\right) - \tau\left(q' + \frac{\kappa\tau}{\kappa^2+\tau^2}\right)}{\kappa^2 + \tau^2} \tag{15}$$

Here, p denotes the distance between the polar planes $Sp\{V_2, V_3\}$ at the corresponding points, and q denotes the distance between the asymptotic planes $Sp\{V_1, V_2\}$ at the corresponding points [18].

3. Parallel Darboux Equidistant Ruled Surfaces in E^3

In differential geometry, a ruled surface is constructed by the motion of a straight line along a base curve. While standard ruled surfaces are well-defined, this study specifically focuses on the surfaces generated by the Darboux vector of the curve. The primary objective of this section is to investigate the geometric relationship between a ruled surface and its parallel d-equidistant counterpart. Rather than analyzing these surfaces in isolation, we establish transition formulas that transfer the geometric properties of the original surface to the equidistant one. To this end, we commence by defining the shape operator, as it constitutes the fundamental structure encapsulating the local curvature, principal directions, and geometric character of the surface.Building upon this definition, the algebraic relationship derived between the corresponding shape operator matrices reveals the intrinsic correspondence between the two surfaces S and S^*. This correlation demonstrates that the geometric properties of the equidistant surface are not independent but are proportionally related to the main surface, which is a defining characteristic of their parallel and equidistant geometry.

Theorem 1. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces. The relationship between the matrices corresponding to the shape operators is given by:*

$$S^* = \left(\frac{dt}{dt^*}\right)^2 \left(1 + \frac{\alpha'}{\theta'}\right) S. \tag{16}$$

Proof. Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel Darboux equidistant ruled surfaces with the following parameterizations:

$$\mathcal{S} : \varphi(t, v) = \gamma(t) + vC(t) \quad \text{and} \quad \mathcal{S}^* : \varphi^*(t^*, v) = \gamma^*(t^*) + vC^*(t^*). \tag{17}$$

We start by differentiating the unit Darboux vector $C(t) = \sin\theta V_1 + \cos\theta V_3$ given in (2). Using the Frenet Formula (1), the derivative is obtained as:

$$C'(t) = (\theta' \cos\theta + \kappa \sin\theta)V_1 + (\tau \cos\theta - \kappa \sin\theta)V_2 - (\theta' \sin\theta + \tau \cos\theta)V_3.$$

Using the relations $\sin\theta = \frac{\tau}{\|W\|}$ and $\cos\theta = \frac{\kappa}{\|W\|}$, the coefficient of V_2 vanishes ($\kappa \sin\theta - \tau \cos\theta = 0$). Thus, the derivative simplifies to:

$$C'(t) = \theta' \cos\theta V_1 - \theta' \sin\theta V_3.$$

For a ruled surface $\varphi(t, v) = r(t) + vC(t)$, the parameter of the striction curve is given by $v = -\frac{\langle r'(t), C'(t)\rangle}{\langle C'(t), C'(t)\rangle}$. Substituting $r'(t) = V_1$ and the calculated $C'(t)$ into this formula yields:

$$v = -\frac{\langle V_1, \theta' \cos\theta V_1 - \theta' \sin\theta V_3\rangle}{\|\theta' \cos\theta V_1 - \theta' \sin\theta V_3\|^2} = -\frac{\theta' \cos\theta}{(\theta')^2} = -\frac{\cos\theta}{\theta'}.$$

Substituting this parameter into the surface equation, the striction curve $\gamma(t)$ is explicitly determined as:

$$\gamma(t) = r(t) - \frac{\cos\theta}{\theta'} C(t).$$

Substituting $\gamma(t)$ into Equation (17), the ruled surface $\varphi(t, v)$ becomes:

$$\varphi(t, v) = r(t) - \frac{\cos\theta}{\theta'} C(t) + vC(t).$$

Differentiating the surface $\varphi(t,v)$ with respect to t and v, we obtain

$$\begin{aligned}
\varphi_t(t,v) &= \Big(1+\frac{\theta'\sin^2\theta+\theta''\cos\theta\sin\theta}{(\theta')^2}-\cos^2\theta+v\theta'\cos\theta\Big)V_1 \\
&\quad +\Big(\frac{\theta'\sin\theta\cos\theta+\theta''\cos^2\theta}{(\theta')^2}+\cos\theta\sin\theta-v\theta'\sin\theta\Big)V_3, \\
\varphi_v(t,v) &= \sin\theta V_1+\cos\theta V_3.
\end{aligned}$$

The inner product of these vectors is $\langle\varphi_t,\varphi_v\rangle\neq 0$. When the Gram-Schmidt orthogonalization method is applied to the vectors φ_t and φ_v, the vectors X and Y are obtained as

$$X=\sin\theta V_1+\cos\theta V_3,\qquad Y=v\theta'\cos\theta V_1-v\theta'\sin\theta V_3. \tag{18}$$

The set $\{X,Y\}$ forms an orthogonal system, which constitutes a basis for the space $\chi(\mathcal{S})$. If the normal vector field of the surface $\mathcal{S}$ is denoted by, $N=X\wedge Y=v\theta' V_2$, then the unit normal vector of the surface, denoted by N_0, is $N_0=V_2$. The shape operator on the ruled surface $\mathcal{S}$ is defined as a linear transformation $S:\chi(\mathcal{S})\longrightarrow\chi(\mathcal{S})$ such that

$$S(X)=\lambda_1X+\lambda_2Y,\quad S(Y)=\mu_1X+\mu_2Y, \tag{19}$$

for some real numbers $\lambda_1,\lambda_2,\mu_1,\mu_2\in R$. Using relations (3) and (18), we obtain

$$S(X)=0,\quad S(Y)=\kappa V_1+\tau V_3. \tag{20}$$

The relations (20) with the help of scalars $\lambda_1,\lambda_2,\mu_1$ and μ_2 are obtained as

$$\lambda_1=\langle S(X),X\rangle=0,\quad \lambda_2=\langle S(X),Y\rangle=0,$$

$$\mu_1=\langle S(Y),X\rangle=0,\quad \mu_2=\langle S(Y),Y\rangle=-v\theta'\sqrt{\kappa^2+\tau^2}.$$

If the matrix corresponding to the shape operator of the linear transformation S is denoted by S, then it is given by

$$S\ =\ \begin{bmatrix}0 & 0\\ 0 & -v\theta'\sqrt{\kappa^2+\tau^2}.\end{bmatrix} \tag{21}$$

Hence, the principal curvatures of the surface $\mathcal{S}$ are determined as

$$k_1=0,\quad k_2=-v\theta'\sqrt{\kappa^2+\tau^2}. \tag{22}$$

Similarly, the shape operator S^* of the ruled surface $\mathcal{S}^*$ is obtained. If the vector $C^*(t)=\sin\theta^*V_1^*+\cos\theta^*V_3^*$ is differentiated, we have

$$C^{*'}(t^*)=\theta^{*'}\cos\theta^*V_1^*-\theta^{*'}\sin\theta^*V_3^*.$$

If this expression is substituted into the striction equation, it becomes

$$\gamma(t^*)=r^*(t^*)-\frac{\cos\theta^*}{\theta^{*'}}C^*(t^*).$$

Substituting this expression into relation (17) yields the ruled surface $\varphi^*(t^*,v)$ as

$$\varphi^*(t^*,v)=r^*(t^*)-\frac{\cos\theta^*}{\theta^{*'}}C^*(t)+vC^*(t).$$

The derivatives of the surface $\varphi^*(t,v)$ with respect to t^* and v are given by

$$\begin{aligned}\varphi_t^*(t^*,v) &= \Big(1+\frac{\theta^{*'}\sin^2\theta^*+\theta^{*''}\cos\theta^*\sin\theta^*}{(\theta^{*'})^2}-\cos^2\theta+v\theta'\cos\theta^*\Big)V_1^* \\ &\quad+\Big(\frac{\theta^{*'}\sin\theta^*\cos\theta^*+\theta^{*''}\cos^2\theta^*}{(\theta^{*'})^2}+\cos\theta^*\sin\theta^*-v\theta^{*'}\sin\theta^*\Big)V_3^*\end{aligned} \tag{23}$$

and

$$\varphi_v^*(t^*,v)=\sin\theta^* V_1^*+\cos\theta^* V_3^*. \tag{24}$$

The inner product of the vectors $\varphi_t^*(t^*,v)$ and $\varphi_v^*(t^*,v)$ taken on the surface $\varphi^*(t^*,v)$ satisfies $\langle\varphi_{t^*},\varphi_v^*\rangle\neq 0$. Then, if the Gram Schmidt orthogonalization method is applied to the system $\{\varphi_t^*,\ \varphi_v^*\}$, we obtain

$$X^*=\sin\theta^* V_1^*+\cos\theta^* V_3^*,\quad Y^*=v\theta^{*\prime}\cos\theta^* V_1^*-v\theta^{*'}\sin\theta^* V_3^*. \tag{25}$$

Thus, the orthogonal system $\{X^*,Y^*\}$ forms a basis for the space $\chi^*(\mathcal{S}^*)$. Let the normal vector field of the surface $\mathcal{S}^*$ be denoted by N^*. In this case, $N^*=v\theta^{*'}V_2^*$. The unit normal vector of the surface is given by $N_0^*=V_2^*$. For the shape operator $S^*:\chi^*(\mathcal{S}^*)\longrightarrow\chi(\mathcal{S}^*)$ and for $a,b,c,d\in R$, the linear transformation $\mathcal{S}^*$ associated with the ruled surface S^* is defined by

$$S^*(X^*)=aX^*+bY^*,\quad S^*(Y^*)=cX^*+dY^*. \tag{26}$$

From (23), (24) and (26), we obtain

$$S^*(X^*)=0,\qquad S^*(Y^*)=\kappa^* V_1^*+\tau^* V_3^*. \tag{27}$$

Using the relations in (27), the scalar coefficients a,b,c and d are calculated as

$$\begin{gathered}a=\langle S^*(X^*),X^*\rangle=0,\qquad b=\langle S^*(X^*),Y^*\rangle=0,\\ c=\langle S^*(Y^*),X\rangle=0,\qquad d=\langle S^*(Y^*),Y^*\rangle=-v\theta^{*'}\sqrt{\kappa^{*2}+\tau^{*2}}.\end{gathered}$$

If the matrix corresponding to the shape operator of the linear transformation S^* is denoted by S^*, then

$$S^* = \begin{bmatrix}0 & 0\\ 0 & -v\theta^{*\prime}\sqrt{\kappa^{*2}+\tau^{*2}}\end{bmatrix}. \tag{28}$$

Accordingly, the principal curvatures k_1^* and k_2^* of the surface $\mathcal{S}^*$ are found to be

$$k_1^*=0,\quad k_2^*=-v\theta^{*\prime}\sqrt{\kappa^{*2}+\tau^{*2}}. \tag{29}$$

Substituting Equation (14) into the expression for k_2^*, we obtain

$$k_2{}^*=-v(\alpha'+\theta')(\frac{dt}{dt^*})^2\sqrt{\kappa^2+\tau^2}.$$

If this expression is substituted into Equation (28), the matrix of the shape operator S^* is found as

$$S^*=\begin{bmatrix}0 & 0\\ 0 & -v(\alpha'+\theta')(\frac{dt}{dt^*})^2\sqrt{\kappa^2+\tau^2}\end{bmatrix}.$$

Hence, the following relation holds between the shape operator matrices S and S^*: $S^* = \left(\frac{dt}{dt^*}\right)^2\left(1+\frac{\alpha'}{\theta'}\right)S$ □

These algebraic results demonstrate that the shape operators S and S^* are structurally equivalent. Since the unit normals coincide ($N = N^*$), both operators share the same principal directions. Furthermore, the parallelism of the Darboux vectors ($W \parallel W^*$) fixes the instantaneous axis of rotation, ensuring that the ruled character of the surface is preserved. Thus, the transformation describes a uniform rescaling of curvature by the distance d, rather than a fundamental change in the surface's geometric nature.

Theorem 2. *Let K and K^* be the Gaussian curvatures, and H and H^* be the mean curvatures of the parallel d-equidistant ruled surfaces $\mathcal{S}$ and $\mathcal{S}^*$, respectively. Then, the following relations hold:*

$$K^* = K = 0, \quad H^* = H\left(\frac{dt}{dt^*}\right)^2 - \left(\frac{dt}{dt^*}\right)^2 v\alpha'\sqrt{\kappa^2+\tau^2} \tag{30}$$

Proof. From Equations (21) and (28), we obtain $K = detS = 0$, $K^* = detS^* = 0$, which implies that $K = K^* = 0$. Similarly, using Equations (21) and (28), the mean curvatures are found as

$$H = \frac{\text{Trace}S}{2} = -\frac{v\theta'\sqrt{\kappa^2+\tau^2}}{2}, \quad H^* = \frac{\text{Trace}S^*}{2} = \frac{-v{\theta^*}'\sqrt{{\kappa^*}^2+{\tau^*}^2}}{2} \tag{31}$$

If the equations in (14) are substituted into Equation (31), then H^* takes the form given in Equation (30). □

Based on these results, the following conclusions can be drawn:

Conclusion 1. ***(i)*** *The parallel d-equidistant ruled surfaces $\mathcal{S}$ and $\mathcal{S}^*$ are developable surfaces.*
(ii) *If the angle α is constant and the ruled surface $\mathcal{S}$ is minimal, then the ruled surface $\mathcal{S}^*$ is also minimal.*

Theorem 3. *For a vector field Z in $\chi(\mathcal{S})$ to also belong to $\chi(\mathcal{S}^*)$, the following relations must hold:*

$$a^* = a - p\sin\theta - q\cos\theta, \quad z = 0, \quad b^*v{\theta^*}' = bv\theta' - p\cos\theta + q\sin\theta \tag{32}$$

Here, the coefficients correspond to $Z = aX + bY$, $Z^* = a^*X^* + b^*Y^*$ and $\overrightarrow{\gamma\gamma^*} = pV_1 + zV_2 + qV_3$.

Proof. Let us determine the conditions under which a vector field $Z \in \chi(\mathcal{S})$ also lies in $\chi(\mathcal{S}^*)$. Since $\chi(\mathcal{S}) = Sp\{X, Y\}$ and $\chi(\mathcal{S}^*) = Sp\{X^*, Y^*\}$ we have

$$X = \sin\theta V_1 + \cos\theta V_3, \quad Y = v\theta'\cos\theta V_1 - v\theta'\sin\theta V_3,$$

and

$$X^* = \sin\theta^* V_1^* + \cos\theta^* V_3^*, \quad Y^* = v{\theta^*}'\cos\theta^* V_1^* - v{\theta^*}'\sin\theta^* V_3^*.$$

If we consider $\alpha = \theta^* - \theta$, and use the trigonometric identities

$$\sin\theta^* = \sin(\theta+\alpha), \quad \cos\theta^* = cos(\theta+\alpha). \tag{33}$$

the corresponding transformations between the vector systems can be obtained. From relations [18], the vectors (33), X^* and Y^* are found as

$$X^* = X, \quad Y^* = \frac{dt}{dt^*}\big(Y + v\alpha'(\cos\theta V_1 - \sin\theta V_3)\big). \tag{34}$$

A vector $Z \in \chi(\mathcal{S})$ can then be expressed as

$$Z = (a \sin\theta + bv\theta' \cos\theta)V_1 + (a\cos\theta - bv\theta' \sin\theta)V_3. \tag{35}$$

Similarly, for the vector $Z^* \in \chi(\mathcal{S}^*)$ with respect to basis $\{X^*, Y^*\}$ Z^*, and using relation (34), the vector Z^* is found to be

$$Z^* = (a^* \sin\theta + b^* v(\theta' + \alpha')\frac{dt}{dt^*}\cos\theta)V_1 + (a^*\cos\theta - b^* v(\theta' + \alpha')\frac{dt}{dt^*}\sin\theta)V_3. \tag{36}$$

From Figure 1, the vector Z^* can be written as

$$Z^* = Z - \gamma\gamma^*. \tag{37}$$

Substituting Equations (35) and (36) using $\overrightarrow{\gamma\gamma^*} = pV_1 + zV_2 + qV_3$ we obtain

$$Z^* = Z - (pV_1 + zV_2 + qV_3)$$

$$\begin{array}{ccc} \big(a^*\sin\theta + b^*v(\theta'+\alpha')\dfrac{dt}{dt^*}\cos\theta\big)V_1 & & \big(a\sin\theta + b\theta' v\cos\theta - p\big)V_1 - zV_2 \\ +\big(a^*\cos\theta - b^*v(\theta'+\alpha')\dfrac{dt}{dt^*}\sin\theta\big)V_3 & = & +\big(a\cos\theta - b\theta' v\sin\theta - q\big)V_3 \end{array}.$$

If the necessary algebraic manipulations are performed, the following system of equations is obtained:

$$\begin{aligned} a^*\sin\theta + b^*v(\theta'+\alpha')\frac{dt}{dt^*}\cos\theta &= a\sin\theta + bv\theta'\cos\theta - p, \\ z &= 0, \\ a^*\cos\theta - b^*v(\theta'+\alpha')\frac{dt}{dt^*}\sin\theta &= a\cos\theta - bv\theta'\sin\theta - q. \end{aligned}$$

From these relations, the coefficients a^* and $b^*v\theta^{*'}$ are obtained as given in Equation (32). □

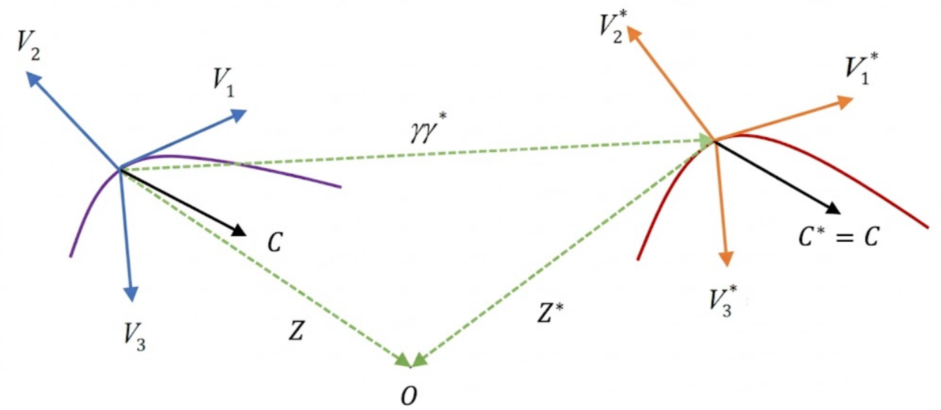

Figure 1. Z and Z^* vectors.

Theorem 4. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces. Then, the following relations hold between the fundamental forms of the ruled surfaces $\mathcal{S}$ and $\mathcal{S}^*$:*

$$I^{*q} = (\frac{dt}{dt^*})^{2q-2}(1 + \frac{\alpha'}{\theta'})^{q-1} I^q, \;\; 1 \le q \le 3. \tag{38}$$

Proof. Before deriving the algebraic relations between the fundamental forms, we first justify the compatibility of the tangent bases $\{X, Y\}$ and $\{X^*, Y^*\}$ under the reparameterization. Since the unit Darboux vectors are parallel ($C \parallel C^*$), the generator vector is invariant, implying $X^* = X$ as derived in (34). Furthermore, since the surfaces are parallel d-equidistant, they share the same unit normal vector field $N = N^*$. Consequently, the orthogonal complement basis vectors Y and Y^* must be collinear. By using the definition of Y^* in (34) and substituting the relation $Y = v\theta'(\cos\theta V_1 - \sin\theta V_3)$ from (18), we establish the linear correspondence:

$$Y^* = \frac{dt}{dt^*}(Y + v\alpha'(\cos\theta V_1 - \sin\theta V_3)) = \frac{dt}{dt^*}\left(1 + \frac{\alpha'}{\theta'}\right)Y.$$

This linear relation confirms that the orthogonality of the frame is preserved ($\langle X^*, Y^*\rangle = 0$), and the transition is governed strictly by the scaling factors. The basic forms I, II, and III of the ruled surface S for $Z, T \in \chi(\mathcal{S})$ are given by the basic forms $I.$, $II.$ and $III.$ of the ruled surface $\mathcal{S}$ for $\forall Z, T \in \chi(\mathcal{S})$ from (4) are given by

$$I(Z,T) = \langle Z, T\rangle, \quad II(Z,T) = \langle S(Z), T\rangle, \quad III(Z,T) = \langle S^2(Z), T\rangle. \tag{39}$$

Similarly, the basic forms of the $\mathcal{S}^*$ ruled surface, denoted by I^*, II^* and III^*, can be expressed using relation (16) as

$$\begin{aligned} I^*(Z,T) &= I(Z,T) \\ II^*(Z,T) &= (\frac{dt}{dt^*})^2(1 + \frac{\alpha'}{\theta'})II(Z,T) \\ III^*(Z^*,T) &= (\frac{dt}{dt^*})^4(1 + \frac{\alpha'}{\theta'})^2 III(Z,T). \end{aligned} \tag{40}$$

From relations (39) and (40), a general relation, such as (38) is obtained. □

Theorem 5. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces. For the principal curvature directions of $\mathcal{S}$ to also be principal curvature directions of $\mathcal{S}^*$, a necessary and sufficient condition is that the angle α is constant.*

Proof. Let X be the principal curvature direction of $\mathcal{S}$. Then, $S(X) = kX$ holds. Using relation (16), we have $S^*(X) = (\frac{dt}{dt^*})^2(1 + \frac{\alpha'}{\theta'})kX$. If α is constant, then $S^*(X) = (\frac{dt}{dt^*})^2 kX$, which shows that X is also a principal curvature direction of $\mathcal{S}^*$. Conversely, let Y be a principal curvature direction of $\mathcal{S}^*$. Then $S^*(Y) = \lambda Y$ holds. From relation (16), we have $S^*(Y) = (\frac{dt^*}{dt})^2(\frac{\theta' + \alpha'}{\theta'})\lambda Y$. If α is constant, then $S^*(Y) = (\frac{dt^*}{dt})^2\lambda Y$ showing that Y is also a principal curvature direction of $\mathcal{S}$. □

Theorem 6. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces.*

(i) *If the angle α is constant, the umbilical points of $\mathcal{S}$ are also umbilical points of $\mathcal{S}^*$.*
(ii) *The flat points of $\mathcal{S}$ are also flat points of $\mathcal{S}^*$.*

Proof. **(i)** Let P be an umbilical point of $\mathcal{S}$. Then the shape operator satisfies $S = \lambda I_2, \lambda \in \mathbb{R}$. Using relation (16), $S^*(X) = \lambda_2\ I_2$, $\lambda_2 \in \mathbb{R}$, which shows that P is also an umbilical point of $\mathcal{S}^*$

(ii) Let P be the flat point of $\mathcal{S}$, i.e., $S = 0$. From relation (16), it follows that $S^*(X) = 0$ so P is also a flat point of $\mathcal{S}^*$. □

Conclusion 2. *(Geometric Interpretation of Theorems 6 and 7): These findings prove that fundamental geometric properties, such as umbilical points and flat points, are symmetrically preserved between different pairs of surfaces, particularly under the condition of constant α. Geometrically, this preservation is guaranteed because the condition $\alpha = const$ implies an isogonal correspondence,* *where the relative angular velocity between the moving frames vanishes ($\alpha' = 0$). In the absence of this relative rotation, the tangent bundle undergoes a rigid transformation without shearing, thereby preserving the isotropic nature of umbilical points and the vanishing curvature of flat points across the parallel ruled surfaces.*

Theorem 7. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces.*

(i) *If $\alpha \neq const.$, the conjugate tangent vectors of $\mathcal{S}$ are also conjugate tangent vectors of $\mathcal{S}^*$.*
(ii) *The asymptotic directions of $\mathcal{S}$ are also the asymptotic directions of $\mathcal{S}^*$.*

Proof. **(i)** For conjugate tangent vectors $X_P, Y_P \in T_{\mathcal{S}} \in (P)$, we have $\langle S(X_P), Y_P\rangle = 0$ From relations (16) and (34), this becomes

$$\langle S(X_P), Y_P\rangle = (\frac{dt^*}{dt})^3(\frac{\alpha'}{\alpha'+\theta'})\left\langle S^*(X_P{}^*), (Y_P{}^*)\right\rangle.$$

Since $\alpha' \neq 0$, it follows that $\left\langle S^*(X_P^*), Y_P^*\right\rangle = 0$, showing that X_P and Y_P are also conjugate tangent vectors of $\mathcal{S}^*$ when α is not constant.

(ii) If X_P is an asymptotic direction of $\mathcal{S}$, then $\left\langle S(X_P), X_P\right\rangle = 0$. Substituting relations (16) and (34) yields

$$\left\langle (\frac{dt^*}{dt})^2(\frac{\alpha'}{\alpha'+\theta'})S^*(X_P), X_P\right\rangle = 0$$

which simplifies to

$$\left\langle S^*(X_P), X_P\right\rangle = 0.$$

Hence, X_P is also an asymptotic direction of $\mathcal{S}^*$. □

Conclusion 3. *The asymptotic lines of $\mathcal{S}$ are also the asymptotic lines of $\mathcal{S}^*$.*

Theorem 8. *Let $P_{\mathcal{S}}(\lambda)$ and $P_{\mathcal{S}^*}(\lambda)$ be the characteristic polynomials of the shape operator matrices corresponding to the parallel d-equidistant ruled surfaces $\mathcal{S}$ and $\mathcal{S}^*$, respectively. Then, the following relation holds:*

$$P_{\mathcal{S}^*}(\lambda) = (\frac{dt}{dt^*})^2 P_{\mathcal{S}}(\lambda) + \lambda^2(1-(\frac{dt}{dt^*})^2) + v\alpha'\lambda\sqrt{\kappa^2+\tau^2} \tag{41}$$

Proof. Let S and S^* be the matrices of the shape operators of $\mathcal{S}$ and $\mathcal{S}^*$, respectively. Their characteristic polynomials are

$$P_{\mathcal{S}}(\lambda) = \lambda^2 - (k_1+k_2)\lambda + (k_1.k_2) \quad \text{and} \quad P_{\mathcal{S}^*}(\lambda) = \lambda^2 - (k_1^*+k_2^*)\lambda + (k_1^*.k_2^*).$$

Setting $H = k_1 + k_2$, $H^* = k_1^* + k_2^*$, $K = k_1k_2 = 0$ and $K^* = k_1^*k_2^* = 0$, we have

$$P_{\mathcal{S}}(\lambda) = -H\lambda + \lambda^2 \quad \text{and} \quad P_{\mathcal{S}^*}(\lambda) = -H^*\lambda + \lambda^2. \tag{42}$$

Substituting relation (30) into this expression gives $P_{\mathcal{S}^*}(\lambda)$ as in (41). □

Theorem 9. *The shape operators of $\mathcal{S}$ and $\mathcal{S}^*$ parallel d-equidistant ruled surfaces satisfy the relation:*

$$P_{\mathcal{S}^*}(S^*) = (\frac{dt}{dt^*})^4(1+\frac{\alpha'}{\theta'})(P_{\mathcal{S}}(S) + Sv\alpha'\sqrt{\kappa^2+\tau^2}). \tag{43}$$

This establishes the relationship between the characteristic polynomials of the corresponding matrices: $(\frac{dt}{dt^*})^4(1+\frac{\alpha'}{\theta'})Sv\alpha'\sqrt{\kappa^2+\tau^2}) = 0$.

Proof. From the relation (42) and the Cayley Hamilton theorem,

$$P_S(S) = 0, \quad P_{S^*}(S^*) = 0. \tag{44}$$

Substituting relations relations (16) and (30) into $P_{S^*}(S^*) = 0$ gives the result in (43). □

Theorem 10. *For the basic forms and mean curvatures of parallel d-equidistant ruled surfaces of $\mathcal{S}$ and $\mathcal{S}^*$, the following relation holds:*

$$III^* - H^*II^* = (\frac{dt}{dt^*})^4(1+\frac{\alpha'}{\theta'})(III - HII) + (\frac{dt}{dt^*})^4(1+\frac{\alpha'}{\theta'})v\alpha'\sqrt{\kappa^2+\tau^2} \equiv 0$$

where $(\frac{dt}{dt^*})^4(1+\frac{\alpha'}{\theta'})v\alpha'\sqrt{\kappa^2+\tau^2} = 0.$

Proof. For the basic forms and mean curvatures of parallel d-equidistant ruled surfaces, we have $III - HII + KI \equiv 0, \quad III^* - H^*II^* + K^*I^* \equiv 0$ From this, it follows that $III^* - H^*II^* = 0$. which implies $\left(\frac{dt}{dt^*}\right)^4(1+\frac{\alpha'}{\theta'})v\alpha'\sqrt{\kappa^2+\tau^2} = 0.$ □

Theorem 11. *The derivative operators of parallel d-equidistant ruled surfaces $\mathcal{S}$ and $\mathcal{S}^*$ satisfy the following relation:*

$$D^*_{X^*}Y^* = (\frac{dt}{dt^*})D_XY + (\frac{dt}{dt^*})\overline{D}_XM - \langle S(X), Y\rangle V_2 + \left(\frac{dt}{dt^*}\right)^3\left(1+\frac{\alpha'}{\theta'}\right)\Big\langle S(X), Y+M\Big\rangle V_2 \tag{45}$$

where $M = v\alpha'(\cos\theta V_1 - \sin\theta V_3)$.

Proof. The unit normals of $\mathcal{S}$ and $\mathcal{S}^*$, which are parallel d-equidistant ruled surfaces, are given by $N_0 = V_2$ and $N_0^* = V_2^*$, respectively. Let the Riemann connection in E^3 be denoted by $\overline{D}$, and the derivative operators on $\mathcal{S}$ and $\mathcal{S}^*$ be denoted by D and D^*, respectively. Then, the Gaussian equations of the surfaces can be expressed as

$$D_XY = \overline{D}_XY + \langle S(X), Y\rangle V_2, \quad D^*_{X^*}Y^* = \overline{D}_{X^*}Y^* + \langle S^*(X^*), Y^*\rangle V_2^*. \tag{46}$$

Considering the relations given in (16) and (34), namely $M = v\alpha'(\cos\theta V_1 - \sin\theta V_3)$ the derivative $D^*_{X^*}Y^*$ can be written as

$$D^*_{X^*}Y^* = (\frac{dt}{dt^*})\overline{D}_XY + (\frac{dt}{dt^*})\overline{D}_XM + (\frac{dt}{dt^*})^3(1+\frac{\alpha'}{\theta'})\langle S(X), Y+M\rangle V_2.$$

Finally, substituting Equation (46) into the above expression leads to the relation given in (45). □

Theorem 12. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces. If an α curve is both asymptotic and geodesic in $\mathcal{S}$, a necessary and sufficient condition for it to be a geodesic in $\mathcal{S}^*$, is $\overline{D}_TT = 0$.*

Proof. Let T be the tangent of the α curve. If the curve is asymptotic and geodesic in $\mathcal{S}$, then $\langle S(T), T\rangle = 0$ and $D_TT = 0$. From Equation (45) we have D^*_TT is

$$D^*_TT = D_TT - \langle S(T), T\rangle V_2 + (\frac{dt}{dt^*})^3(1+\frac{\alpha'}{\theta'})\big(\langle S(T), T\rangle V_2\big) + \overline{D}_TT = 0.$$

Thus, if $\overline{D}_TT = 0$, it follows that $D^*_TT = 0$, completing the proof. □

Theorem 13. *The following relations hold between the arc lengths of the spherical indicatrices of the striction curves of parallel d-equidistant ruled surfaces $\mathcal{S}$ and $\mathcal{S}^*$:*

$$S_{V_1^*} = \cos\alpha S_{V_1} - \sin\alpha S_{V_3}, \quad S_{V_2^*} = S_{V_2}, \quad S_{V_3^*} = \sin\alpha S_{V_1} + \cos\alpha S_{V_3}, \tag{47}$$

where α is constant.

Proof. The arc length of the curve $(V_1{}^*)$ is denoted by $S_{V_1^*}$ and the angle α is constant. Then

$$S_{V_1^*} = \int_0^t \kappa^* dt^* = \int_0^t (\kappa\cos\alpha - \tau\sin\alpha)dt = \cos\alpha\int_0^t \kappa dt - \sin\alpha\int_0^t \tau dt = \cos\alpha S_{V_1} - \sin\alpha S_{V_3}.$$

Similarly, $S_{V_2^*}$ and $S_{V_3^*}$ can be obtained in the same manner (see [3]). □

Theorem 14. *In E^3, let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces. Let the spherical indicatrix curvatures and radii of curvature of the generator vectors V_1 and $V_1{}^*$ be denoted by κ_{V_1}, $\kappa_{V_1^*}$ and ρ_{V_1}, $\rho_{V_1{}^*}$, respectively. Then the following relations hold:*

$$\kappa_{V_1^*} = \frac{\kappa\kappa_{V_1}}{\kappa^*}\frac{dt}{dt^*} \quad \text{and} \quad \rho_{V_1{}^*} = \frac{\rho_{V_1}\kappa^*}{\kappa}\frac{dt^*}{dt}. \tag{48}$$

Proof. Let $r^* = r^*(t)$ be the underlying curve of the ruled $\mathcal{S}^*$. Differentiating, we obtain $V_1^* = \frac{dr^*}{dt^*}$, $V_1^{*\prime} = \kappa^* V_2^*$, $V_1^{*\prime\prime} = -\kappa^{*2}V_1{}^* + \kappa^{*\prime}V_2{}^* + \kappa^*\tau^*V_3{}^*$ and
$\|V_1^{*\prime} \wedge V_1^{*\prime\prime}\| = \kappa^{*2}\sqrt{\kappa^{*2} + \tau^{*2}}$. Denoting the curvature function by $\kappa_{V_1{}^*}$ and the radius of curvature by $\rho_{V_1{}^*}$, from [32], we have $\kappa_{V_1{}^*}$

$$\kappa_{V_1{}^*} = \frac{\sqrt{\kappa^{*2} + \tau^{*2}}}{\kappa^*} = \frac{\sqrt{\begin{array}{c}\kappa^2\cos^2\alpha - 2\kappa\tau\cos\alpha\sin\alpha + \tau^2\sin^2\alpha \\ +\kappa^2\sin^2\alpha + 2\kappa\tau\sin\alpha\cos\alpha + \tau^2\cos^2\alpha\end{array}}}{(\kappa\cos\alpha - \tau\sin\alpha)\frac{dt}{dt^*}} = \frac{\sqrt{\kappa^2 + \tau^2}}{(\kappa\cos\alpha - \tau\sin\alpha)\frac{dt}{dt^*}}$$

Finally, $\kappa_{V_1{}^*}$ and $\rho_{V_1{}^*}$ are given by (48). □

Theorem 15. *In E^3, let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces. Let the spherical indicatrix curvature and radius of curvature of the central normal vectors V_2 and $V_2{}^*$ be denoted by κ_{V_2}, $\kappa_{V_2^*}$ and ρ_{V_2}, $\rho_{V_2{}^*}$, respectively. Then*

$$\kappa_{V_2{}^*} = \kappa_{V_2} \quad \text{and} \quad \rho_{V_2^*} = \rho_{V_2} \tag{49}$$

Proof. Taking the first and second derivatives of the spherical indicatrix curve of the central normal vector $(V_2{}^*)$, we have $V_2{}^{*\prime} = -\kappa^*V_1{}^* + \tau^*V_3{}^*$,
$V_2{}^{*\prime\prime} = -\kappa^{*\prime}V_1{}^* - (\kappa^{*2} + \tau^{*2})V_2{}^* + \tau^{*\prime}V_3{}^*$. Then $\|V_2{}^{*\prime}\|^3 = (\kappa^{*2} + \tau^{*2})^{\frac{3}{2}}$ and $|W^*\| = \sqrt{\kappa^{*2} + \tau^{*2}}$ Denoting the curvature function of the spherical indicatrix by $\kappa_{V_2{}^*}$, we obtain

$$\kappa_{V_2{}^*} = \frac{\|V_2{}^{*\prime} \wedge V_2{}^{*\prime\prime}\|}{\|V_2{}^{*\prime}\|^3} = \frac{\sqrt{\|W^*\|^6 + (-\kappa^{*\prime}\tau^* + \kappa^*\tau^{*\prime})^2}}{\|W^*\|^3}.$$

If $\frac{dt}{dt^*}$ is constant, this equation becomes

$$\kappa_{V_2{}^*} = \frac{\sqrt{\|W^*\|^6(\frac{dt^*}{dt})^6 + (\frac{dt^*}{dt})^6(-\kappa^{*\prime}\tau^* + \kappa^*\tau^{*\prime})^2}}{(\frac{dt^*}{dt})^3\|W^*\|^3} \tag{50}$$

Here, the quantities W^*, ${\kappa^*}'$, ${\tau^*}'$ and ${\kappa^*}'\tau^* + \kappa^*{\tau^*}'$ are expressed as

$$\begin{aligned} \|W^*\| &= \sqrt{\kappa^{*2}+\tau^{*2}} = \sqrt{\kappa^2+\tau^2}\frac{dt}{dt^*} = \|W\|\frac{dt}{dt^*}, \\ -{\kappa^*}'\tau^* + {\tau^*}'\kappa^* &= (\frac{dt}{dt^*})^3(\kappa'\tau+\kappa^2\alpha'+\kappa\tau'+\tau^2\alpha'). \end{aligned} \tag{51}$$

If these expressions are substituted into (50) and the special case where α' is constant is considered, we obtain

$$\kappa_{{V_2}^*} = \frac{\sqrt{\|W\|^6 + (-\kappa'\tau+\tau'\kappa)^2}}{\|W\|^3}.$$

From [32], it follows that $\kappa_{{V_2}^*} = \kappa_{V_2}$. If the radius of curvature is denoted by $\rho_{{V_2}^*}$, then $\rho_{{V_2}^*} = \frac{1}{\kappa_{{V_2}^*}} = \rho_{V_2}$. This completes the proof. $\square$

Theorem 16. *Let $\mathcal{S}$ and $\mathcal{S}^*$ be parallel d-equidistant ruled surfaces in E^3. Let V_3 and ${V_3}^*$ denote the central tangent vectors, and let their spherical indicatrix curvatures and radius of curvature be represented by κ_{V_3}, $\kappa_{V_3^*}$ and ρ_{V_3}, $\rho_{{V_3}^*}$, respectively. Then the following relations hold:*

$$\kappa_{{V_3}^*} = \frac{\kappa_{V_3}\tau}{\tau^*}\frac{dt}{dt^*} \quad \textit{and} \quad \rho_{{V_3}^*} = \frac{\rho_{{V_3}^*}\tau^*}{\tau}\frac{dt^*}{dt} \tag{52}$$

Proof. Taking the first and second derivatives of the spherical indicatrix curve $({V_3}^*)$, we obtain ${{V_3}^*}' = \tau'{V_2}^*$ and ${{V_3}^*}'' = \kappa^*\tau^*{V_1}^* - {\tau^*}'{V_2}^* - \tau^{*2}{V_3}^*$. Here, ${{V_3}^*}' \wedge {{V_3}^*}'' = \tau^{*3}{V_1}^* + \kappa^*\tau^{*2}{V_3}^*$ and $\|{V_3}^* \wedge {{V_3}^*}''\| = \tau^{*2}\|W^*\|$, Therefore, the curvature function of the spherical indicatrix $({V_3}^*)$ is $\kappa_{{V_3}^*} = \dfrac{\|W^*\|}{\tau^*}$. Using the relations in (51) $\kappa_{{V_3}^*}$ and $\rho_{{V_3}^*}$ can be expressed as in (52). $\square$

Example 1. *Let the base curves of ruled surfaces $\varphi_C(t,v)$ and $\varphi_{C^*}(t,v)$ be defined respectively as*

$$r(t) = (\frac{\sqrt{3}}{3}\sin(\sqrt{3}t), \sqrt{2}, \frac{\sqrt{3}}{3}\cos(\sqrt{3}t)), \quad r^*(t) = (\frac{2\sqrt{3}}{3}\sin(\sqrt{3}t), \frac{\sqrt{6}}{3}, 2\frac{\sqrt{3}}{3}\cos(\sqrt{3}t)),$$

The Darboux vectors and the corresponding curvatures of r and r^* are obtained as

$$D = (0,\sqrt{3},0), \quad C = (0,1,0), \quad \kappa(t) = \sqrt{3}, \quad \tau(t) = 0,$$

$$D^* = (0,\frac{\sqrt{3}}{2},0), \quad C^* = (0,1,0), \quad \kappa^*(t) = \frac{\sqrt{3}}{2}, \quad \tau^*(t) = 0$$

Hence, the ruled surfaces generated by these vectors are

$$\varphi_C(t,v) = (\frac{\sqrt{3}}{3}\sin(\sqrt{3}t), \sqrt{2}, \frac{\sqrt{3}}{3}\cos(\sqrt{3}t)) + v(0,\sqrt{3},0),$$

$$\varphi_{C^*}(t,v) = (\frac{2\sqrt{3}}{3}\sin(\sqrt{3}t), \frac{\sqrt{6}}{3}, 2\frac{\sqrt{3}}{3}\cos(\sqrt{3}t)) + v(0,\frac{\sqrt{3}}{2},0).$$

The striction lines $\gamma(t)$ and $\gamma^*(t)$ of the ruled surfaces $\varphi_C(t,v)$ and $\varphi_{C^*}(t,v)$ are given by

$$\gamma(t) = \Big(\frac{\sqrt{3}}{3}\sin(\sqrt{3}t), \sqrt{2} - \sqrt{3}\cos\theta\frac{dt}{d\theta}, \frac{\sqrt{3}}{3}\cos(\sqrt{3}t)\Big),$$

$$\gamma^*(t) = \Big(\frac{2\sqrt{3}}{3}\sin(\sqrt{3}t), \frac{\sqrt{6}}{3} - \frac{\sqrt{3}}{2}\cos\theta^*\frac{dt}{d\theta^*}, \frac{2\sqrt{3}}{3}\cos(\sqrt{3}t)\Big)$$

Since $\gamma\gamma^* = \gamma^* - \gamma$ *and* $\cos\theta = \dfrac{\kappa}{\sqrt{\kappa^2+\tau^2}} = 1$ *and* $\cos\theta^* = \dfrac{\kappa^*}{\sqrt{\kappa^{*2}+\tau^{*2}}} = 1$, *we obtain* $\gamma\gamma^*$ *are found as*

$$\gamma\gamma^*(t) = \left(\frac{\sqrt{3}}{3}\sin(\sqrt{3}t), \frac{\sqrt{6}}{3} - \sqrt{2} - \frac{\sqrt{3}}{2}\frac{dt}{d\theta^*} + \sqrt{3}\frac{dt}{d\theta}, \frac{\sqrt{3}}{3}\cos(\sqrt{3}t)\right).$$

Let p *and* q *denote respectively the distances between the polar and asymptotic planes. Then,* $p = \dfrac{\sqrt{3}}{3}\sin(\sqrt{3}t)$, $q = \dfrac{\sqrt{3}}{3}\cos(\sqrt{3}t)$. *From Equation (15), the distance* d *between the central planes is*

$$d = \frac{\sqrt{3}}{3}\cos(\sqrt{3}t).$$

Let the central planes be $H = Sp\{V_1, V_3\}$ *and* $H^* = Sp\{V_1{}^*, V_3{}^*\}$. *For any* $X \in H$ *and* $X^* \in H^*$, *from* $\langle \overrightarrow{\gamma X}, \overrightarrow{C} \rangle = 0$ *and* $\langle \overrightarrow{\gamma^* X^*}, \overrightarrow{C^*} \rangle = 0$ *the corresponding plane equations are:*

$$H ... y\sqrt{3} - \sqrt{6} + 3\frac{dt}{d\theta} = 0,$$

$$H^* ... y^* 2\sqrt{3} - 2\sqrt{2} + 3\frac{dt}{d\theta^*} = 0.$$

Case 1: $t = 0$

At $t = 0$ the striction points and distance d are

$$\gamma(0) = (0, \sqrt{2} - \sqrt{3}\frac{dt}{d\theta}, \frac{\sqrt{3}}{3}), \quad \gamma^*(0) = (0, \frac{\sqrt{6}}{3} - \frac{\sqrt{3}}{2}\frac{dt}{d\theta^*}, \frac{2\sqrt{3}}{3}), \quad d = \frac{\sqrt{3}}{3}.$$

The Darboux vectors are parallel at the striction point. Let A and B be points on the central planes H and H^*, respectively:

$$A(0, \sqrt{2} - \sqrt{3}\frac{dt}{d\theta}, 0) \in H, \quad B(0, \frac{\sqrt{6}}{3} - \frac{\sqrt{3}}{2}\frac{dt}{d\theta^*}, 0) \in H^*.$$

Since the distance between A and B is $\dfrac{\sqrt{3}}{3}$, if $\dfrac{dt}{d\theta} = \dfrac{1}{3} - \dfrac{\sqrt{2}}{3}$ and $\dfrac{dt}{d\theta^*} = -\dfrac{2\sqrt{6}}{3}$ are chosen, the points A and B are obtained.

$$A(0, \sqrt{2} - \frac{\sqrt{3}}{3} + \frac{\sqrt{6}}{3}, 0), \quad B(0, \frac{\sqrt{6}}{3} + \sqrt{2}, 0).$$

Substituting into the striction lines gives

$$\gamma(0) = (0, \sqrt{2} - \frac{\sqrt{3}}{3} + 1, \frac{\sqrt{3}}{3}), \quad \gamma^*(0) = (0, \frac{\sqrt{6}}{3} + \sqrt{2}, \frac{2\sqrt{3}}{3}).$$

For $t = 0$ and $v \in (-4, 4)$, the striction curve and the ruled surface are shown in Figure 2, and the distance d between the central planes at the corresponding points is illustrated in Figure 3.

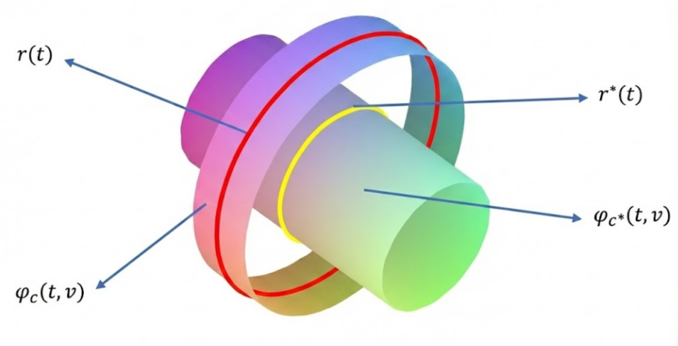

Figure 2. d-equidistant ruled surfaces.

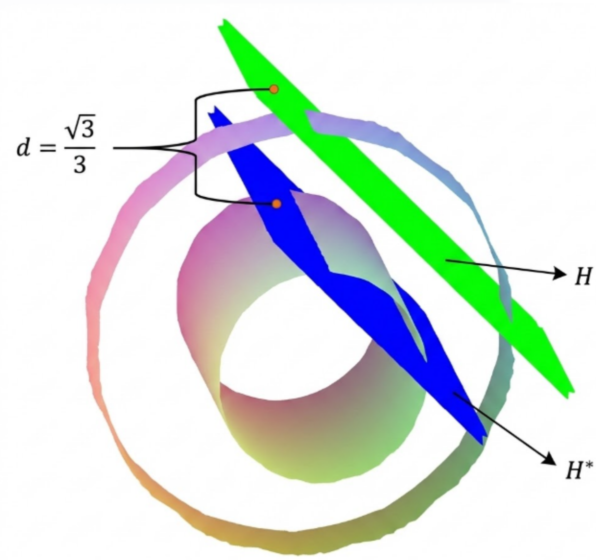

Figure 3. Distance d at appropriate points for $t = 0$.

Case 2: $t = \dfrac{\pi\sqrt{3}}{18}$

For $t = \dfrac{\pi\sqrt{3}}{18}$ striction points γ, γ^* and distance d are

$$\gamma\Big(\frac{\pi\sqrt{3}}{18}\Big) = \Big(\frac{\sqrt{3}}{6}, \sqrt{2} - \sqrt{3}\frac{dt}{d\theta}, \frac{1}{2}\Big), \quad \gamma^*\Big(\frac{\pi\sqrt{3}}{18}\Big) = \Big(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3} - \frac{\sqrt{6}}{2}\frac{dt}{d\theta^*}, 1\Big), \quad d = \frac{1}{2}.$$

The Darboux vectors are again parallel at the striction point. Let A and B be two points of the central planes. These points are

$$A\Big(0, \sqrt{2} - \sqrt{3}\frac{dt}{d\theta}, 0\Big) \in H, \quad B\Big(0, \frac{\sqrt{6}}{3} - \frac{\sqrt{3}}{2}\frac{dt}{d\theta^*}, 0\Big) \in H^*.$$

Since the distance between A and B is $\frac{1}{2}$, if $\dfrac{dt}{d\theta} = \dfrac{6}{3} - \dfrac{\sqrt{2}}{3}$ and $\dfrac{dt}{d\theta^*} = -\dfrac{\sqrt{3}}{3}$ are chosen, the points A and B are obtained as follows

$$A\Big(0, \frac{\sqrt{6}}{3}, 0\Big), \quad B\Big(0, \frac{\sqrt{6}}{3} + \frac{1}{2}, 0\Big).$$

Substituting in the striction equations gives

$$\gamma\Big(\frac{\sqrt{3}\pi}{18}\Big)\Big(\frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}, \frac{1}{2}\Big), \quad \gamma^*\Big(\frac{\sqrt{3}\pi}{18}\Big) = \Big(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3} - \sqrt{2} + \frac{1}{2}, 1\Big).$$

For $t = \dfrac{\sqrt{3}\pi}{18}$ and $v \in (-4.4)$, the distance d of the central planes at the corresponding points is shown in Figure 4.

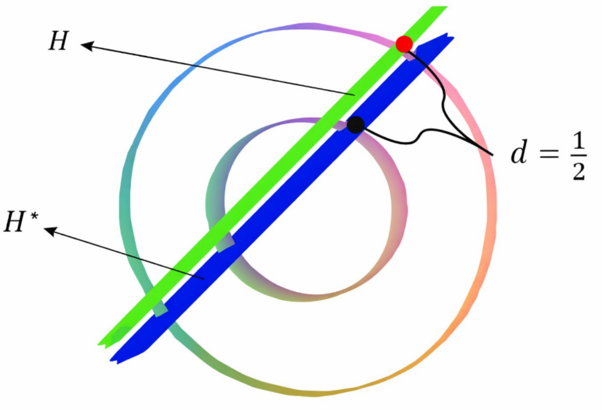

Figure 4. Distance d at appropriate points for $t = \dfrac{\sqrt{3}\pi}{18}$.

Maple visualizations confirm the theoretical findings for parallel d-equidistant ruled surfaces. The nested cylinders in Figure 2 result directly from the vanishing torsion ($\tau = 0$) of the base curves. Since $\tau = 0$, the Darboux vectors align with the constant binormal direction, creating generalized cylinders with translational symmetry. The calculated principal curvatures ($k_2 = 0$) analytically confirm this structure. Since $K = 0$, the surfaces are developable, but they are not minimal because the mean curvature is non-zero ($H \neq 0$). Additionally, Figures 3 and 4 display the Central Planes (H and H^*) rather than the surfaces themselves. These planes act as reference frames for the d-equidistant condition. The distance between these planes in the figures matches the calculated values ($d = \frac{\sqrt{3}}{3}$ and $d = \frac{1}{2}$) exactly. Thus, the visual results align with the algebraic findings regarding parallelism and the fixed distance between the central planes.

4. Conclusions

In this study, equidistant ruled surfaces generated by Darboux vectors in three-dimensional Euclidean space were analyzed in in detail, and their geometric and differential properties were systematically examined. By deriving transition formulas for shape operators and curvatures, we established that the geometry of the equidistant surface is a scaled transformation of the original one. Our analysis proved that both surfaces are developable ($K = K^* = 0$) but generally not minimal ($H \neq 0$). A central theme throughout our investigation was the exploration of the intrinsic geometric correspondence between the two surfaces. Specifically, we demonstrated that asymptotic lines are essentially invariant, while principal directions and umbilical points are preserved under the condition of a constant angle α. The geometric interpretation of these theoretical results, supported by Maple visualizations, bridges the gap between abstract algebraic invariants and tangible spatial properties.

Building upon these concrete geometric interpretations established in Euclidean 3-space (E^3), future research aims to expand this framework in several strategic directions. Primarily, since the Darboux vector is defined only in odd-dimensional spaces, we aim to extend these findings to higher-dimensional Euclidean spaces (E^{2n+1}). In addition, the Darboux vector obtained using the Frenet frame can be reconsidered by employing the Bishop and alternate moving frames, and the equidistant ruled surfaces generated by this vector can be examined in detail. Moreover, the proposed approach can be extended to non-Euclidean geometries, including Lorentz–Minkowski, Dual, and Galilean spaces. Comparative analyses carried out in these different metric spaces are expected to reveal how the geometric properties of equidistant ruled surfaces vary depending on the chosen frame and the underlying metric structure. The concrete E^3 model developed in this paper will serve as a fundamental benchmark for these analytical generalizations, facilitating the interpretation of complex geometric structures in future studies.

Author Contributions: Conceptualization, S.Ş. and C.C.Y.; methodology, L.G.; software, C.C.Y.; validation, C.C.Y., S.Ş. and L.G.; formal analysis, C.C.Y.; investigation, C.C.Y.; resources, C.C.Y.; data curation, L.G.; writing—original draft preparation, C.C.Y.; writing—review and editing, S.Ş.; visualization, C.C.Y.; supervision, C.C.Y.; project administration, S.Ş.; funding acquisition, L.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

Symbol	Description
E^3	3-Dimensional Euclidean space
$r(t)$	Unit speed curve (Base curve)
$\{V_1, V_2, V_3\}$	Frenet frame field (Tangent, Normal, Binormal vectors)
κ, τ	Curvature and Torsion of the curve $r(t)$
$r^*(t^*)$	Unit speed base curve of the parallel ruled surface $\mathcal{S}^*$
$\{V_1^*, V_2^*, V_3^*\}$	Frenet frame field of the curve r^* (Tangent, Normal, Binormal)
κ^*, τ^*	Curvature and Torsion of the curve r^*
$W(t)$	Darboux vector
$C(t)$	Unit (normalized) Darboux vector
θ	Angle between the Darboux vector W and the binormal vector V_3
$\mathcal{S}$	Ruled surface generated by the Darboux vector $C(t)$
$\mathcal{S}^*$	Parallel d-equidistant ruled surface to $\mathcal{S}$
$\varphi(t, v)$	Parametrization of the ruled surface
$\gamma(t)$	Striction curve of the ruled surface
d	Constant distance between the central planes of the surfaces
α	Angle difference function ($\alpha = \theta^* - \theta$)
S	Shape operator (Weingarten map) or its matrix representation
K, H	Gaussian and Mean curvatures
k_1, k_2	Principal curvatures
I, II, III	First, Second, and Third Fundamental Forms
$D, \overline{D}$	Riemannian connection operators
N, N_0	Normal vector field and Unit normal vector of the surface
$\frac{dt}{dt^*}$	Derivative ratio between the parameters of the curves
$Sp\{V_1, V_3\}$	Central plane spanned by Tangent and Binormal vectors
κ_{V_i}, ρ_{V_i}	Curvature and radius of curvature of the spherical indicatrix curves
$*$	Indicates parameters associated with the parallel d-equidistant ruled surface

References

1. Atalay, G.Ş. Smarandache Curved Ruled Surfaces and Their Characterizations According to Modified Orthogonal Frame in E^3. *Appl. Appl. Math.* **2025**, *20*, 1–19.
2. Damar, E. A Novel Approach to Ruled Surfaces Using Adjoint Curve. *Symmetry* **2025**, *17*, 1018. [CrossRef]
3. do Carmo, M.P. *Differential Geometry of Curves and Surfaces*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1976.
4. Li, Y.; Çalışkan, A. Quaternionic Shape Operator and Rotation Matrix on Ruled Surfaces. *Axioms* **2023**, *12*, 486. [CrossRef]
5. Orbay, K.; Kasap, E.; Aydemir, I. Mannheim offsets of ruled surfaces. *Math. Probl. Eng.* **2009**, *2009*, 160917. [CrossRef]

6. Ali, A.T. Non-lightlike constant angle ruled surfaces in Minkowski 3-space. *J. Geom. Phys.* **2020**, *157*, 103833. [CrossRef]
7. Ali, A.T. A constant angle ruled surfaces. *Int. J. Geom.* **2018**, *7*, 69–80.
8. Ali, A.T. Non-lightlike ruled surfaces with constant curvatures in Minkowski 3-space. *Int. J. Geom. Methods Mod. Phys.* **2018**, *15*, 1850068. [CrossRef]
9. Hunt, K.H. The Geometry of the Watt Six-bar Mechanisms. In *Kinematic Geometry of Mechanisms*; Oxford University Press: Oxford, UK, 1978.
10. Wang, Z.; Tao, Y.; Zhang, S.; Liu, X.; Lin, Y.; Li, L.; Li, Z. Spatial spiral tube multi-roller bending: Accurate axial prediction utilizing AWPSO-FECAM-LSTM framework. *Expert Syst. Appl.* **2026**, *296*, 128960. [CrossRef]
11. Chen, J.; Sun, L.; Song, Y.; Geng, Y.; Xu, H.; Xu, W. 3D Surface Highlight Removal Method Based on Detection Mask. *Arab. J. Sci. Eng.* **2025**, 1–13.. [CrossRef]
12. Ali, A.T.; Hamdoon, F.M. Surfaces foliated by ellipses with constant Gaussian curvature in Euclidean 3-space. *Korean J. Math.* **2017**, *25*, 537–554.
13. Ali, A.T.; Abdel Aziz, H.S.; Sorour, A.H. On some geometric properties of quadric surfaces in Euclidean space. *Honam Math. J.* **2016**, *38*, 593–611. [CrossRef]
14. Ali, A.T.; Abdel Aziz, H.S.; Sorour, A.H. On curvatures and points of the translation surfaces in Euclidean 3-space. *J. Egypt. Math. Soc.* **2015**, *23*, 167–172. [CrossRef]
15. Valeontis, I. Paralel p-Aquidistante Regelflachen. *Manuscripta Math.* **1986**, *54*, 391–404. [CrossRef]
16. Masal, M.; Kuruoğlu, N. Some Characteristic Properties of the Shape Operators of Parallel p-Equidistant Ruled Surfaces. *Bull. Pure Appl. Sci. E* **2000**, *19*, 361–364.
17. Masal, M.; Kuruoğlu, N. Some Characteristic properties of the Spherical Indicatrix Leading Curves of Parallel p-Equidistant Ruled Surfaces. *Bull. Pure Appl. Sci. E* **2000**, *19*, 405–410.
18. Özduran, A. Equidistante Regle Yüzeylerin Bazı Yeni Özellikleri. Master's Thesis, Ordu University, Ordu, Turkey, 2019. (In Turkish)
19. Şenyurt, S. Integral Invariants of Parallel P-Equidistant Ruled Surfaces Which are Generated by Instantaneous Pfaff Vector. *Ordu Univ. J. Sci. Tecnol.* **2012**, *2*, 13–22.
20. Sarıaydın, M.T.; Korpınar, V.; Asil, V. On Parallel p-equidistant Ruled Surfaces by Using Modified Orthogonal Frame with Curvature in E^3. *Bol. Soc. Paran. Mat.* **2022**, *40*, 1–7. [CrossRef]
21. As, E.; Şenyurt, S. Some Characteristic Properties of Parallel z-Equidistant Ruled Surfaces. *Math. Probl. Eng.* **2013**, *2013*, 587289. [CrossRef]
22. Li, Y.; Şenyurt, S.; Özduran, A.; Canlı, D. The characterizations of parallel q-equidistant ruled surfaces. *Symmetry* **2022**, *14*, 1879. [CrossRef]
23. Masal, M.; Kuruoğlu, N. Generalized parallel p_i-equidistant ruled surfaces. *Acta Comment. Univ. Tartu. Math.* **2013**, *17*, 3–18.
24. Gür Mazlum, S.; Şenyurt, S.; Grilli, L. The Dual Expression of Parallel Equidistant Ruled Surfaces in Euclidean 3-Space. *Symmetry* **2022**, *14*, 1062. [CrossRef]
25. Masal, M.; Kuruoğlu, N. Timelike parallel p_i-equidistant ruled surfaces by a time-like base curve in the Minkowski 3-space ${R_1}^3$. *Acta Comment. Univ. Tartu. Math.* **2007**, *11*, 1–13.
26. Azak, A.Z.; Tosun, M.; Masal, M. Null parallel p-equidistant b scrolls. *Bol. Soc. Parana.* **2014**, *32*, 23–34. [CrossRef]
27. Masal, M.; Kuruoğlu, N. Timelike parallel equidistant ruled surfaces by a time-like base curve in the Minkowski 3-space ${R_1}^3$. *Acta Comment. Univ. Tartu. Math.* **2009**, *13*, 3–9.
28. Ekici, C.; Dede, M. On the Darboux Vector of Ruled Surfaces in Pseudo-Galilean Space. *Math. Comput. Appl.* **2011**, *16*, 830–838. [CrossRef]
29. Kocayiğit, H.; Öztürk, G.; Bayram, B.K.; Bulca, B.; Arslan, K. Characterization of Curves in E^{2n+1} with 1-Type Darboux Vector. *Math. Morav.* **2013**, *17*, 29–37. [CrossRef]
30. Fenchel, W. On the Differential Geometry of Closed Space Curves. *Bull. Am. Math. Soc.* **1951**, *57*, 44–54. [CrossRef]
31. Thas, C. Minimal Monosystems. *Yokohama Math. J.* **1978**, *26*, 157–167.
32. Zenciroğlu, E. Çizgiler Geometrisinde Meusnier Teoremi ve Sonuçları. Ph.D. Thesis, Anadolu University, Eskişehir, Turkey, 1988. (In Turkish)