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# Different types of soliton solutions for the resonant nonlinear Schrödinger equation with parabolic law nonlinearity via Kumar–Malik approach

Fatma Nur Kaya Sağlam[a], Zabihullah Movaheedi[b], Abdallah M. Talafha[c], Bahadır Kopçasız[d] and Sandeep Malik[e]

[a]Department of Information Security Technology, Faculty of Computer and Information Technologies, Cappadocia University, Nevşehir, Türkiye; [b]Department of Mathematics, Faculty of Science, Herat University, Herat, Afghanistan; [c]Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd university (PMU), Khobar, Saudi Arabia; [d]Department of Basic Sciences, Faculty of Engineering and Architecture, Istanbul Gelisim University, Istanbul, Türkiye; [e]Department of Mathematics, Akal University, Talwandi Sabo, Bathinda, Punjab, 151302, India

**ABSTRACT**
This paper investigates the resonant nonlinear Schrödinger equation (RNSE) with parabolic law nonlinearity, modeling optical pulse propagation in nonlinear optical fibers. By employing the Kumar–Malik approach, we have derived some analytical soliton solutions for the considered equation. These solutions are in the form of Jacobi elliptic, hyperbolic, trigonometric, exponential functions are obtained by this analytical approach. Dark, bright, singular, and periodic wave solitons are created by selecting proper values for the parameters. The new results are compared with previously obtained results. In addition, the physical properties of the presented solutions are represented by 2d, contour and 3d graphs created by selecting appropriate constant parameters. The findings of this study are novel. The acquired results highlight the simplicity, efficacy, and dependability of this method in the analysis of various nonlinear models encountered in the fields of mathematical physics and engineering.

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## 1. Introduction

Mathematical models, which are usually nonlinear partial differential equations (NPDEs), used to various complicated phenomenon in nature, quantum mechanics, optics, control theory, signal processing, plasma physics, electrical chemistry, system identification, image processing, medicine, control systems, and so on (Ala et al., 2020; Ding et al., 2022; Farooq et al., 2025; Izgi et al., 2024; Kopçasız & Yaşar, 2023; Murad et al., 2025; Murad & Omar, 2025; Taşbozan & Kurt, 2023). Therefore, obtaining the exact solutions of the NPDEs are especially important (Khater, 2023; Raissi et al., 2019; Zhou & Yan, 2021). Many authors have studied different models using different methods, such as Riccati-Bernoulli sub-ODE method (Izgi, 2023; Sağlam & Ahmad, 2025), the Kumar–Malik approach (Kumar & Malik, 2024; Sağlam & Malik, 2024), the Kudryashov-expansion method (Alquran et al., 2021), the sine-cose method (Arshed et al., 2018), the simplest equation method (Kudryashov, 2022), the extended $\left(\frac{G'}{G}\right)$-expansion method (Kopçasız et al., 2022), Jacobi elliptic functions approach (Shamseldeen et al., 2017), bifurcation theory (Younas et al., 2025), Adomian's decomposition method (Behera et al., 2022) and the exponential method (Razzaq et al., 2024).

In the study of NPDEs, lump, travelling wave, and soliton solutions are essential tools with wide applications in materials science, optics, and plasma physics. Different types of soliton solutions have been observed in the literature, such as bright solitons, dark solitons, M-W shape solitons. For example, Wang

**CONTACT** Zabihullah Movaheedi z.movaheedi2@gmail.com Department of Mathematics, Faculty of Science, Herat University, Herat, Afghanistan

et al. obtained bright soliton solutions for the $(2+1)$ dimensional general nonlinear Schrödinger equation system using the Hirota technique (Wang et al., 2021). Bekir and Zahran observed dark soliton solutions for the Kundu-Eckhaus equation by means of the method of the extended simple equation (Bekir & Zahran, 2020). Qiao, found M-W soliton solutions for the two-dimensional Euler equation using the approximation method (Qiao, 2007).

Many mathematical equations, including the Sasa-Satsuma model (Murad et al., 2024), the Schrödinger model (Hao et al., 2004), the Kadomtsev-Petviashvili II (KPII) model (Chakravarty & Kodama, 2008), the Bogoyavlenskii-Schiff (BS) model (Yu et al., 1998), and the Korteweg-de Vries model (Hirota & Satsuma, 1981), are known to model the behavior of optical solitons.

In quantum mechanics, a mathematical model known as the Schrödinger equation is used to explain how quantum states behave over time. The diverse range of nonlinear Schrödinger equations (NLSEs) has proven essential in physics, mathematics, biology, and other domains. Natural phenomena such as waves in deep surface water, Langmuir waves in thermal plasma, rough waves in oceans, and light transmission in optical fiber cables can be modeled and explained using them. Furthermore, they have been used in a wide range of fields, including molecular biology, stochastic mechanics, nuclear physics, fluid dynamics, water surfaces, plasma physics, elastic media, and many more (Chakravarty & Kodama, 2008; Ibrahim et al., 2023; Pan et al., 2024; Younas et al., 2020; Yu et al., 1998; Zayed et al., 2008). Researchers introduce RNSE, a new class of nonlinear Schrödinger equations (Zayed et al., 2008). RNSE is used to describe the propagation of an optical pulse in nonlinear optical fibers.

The aim of this paper is to obtain new analytic solutions for the RNSE with parabolic law nonlinearity (Zayed et al., 2008) having the form

$$i\psi_t + a\psi_{xx} + \left(b|\psi|^2 + c|\psi|^4\right)\psi + d\left(\frac{(|\psi|)_{xx}}{|\psi|}\right)u = 0, \quad i = \sqrt{-1}, \tag{1}$$

where $\psi = \psi(x,t)$ is a complex function describing optical pulse propagation in nonlinear optical fibers and represents the normalized complex pulse envelope amplitude in nonlinear optical fibers, as well as $a$, $b$, $c$, and $d$ are non-zero real constants. Different solutions for Eq. (1) are obtained using the extended auxiliary equation method (Zayed & Alurrfi, 2016), the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method (Zayed & Alurrfi, 2015), the new mapping method (Zayed & Al-Nowehy, 2017), modified Kudryashov method (Seadawy et al., 2021), the $\left(m + \frac{G'}{G}\right)$ improved expansion method (Gao et al., 2019), modified simple equation method (Ali et al., 2017), the new extended direct algebraic method (Vahidi et al., 2021) and the $\phi^6$ model expansion method (Zayed et al., 2008) and so on.

In the present research, we investigate Eq. (1) using the Kumar–Malik approach. This method has not been examined by other researchers, according to a study of earlier research on RNSE with a parabolic law nonlinearity model. Additionally, this method is not used to derive the soliton solutions reported in this study, and the effect of parameters has not been investigated in the literature to date. This highlights a significant gap in the literature. By presenting this approach and providing a comprehensive model analysis, our work fills this gap. In the field of optics, these approaches are very effective and powerful for obtaining soliton solutions to NPDEs. We obtains a variety of optical soliton patterns, such as bright and dark.

The paper is divided into the following sections: Section 2 describes the method to be used. We obtained soliton solutions of the mentioned equation in Section 3. The graphical representation of these calculations and the results to provide a physical understanding of the results and some discussions are presented in Section 4. Finally, Section 5 presents the concluding remarks of the study.

## 2. The Kumar–Malik approach

Assume that we have a NPDE as follows:

$$P(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{xt}, \psi_{tt}, \ldots) = 0, \tag{2}$$

where $u = u(x,t)$ is function of $x$ and $t$ and containing its partial derivatives.

We summarized the essential components of Kumar–Malik approach (Kumar & Malik, 2024; Sağlam & Malik, 2024).

**Phase 1.** We use the following transformation

$$\psi(x,t) = U(\xi), \quad \xi = (x - \nu t), \tag{3}$$

where $\nu$ represents the wave speed constant. Inserting Eq. (3) into Eq. (2) gives an ODE as

$$O(U, U', U'', U''', \ldots) = 0. \tag{4}$$

**Phase 2.** Suppose that Eq. (4) has the following solution as

$$U(\xi) = A_0 + A_1\phi(\xi) + A_2\phi(\xi)^2 + \ldots + A_N\phi(\xi)^N, \tag{5}$$

where the constants $A_i$'s (i = 1, 2, ..., N) and the function $\Phi(\xi)$ accomplishes the first order differential equation

$$[\phi'(\xi)]^2 = \left[\beta_1\phi(\xi)^4 + \beta_2\phi(\xi)^3 + \beta_3\phi(\xi)^2 + \beta_4\phi(\xi) + \beta_5\right]. \tag{6}$$

Here $\beta_i'$s, $(i = 1, 2, ..., 5)$ are arbitrary constants. In Subsection 2.1 the use of Eq. (6) was given.

**Phase 3.** Using the balance principle in Eq. (4), the value of $N$ is determined.

**Phase 4.** The polynomial of $\phi(\xi)\phi'(\xi)$ is obtained by inserting Eq. (5) and its derivatives according to Eq. (6) into Eq. (4). Collecting all coefficients of different powers and setting them equal to zero gives an algebraic system in the u nknown parameters $\nu, A_i$'s (i = 1, 2, ..., N), $\beta_j$(j = 1, 2, ..., 5). The solutions of Eq. (4) are achieved by solving the obtained system.

**Phase 5.** Finally, using the solutions of Eq. (4) and the transformation Eq. (3), we can construct the analytic solutions of the NPDE Eq. (2).

### *2.1. Solutions of Eq. (6)*

Considering the following four different cases, we offer the analytic solutions of Eq. (6). The following notations are used throughout the manuscript.

$$s_1 = (4\beta_1\beta_3 - \beta_2^2), \quad s_2 = (16\beta_1\beta_3 - 5\beta_2^2), \quad s_3 = (8\beta_1\beta_3 - 3\beta_2^2). \tag{7}$$

**Family 1.** If $\beta_4 = \frac{\beta_2 s_1}{8\beta_1^2}, \quad \beta_5 = 0$, then we get the Jacobi elliptic solutions of the Eq. (6) as follows:

Sub-family 1.1. When $\beta_1 < 0, \quad s_1 > 0$, then

$$\phi_{01}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\beta_2}{4\beta_1}\ \mathrm{cn}\left(\frac{\sqrt{-\beta_1 s_1}}{2\beta_1}\ \xi, \frac{\beta_2}{2\sqrt{s_1}}\right), \tag{8}$$

$$\phi_{02}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\beta_2}{4\beta_1}\ \mathrm{dn}\left(\frac{\beta_2}{4\sqrt{-\beta_1}}\ \xi, \frac{2\sqrt{s_1}}{\beta_2}\right). \tag{9}$$

Sub-family 1.2. When $\beta_1 < 0, \quad s_1 < 0, \quad s_2 < 0$, then

$$\phi_{03}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_2}}{4\beta_1}\ \mathrm{cn}\left(\frac{\sqrt{\beta_1 s_1}}{2\beta_1}\ \xi, \frac{\sqrt{s_1 s_2}}{2s_1}\right), \tag{10}$$

$$\phi_{04}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_2}}{4\beta_1}\ \mathrm{dn}\left(\frac{\sqrt{\beta_1 s_2}}{4\beta_1}\ \xi, \frac{2\sqrt{s_1 s_2}}{s_2}\right). \tag{11}$$

Sub-family 1.3. When $\beta_1 < 0, \quad s_1 > 0$ and $s_2 < 0$, then

$$\phi_{05}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_2}}{4\beta_1}\ \mathrm{nc}\left(\frac{\sqrt{-\beta_1 s_1}}{2\beta_1}\ \xi, \frac{\beta_2}{2\sqrt{s_1}}\right), \tag{12}$$

$$\phi_{06}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_2}}{4\beta_1}\ \mathrm{nd}\left(\frac{\beta_2}{4\sqrt{-\beta_1}}\ \xi, \frac{2\sqrt{s_1}}{\beta_2}\right). \tag{13}$$

Sub-family 1.4. When $\beta_1 s_1 > 0$ and $s_1 s_2 > 0$, then

$$\phi_{07}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\beta_2}{4\beta_1}\ \mathrm{nc}\left(\frac{\sqrt{\beta_1 s_1}}{2\beta_1}\ \xi, \frac{\sqrt{s_1 s_2}}{2s_1}\right), \tag{14}$$

$$\phi_{08}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\beta_2}{4\beta_1}\ \mathrm{nd}\left(\frac{\sqrt{\beta_1 s_2}}{4\beta_1}\ \xi, \frac{2\sqrt{s_1 s_2}}{s_2}\right). \tag{15}$$

Sub-family 1.5. When $\beta_1 > 0, \quad s_2 < 0$, then

$$\phi_{09}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\beta_2}{4\beta_1}\ \mathrm{ns}\left(\frac{\beta_2}{4\sqrt{\beta_1}}\ \xi, \frac{\sqrt{-s_2}}{\beta_2}\right), \tag{16}$$

$$\phi_{10}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_2}}{4\beta_1}\ \mathrm{ns}\left(\frac{\sqrt{-\beta_1 s_2}}{4\beta_1}\ \xi, \frac{\beta_2}{\sqrt{-s_2}}\right), \tag{17}$$

$$\phi_{11}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_2}}{4\beta_1}\ \mathrm{sn}\left(\frac{\beta_2}{4\sqrt{\beta_1}}\ \xi, \frac{\sqrt{-s_2}}{\beta_2}\right), \tag{18}$$

$$\phi_{12}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\beta_2}{4\beta_1}\ \mathrm{sn}\left(\frac{\sqrt{-\beta_1 s_2}}{4\beta_1}\ \xi, \frac{\beta_2}{\sqrt{-s_2}}\right). \tag{19}$$

**Family 2.** If $\beta_4 = \frac{\beta_2 s_1}{8\beta_1^2}, \quad \beta_5 = \frac{s_1^2}{64\beta_1^3}$, then we reach the hyperbolic and trigonometric solutions of the Eq. (6) as:

Sub-family 2.1. When $\beta_1 > 0, \quad s_3 < 0$, then

$$\phi_{13}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_3}}{4\beta_1}\ \tanh\left(\frac{\sqrt{-\beta_1 s_3}}{4\beta_1}\ \xi\right), \tag{20}$$

$$\phi_{14}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-s_3}}{4\beta_1}\ \coth\left(\frac{\sqrt{-\beta_1 s_3}}{4\beta_1}\ \xi\right). \tag{21}$$

Sub-family 2.2. When $\beta_1 > 0, \quad s_3 > 0$, then

$$\phi_{15}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{s_3}}{4\beta_1}\ \tan\left(\frac{\sqrt{\beta_1 s_3}}{4\beta_1}\ \xi\right), \tag{22}$$

$$\phi_{16}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{s_3}}{4\beta_1}\ \cot\left(\frac{\sqrt{\beta_1 s_3}}{4\beta_1}\ \xi\right). \tag{23}$$

**Family 3.** If $\beta_4 = \frac{\beta_2 s_1}{8\beta_1^2}, \quad \beta_5 = \frac{\beta_2^2 s_2}{256\beta_1^3}$, then we obtain hyperbolic and trigonometric solutions of the Eq. (6) as follows:

Sub-family 3.1. When $\beta_1 < 0, \quad s_3 < 0$, then

$$\phi_{17}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-2s_3}}{4\beta_1}\ \mathrm{sech}\left(\frac{\sqrt{2\beta_1 s_3}}{4\beta_1}\ \xi\right). \tag{24}$$

Sub-family 3.2. When $\beta_1 > 0, \quad s_3 > 0$, then

$$\phi_{18}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{2s_3}}{4\beta_1}\ \mathrm{csch}\left(\frac{\sqrt{2\beta_1 s_3}}{4\beta_1}\ \xi\right). \tag{25}$$

Sub-family 3.3. When $\beta_1 > 0, \quad s_3 < 0$, then

$$\phi_{19}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-2s_3}}{4\beta_1}\ \sec\left(\frac{\sqrt{-2\beta_1 s_3}}{4\beta_1}\ \xi\right), \tag{26}$$

$$\phi_{20}(\xi) = -\frac{\beta_2}{4\beta_1} \pm \frac{\sqrt{-2s_3}}{4\beta_1} \csc\left(\frac{\sqrt{-2\beta_1 s_3}}{4\beta_1}\xi\right). \tag{27}$$

**Family 4.** If $\beta_2 = \beta_4 = \beta_5 = 0, \quad \beta_3 > 0$, then we get solution of the Eq. (6) in the following form

$$\phi_{21}(\xi) = \frac{4\rho\beta_3}{\left(4\rho^2 \ e^{\sqrt{\beta_3}\xi} - \beta_1\beta_3 \ e^{-\sqrt{\beta_3}\xi}\right)}. \tag{28}$$

## 3. Soliton solutions for Eq. (1)

Let us consider the following complex wave transformation,

$$\psi(x,t) = U(\xi) \times \exp(i(-\kappa x + \omega t)), \quad \xi = x - \sigma t, \quad \sigma \neq 0. \tag{29}$$

Here $\kappa$ and $\omega$ are constants. Plugging Eq. (29) into Eq. (1) and separating the imaginary part from the real part, we get the following equation

$$\sigma = -2a\kappa, \tag{30}$$

$$(a+d)U'' - (a\kappa^2 + \omega)U + bU^3 + cU^5 = 0, \tag{31}$$

where $('')$ is a second order derivative. Using the principle of balancing between the terms $U^5$ and $U''$, we obtain $N = \frac{1}{2}$. So, we get the solution of Eq. (31),

$$U(\xi) = \sqrt{V(\xi)}, \tag{32}$$

and $V(\xi)$ is a function of $\xi$. Inserting Eq. (32) into Eq. (31) then we gain,

$$\frac{1}{4}(a+d)(2V(\xi)V''(\xi) - V'^2(\xi) - (a\kappa^2+\omega)V^2(\xi) + bV^3(\xi) + cV^4(\xi). \tag{33}$$

Balancing $V(\xi)V''(\xi)$ with $V^4(\xi)$ in Eq. (33), we get $N = 1$. Therefore, Eq. (33) has the following solution:

$$V(\xi) = A_0 + A_1\phi(\xi) \tag{34}$$

The following algebraic system is obtained by inserting Eq. (34) into Eq. (33),

$$\begin{cases} \phi(\xi)^4 : \dfrac{3(a+d)A_1^2\alpha_1}{4} + cA_1^4 = 0, \\ \phi(\xi)^3 : \dfrac{(a+d)\left(4A_0A_1\beta_1 + 2A_1^2\beta_2\right)}{4} + bA_1^3 + 4cA_0A_1^3 = 0, \\ \phi(\xi)^2 : \dfrac{(a+d)\left(3A_0A_1\beta_2 + A_1^2\beta_3\right)}{4} - (a\kappa^2+\omega)A_1^2 + 3bA_0A_1^2 + 6cA_0^2A_1^2 = 0, \\ \phi(\xi)^1 : \dfrac{(a+d)A_0A_1\beta_3}{2} - 2(a\kappa^2+\omega)A_0A_1 + 3bA_0^2A_1 + 4cA_0^3A_1 = 0, \\ \phi(\xi)^0 : \dfrac{(a+d)\left(A_0A_1\beta_4 - A_1^2\beta_5\right)}{4} - (a\kappa^2+\omega)A_0^2 + bA_0^3 + cA_0^4 = 0, \end{cases} \tag{35}$$

By solving the above algebraic system of equations, the following sets of solutions are obtained for each case.

**Family 1.** Given $\beta_4 = \frac{\beta_2(4\beta_1\beta_3 - \beta_2^2)}{(8\beta_1^2)}, \quad \beta_5 = 0$, then we get the following sets of solutions:

Set 1

$$A_0 = 0, \quad A_1 = \frac{3b\beta_1}{2\beta_2 c}, \quad \kappa = \frac{\sqrt{-\frac{3b^2\beta_3\beta_1 + 4\omega c\beta_2^2}{4ac}}}{\beta_2}, \quad d = \frac{ac\beta_2^2 + 3b^2\beta_1}{c\beta_2^2}. \tag{36}$$

Application of the methodology, the Jacobi elliptic solutions of Eq. (1) are as follows:

Sub-family 1.1. If $\beta_1 < 0, \quad s_1 > 0$, then

$$\psi_1(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{cn}\left(\frac{\sqrt{-\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\alpha_2}{2\sqrt{s_1}}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{37}$$

$$\psi_2(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\ \mathrm{dn}\left(\frac{\beta_2}{4\sqrt{-\beta_1}}(x-\sigma t), \frac{2\sqrt{s_1}}{\beta_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{38}$$

Sub-family 1.2. If $\beta_1 < 0, \quad s_1 < 0, \quad s_2 < 0,$ then

$$\psi_3(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{\beta_2}\mathrm{cn}\left(\frac{\sqrt{\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\sqrt{s_1 s_2}}{2s_1}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{39}$$

$$\psi_4(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{\beta_2}\mathrm{dn}\left(\frac{\sqrt{\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{2\sqrt{s_1 s_2}}{s_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{40}$$

Sub-family 1.3. If $\beta_1 < 0, \quad s_1 > 0, \quad s_2 < 0$, then

$$\psi_5(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{\beta_2}\mathrm{nc}\left(\frac{\sqrt{-\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\alpha_2}{2\sqrt{s_1}}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{41}$$

$$\psi_6(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{\beta_2}\mathrm{nd}\left(\frac{\beta_2}{4\sqrt{-\beta_1}}(x-\sigma t), \frac{2\sqrt{s_1}}{\beta_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{42}$$

Sub-family 1.4. If $\beta_1 s_1 > 0, \quad s_1 s_2 > 0$, then

$$\psi_7(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{nc}\left(\frac{\sqrt{\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\sqrt{s_1 s_2}}{2s_1}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{43}$$

$$\psi_8(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{nd}\left(\frac{\sqrt{\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{2\sqrt{s_1 s_2}}{s_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{44}$$

Sub-family 1.5. If $\beta_1 > 0, \quad s_2 < 0$, then

$$\psi_9(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{ns}\left(\frac{\beta_2}{4\sqrt{\beta_1}}(x-\sigma t), \frac{\sqrt{-s_2}}{\beta_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{45}$$

$$\psi_{10}(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{\beta_2}\mathrm{ns}\left(\frac{\sqrt{-\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{\beta_2}{\sqrt{-s_2}}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{46}$$

$$\psi_{11}(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{\beta_2}\mathrm{sn}\left(\frac{\beta_2}{4\sqrt{\beta_1}}(x-\sigma t), \frac{\sqrt{-s_2}}{\beta_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{47}$$

$$\psi_{12}(x,t) = \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{sn}\left(\frac{\sqrt{-\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{\beta_2}{\sqrt{-s_2}}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{48}$$

Set 2.

$$\begin{aligned} A_0 &= \frac{3b\left(-s_2 + \sqrt{-s_2\beta_2^2}\right)}{8cs_2}, \quad A_1 = \frac{3b\beta_1\left(\frac{\left(-s_2+\sqrt{-s_2\beta_2^2}\right)}{s_2} + 1\right)}{2\beta_2 c}, \\ \kappa &= \sqrt{-\frac{30b^2\beta_3\beta_1 - 9b^2\beta_2^2 + 8c\omega s_2}{8acs_2}}, \quad d = -\frac{acs_2 - 3b^2\beta_1}{cs_2}. \end{aligned} \tag{49}$$

With the parameters given above, the Jacobi elliptic solutions to the Eq. (1) are given as follows:
Sub-family 1.1. If $\beta_1 < 0, \quad s_1 > 0$, then

$$\psi_{13}(x,t) = -\frac{3b}{8c}\left(1 + \mathrm{cn}\left(\frac{\sqrt{-\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\beta_2}{2\sqrt{s_1}}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{50}$$

$$\psi_{14}(x,t) = -\frac{3b}{8c}\left(1 + \mathrm{dn}\left(\frac{\beta_2}{4\sqrt{-\beta_1}}(x-\sigma t), \frac{2\sqrt{s_1}}{\beta_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{51}$$

Sub-family 1.2. If $\beta_1 < 0, \quad s_1 < 0, \quad s_2 < 0$, then

$$\psi_{15}(x,t) = -\frac{3b}{8c\beta_2}\left(\beta_2 + \sqrt{-s_2}\mathrm{cn}\left(\frac{\sqrt{\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\sqrt{s_1 s_2}}{2s_1}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{52}$$

$$\psi_{16}(x,t) = -\frac{3b}{8c\beta_2}\left(\beta_2 + \sqrt{-s_2}\mathrm{dn}\left(\frac{\sqrt{\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{2\sqrt{s_1 s_2}}{s_2}\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{53}$$

Sub-family 1.3. If $\beta_1 < 0, \quad s_1 > 0, \quad s_2 < 0$, then

$$\psi_{17}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{4\beta_1}\mathrm{nc}\left(\frac{\sqrt{-\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\beta_2}{2\sqrt{s_1}}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{54}$$

$$\psi_{18}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{4\beta_1}\mathrm{nd}\left(\frac{\beta_2}{4\sqrt{-\beta_1}}(x-\sigma t), \frac{2\sqrt{s_1}}{\beta_2}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{55}$$

Sub-family 1.4. If $\beta_1 s_1 > 0, \quad s_1 s_2 > 0$, then

$$\psi_{19}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{nc}\left(\frac{\sqrt{\beta_1 s_1}}{2\beta_1}(x-\sigma t), \frac{\sqrt{s_1 s_2}}{2s_2}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{56}$$

$$\psi_{20}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{nd}\left(\frac{\sqrt{\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{2\sqrt{s_1 s_2}}{s_2}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{57}$$

Sub-family 1.5. If $\beta_1 > 0, \quad s_2 < 0$, then

$$\psi_{21}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{ns}\left(\frac{\beta_2}{4\sqrt{\beta_1}}(x-\sigma t), \frac{\sqrt{-s_2}}{\beta_2}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{58}$$

$$\psi_{22}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{4\beta_1}\mathrm{ns}\left(\frac{\sqrt{-\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{\beta_2}{\sqrt{-s_2}}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{59}$$

$$\psi_{23}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\sqrt{-s_2}}{4\beta_1}\mathrm{sn}\left(\frac{\beta_2}{4\sqrt{\beta_1}}(x-\sigma t), \frac{\sqrt{-s_2}}{\beta_2}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{60}$$

$$\psi_{24}(x,t) = \left(-\frac{3b}{4c} - \frac{3b\beta_1}{2c\beta_2}\left(-\frac{\beta_2}{4\beta_1} + \frac{\beta_2}{4\beta_1}\mathrm{sn}\left(\frac{\sqrt{-\beta_1 s_2}}{4\beta_1}(x-\sigma t), \frac{\beta_2}{\sqrt{-s_2}}\right)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{61}$$

The parameters $s_1$, $s_2$, and $s_3$ are given by Eq. (7).

**Remark:** By substituting the following parameters into the equation, different solutions obtained:

$$A_0 = -\frac{3b}{4c}, \quad A_1 = -\frac{3b\beta_1}{2\beta_2 c}, \quad \kappa = \frac{\sqrt{-\frac{3b^2\beta_3\beta_1+4\omega c\beta_2^2}{4ac}}}{\beta_2}, \quad d = \frac{ac\beta_2^2+3b^2\beta_1}{c\beta_2^2}. \tag{62}$$

**Family 2.** Given $\beta_4 = \frac{\beta_2 s_1}{8\beta_1^2}$, $\beta_5 = \frac{{s_1}^2}{64\beta_1^3}$, then we obtained the following set:

$$A_0 = \frac{3b\left(-s_3+\sqrt{-s_3\beta_2^2}\right)}{8cs_3}, \quad A_1 = \frac{3b\beta_1\left(\frac{\left(-s_3+\sqrt{-s_3\beta_2^2}\right)}{s_3}+1\right)}{2\beta_2 c},$$
$$\kappa = \sqrt{-\frac{3b^2+16c\omega}{16ac}}, \quad d = -\frac{acs_3-3b^2\beta_1}{cs_3}. \tag{63}$$

Using the method's procedure, the following hyperbolic and trigonometric solutions exist for Eq. (1):
Sub-family 2.1. If $\beta_1 > 0$ and $s_3 < 0$, then

$$\psi_{25}(x,t) = -\frac{3b}{8c}\left(1+\tanh\left(\frac{\sqrt{-\beta_1 s_3}}{4\beta_1}(x-\sigma t)\right)\right) \times \exp\left(i(-\kappa x+\omega t)\right), \tag{64}$$

$$\psi_{26}(x,t) = -\frac{3b}{8c}\left(1+\coth\left(\frac{\sqrt{-\beta_1 s_3}}{4\beta_1}(x-\sigma t)\right)\right) \times \exp\left(i(-\kappa x+\omega t)\right). \tag{65}$$

*Sub-family 2.2.* If $\beta_1 > 0$ and $s_3 > 0$, then

$$\psi_{27}(x,t) = -\frac{3b}{8c}\left(1+\tan\left(\frac{\sqrt{\beta_1 s_3}}{4\beta_1}(x-\sigma t)\right)\right) \times \exp\left(i(-\kappa x+\omega t)\right), \tag{66}$$

$$\psi_{28}(x,t) = -\frac{3b}{8c}\left(1+\cot\left(\frac{\sqrt{\beta_1 s_3}}{4\beta_1}(x-\sigma t)\right)\right) \times \exp\left(i(-\kappa x+\omega t)\right). \tag{67}$$

In the solutions Eqs. (64)–(67) the soliton velocity is given by Eq. (63) while the parameters $s_1$, $s_2$, and $s_3$ are represented by Eq. (7).

**Family 3.** Given $\beta_4 = \frac{\beta_2 s_1}{8\beta_1^2}$, $\beta_5 = \frac{s_2{\beta_2}^2}{256\beta_1^3}$ and applying the method we have the solutions set as follows:

$$A_0 = \frac{3b\left(-2s_3+\sqrt{-2s_3\beta_2^2}\right)}{16cs_3}, \quad A_1 = \frac{3b\beta_1\left(\frac{\left(-2s_3+\sqrt{-2s_3\beta_2^2}\right)}{2s_3}+1\right)}{2\beta_2 c},$$
$$\kappa = \sqrt{-\frac{15b^2+64c\omega}{64ac}}, \quad d = -\frac{2acs_3-3b^2\beta_1}{2cs_3}. \tag{68}$$

With above parameters, the hyperbolic and trigonometric solutions to Eq. (1) are given as follows:
Sub-family 3.1. If $\beta_1 < 0$ and $s_3 < 0$, then

$$\psi_{29}(x,t) = -\frac{3b}{8c}\left(1+\operatorname{sech}\left(\frac{\sqrt{2\beta_1 s_3}}{4\beta_1}(x-\sigma t)\right)\right) \times \exp\left(i(-\kappa x+\omega t)\right). \tag{69}$$

Sub-family 3.2. If $\beta_1 > 0$ and $s_3 > 0$, then

$$\psi_{30}(x,t) = -\frac{3b}{8c}\left(1+\operatorname{csch}\left(\frac{\sqrt{2\beta_1 s_3}}{4\beta_1}(x-\sigma t)\right)\right) \times \exp\left(i(-\kappa x+\omega t)\right). \tag{70}$$

Sub-family 3.3. If $\beta_1 > 0$ and $s_3 < 0$, then

$$\psi_{31}(x,t) = -\frac{3b}{8c}\left(1 + \sec\left(\frac{\sqrt{-2\beta_1 s_3}}{4\beta_1}(x - \sigma t)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right), \tag{71}$$

$$\psi_{32}(x,t) = -\frac{3b}{8c}\left(1 + \csc\left(\frac{\sqrt{-2\beta_1 s_3}}{4\beta_1}(x - \sigma t)\right)\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{72}$$

For solutions Eqs. (69)–(72), the soliton velocity is given by Eq. (68) while the parameters $s_1$, $s_2$, and $s_3$ are represented by Eq. (7).

**Family 4.** Given $\beta_2 = 0, \quad \beta_4 = 0, \quad \beta_5 = 0$ and then we achieve

$$A_0 = -\frac{3b}{8c}, \quad A_1 = \frac{3b\sqrt{-\frac{\beta_1}{64\beta_3}}}{c}, \quad \kappa = \sqrt{-\frac{15b^2 + 64c\omega}{64ac}}, \quad d = -\frac{16ac\beta_3 - 3b^2}{16c\beta_3}. \tag{73}$$

Adhering the method, we have the following solution of the Eq. (1):

$$\psi_{33}(x,t) = \left(-\frac{3b}{8c} + \frac{3b\rho\beta_3\sqrt{-\frac{\beta_1}{\beta_3}}}{2c\left(4\rho^2 \exp\left(\sqrt{\beta_3}(x - \sigma t)\right) - \beta_1\beta_3 \exp\left(-\sqrt{\beta_3}(x - \sigma t)\right)\right)}\right) \times \exp\left(i(-\kappa x + \omega t)\right). \tag{74}$$

For solutions Eq. (74) the soliton velocity is given by Eq. (73) while the parameters $s_1$, $s_2$, and $s_3$ are described by Eq. (7).

## 4. Graphical representation and discussion

This section displays a graph of a few soliton solutions that have been obtained and express various behaviors for different values of the unknown constants. In Zayed et al. have discovered different type soliton solutions for RNLSE with parabolic law nonlinearity using the new $\phi^6$-model expansion method (Zayed et al., 2008). However, the RNSE with the parabolic law nonlinearity has been applied to the Kumar–Malik approach in this study. We obtained the solutions in the periodic, bright, anti-kink, and dark types, and we conducted a graphic analysis on them.

Figure 1 shows the solution to nonlinear wave equations with amplitude and phase periodic waves. They are defined by the frequency and amplitude modulation that is controlled by the system parameters. Periodic solitons are features of many physical systems, such as optical materials, photonic waveguides, and nonlinear lattices.

Figure 2 anti-kink soliton solutions represent a structure in systems described by nonlinear differential equations, in particular of the opposite sign of kink solitons. These solutions are often used to understand topological and phase transition properties of systems. For example, anti-kink solitons arising in sine-Gordon equations or nonlinear Schrödinger equations can be important for modeling a phase

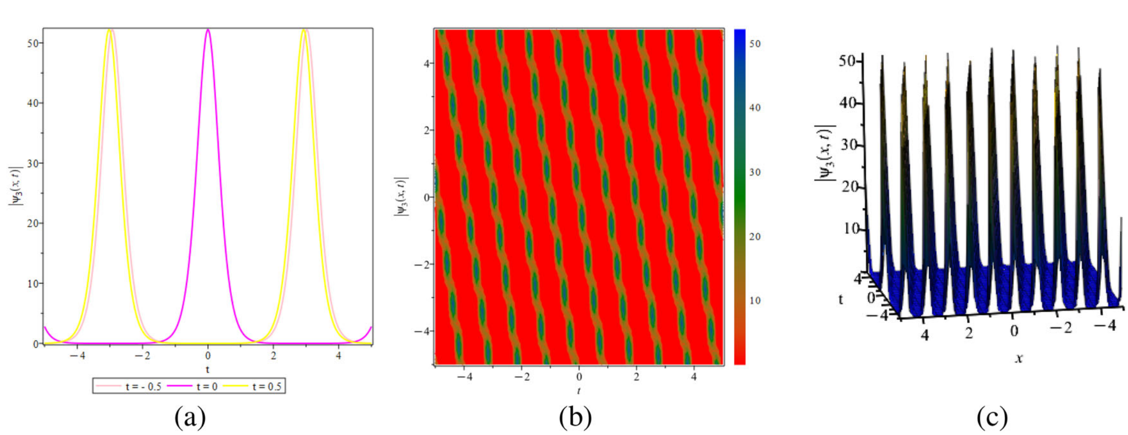


Figure 1. View of travelling waves Eq. (39) for $a = 2,\ \beta_1 = -3,\ \beta_2 = 0.08,\ \beta_3 = 5,\ \omega = 0.7,\ b = 0.03,\ c = 0.3$.

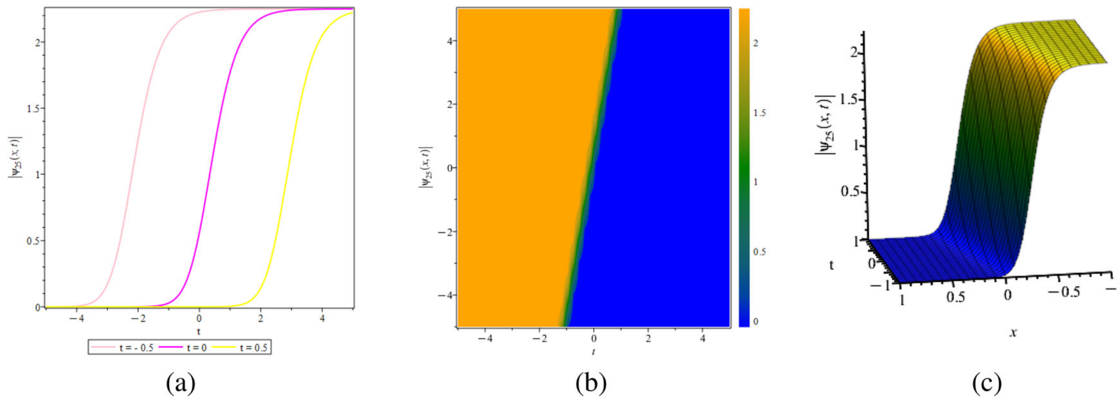


**Figure 2.** 2d, contour and 3d plots anti-kink soliton solution of $|\psi_{25}(x,t)|$ with $b = 4,\ \beta_1 = 3,\ \beta_2 = 1,\ \beta_3 = -2,$ $a = -2,\ \rho = 0.7,\ c = 2,\ \omega = 1.7.$

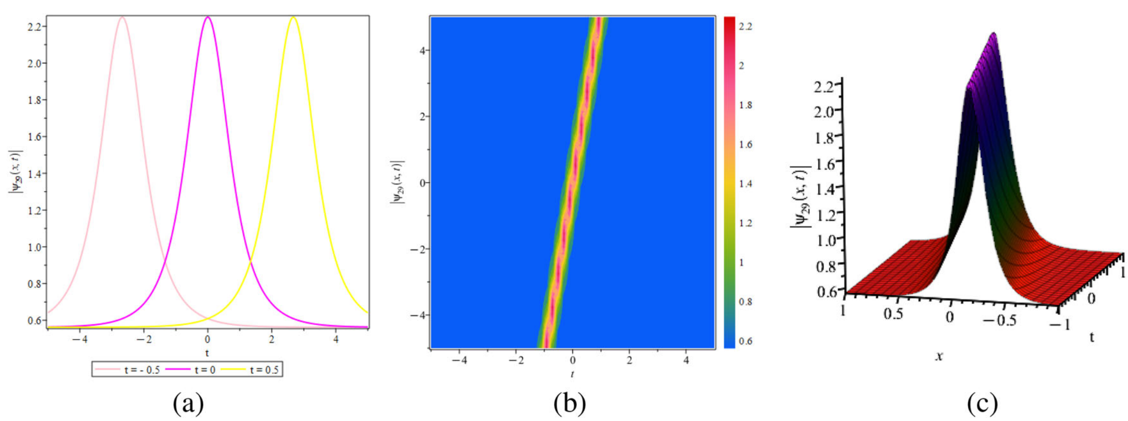


**Figure 3.** Bright soliton solutions in the Eq. (65) with $a = -2,\ \beta_1 = -3,\ \beta_2 = 1,\ \beta_3 = 2,\ b = 4,\ c = 2,\ \omega = 1.7.$

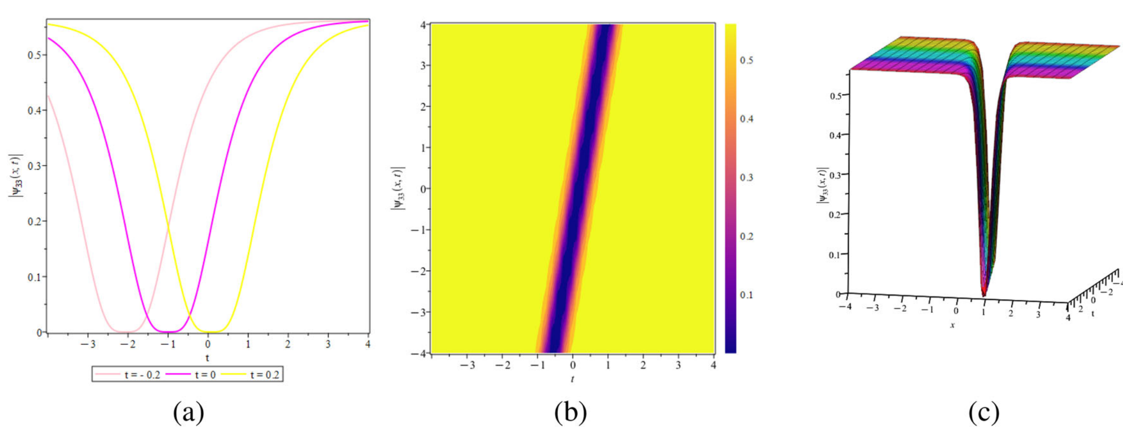


**Figure 4.** Dark soliton solutions in $|\psi_{33}(x,t)|$ with $a = -2,\ \beta_1 = -3,\ \beta_3 = 2,\ b = 4,\ c = 2,\ \omega = 1.7,\ \rho = 5.$

transition in physical systems in the opposite direction. These solutions play a critical role in understanding the dynamics and structure of systems ranging from quantum field theory to solid state physics.

Figure 3 bright soliton solutions are solutions to finite energy nonlinear wave equations exhibiting localized behavior. They are identified by their phase change, peak amplitude, and propagation velocity. Bright solitons are found in a variety of physical systems, such as optical materials.

Figure 4 a wave type known as a dark type soliton has a localized intensity notch or dip. It is identified by high amplitude regions enclosing low amplitude regions. Dark solitons are commonly encountered in nonlinear optics and Bose-Einstein condensates. They are also present in many other physical systems, such as water waves and plasma physics.

Solitons are generally remarkable occurrences that have been studied for a long time in various fields of physics and engineering. They are also widely used in optical metamaterials, signal processing, photonic waveguides, ultrafast lasers, data transfer, and quantum computing.

## 5. Conclusion

This work has investigated optical soliton for the RNLSE with power law of nonlinearity in metamaterials or fiber. The Kumar–Malik approach has been used to obtain numerous solutions of optical solitons. These solutions are expressed of Jacobi elliptic, trigonometric, exponential, and hyperbolic functions. Numerous exact wave solutions, such as periodic Figure 1 and anti-kink Figure 2 and bright Figure 3 and bright Figure 4, have been derived. These findings are potentially useful in the fields of materials science, photonic waveguide, optical laser, and ultrafast laser. They also have an important impact on our knowledge and understanding of electromagnetic wave propagation in metamaterials. With a power law of nonlinearity, the discovered solutions for the RNLSE are completely new and have not been published in any previous study. With the aid of Maple, we verified the results by substituting them back into the given model. Future research will examine the RNLSE with power law nonlinearity's more exact solution form.

## Authors contribution

All authors contributed equally in the preparation, drafting, editing, and reviewing of the manuscript.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

## Data availability statement

No data was used for the research described in the article.

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