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ORIGINAL RESEARCH

On the solution structure of fractional integro-differential equations via a new (ω, φ)-Caputo derivative with the modified generalized Mittag-Leffler kernel

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Abstract
This article extends new left-sided (L.S) fractional derivatives and integrals to higher orders with respect to another (wrta) function φ, which involves a weighted function ω and the modified generalized Mittag-Leffler function (GMLF) in the context of Riemann-Liouville (RL) and Caputo operators. Additionally, several operators of non-singular kernels are deduced as special cases of the study. Also, some important properties of the proposed operators are proved. Building on the new general Q-operator of a function wrta function φ, we introduce its reciprocal operator and present the general fundamental relations between the left and right (L&R) sides to any (ω, φ)-fractional integrals and derivatives. As a consequence, we have derived the right-sided (R.S) definitions of our proposed operators, along with their characteristics. Furthermore, fixed point theorems (FPTs) of the Banach and Schaefer are utilized to examine the qualitative results of solutions for (ω, φ)-Caputo fractional integro-differential equations (IDEs). Finally, two mathematical applications are demonstrated to confirm the effectiveness of our outcomes.

Keywords (ω, φ)-fractional derivative · Generalized Mittag-Leffler function · General Q-operator · Existence and uniqueness · Fixed point theorem · Fractional integro-differential equation

AMS Subject Classification: 34A08 · 34B15

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1 Introduction

The concept of fractional calculus is the generalization of derivatives and integrals of non-integer orders, and it dates back to L'Hopital, who raised the question about the derivative of order $\frac{1}{2}$ [1–3]. After that, a lot of studies in these fields were conducted by more authors; we refer the researcher to [4, 5]. The non-local fractional derivatives categories are two types depending on the kernel kind: a singular kernel, such as RL [1], Caputo [1], Hadamard [6], and Hilfer [4] derivatives, and a nonsingular kernel type, such as Caputo-Fabrizio (CF) with exponential kernel [7], and Atangana-Baleanu (AB) [8]; for their applications, we indicate the readers these works [9–15].

Recently, the scope of fractional operators has been expanded, including tempered and weighted fractional operators, see [16, 17]. Furthermore, the fractional operator wrta function is an important class of fractional operators, and it was discussed in [18]. Also, another important aspect of the fractional operator is a weighted fractional calculus wrta function, it is appeared for the first time in 2012 by Agrawal [19, 20]. He derived general class of operators called the weighted fractional calculus wrta function. After that, the new operator was generalized to differointegration wrta function in 2018 by Fernandez and Baleanu [21], and it was extended to the higher fractional order by Abdeljawad et al. [22].

These (ω, φ)-fractional derivatives/integrals are applied practically since they enable the researcher or model developer to select the kernel of memory as well as the time or space scale that better suits real data phenomena than power-law kernels. Moreover, they provide a framework that could unify many other classic and recent fractional derivatives/integrals as special forms, so that a universal theory could tackle various demands in applications [23, 24]. These fractional operators with a kernel of any type could be applied in the domains where the classical models of ordinary or partial differential equations seem to erroneously capture real phenomena, such as biology, transport phenomena, climate phenomena, or any other type of complex systems that depend on memory. From the point of analysis of applications, the formulations with any kernel are applied for formulating, analysis, as well as numerical solutions for fractional ordinary differential equations [25] and references cited therein.

In the literature, several research papers have been proposed to explain the existence and uniqueness (EaU) results of the solutions by using FPTs, and they have been applied in many different scopes such as geometry, topology, engineering, analysis, and physics. On the other hand, FPT is a powerful technique to investigate the EaU of the solution of nonlinear problems and systems. Particularly, in 2019, Al-Refai and Jarrah [26] established a unique solution to a nonlinear fractional initial problem via the weighted Caputo-Fabrizio fractional derivative. Gudade et al. [27] proved the EaU of a non-negative solution for non-linear differential problem via the fractional AB operator. In 2023, Thabet et al. [28] investigated the EaU results of the $k-$tempered fractional implicit differential problem wrta function by utilizing the Banach FPT. Furthermore, Khan et al. [6] discussed the sufficient criteria for the existence of a solution of a modified AB fractional hybrid system. Zahra et al. [29] presented appropriate constraints for the EaU of a solution to an initial value problem involving weighted Caputo derivatives wrta function in a Sobolev space. The Banach FPT and a successive approximation method used by Othmane et al. [25] to show the EaU of the solution to a Volterra integral problem containing the weighted RL fractional derivative wrta function. Benia et al. [30] studied a weighted fractional boundary value problem of variable order wrta function. Authors in [31] investigated the sufficient criteria that confirm the solutions to an (ω, ψ)-fractional impulsive differential inclusions in an infinite dimensional space.

Many researchers are developing new definitions of fractional derivatives based on the complexity of application forces. In 2016, Caputo and Fabrizio introduced a nonsingular fractional operator with exponential kernel [7]. After that, Atangana and Baleanu derived another nonsingular fractional operator involving MLF kernel [8]. In 2018, the operator involving the GMLF with three parameters was derived by [32, 33]. Then, authors [26] established fractional Caputo-Fabrizio-Caputo (CFC) operator wrta function involved the weighted function. Also, Al-Refai drove the fractional Atangana-Baleanu-Caputo (ABC) derivative included the weighted function [34]. Very recently, Thabet et al. presented a new generalized weighted fractional derivative and integral wrta function via modified GMLF of fractional order $\alpha \in (0, 1)$ [35], which unifies aforementioned fractional operators in one form.

Motivated by the above fractional operator [35], the main objective of this study is to extend the L.S (ω, φ)-fractional derivatives and integrals, which involve the modified GMLF to higher orders $\alpha \in (k, k+1), k \in \mathbb{N}$. Furthermore, we investigate various results related to the new fractional operators, including the series forms of these operators, the relationship between Caputo and RL versions, the fundamental calculus theorem for these operators, and their (ω, φ)-Laplace transformation. Additionally, we introduce reciprocal operator of the new general Q-operator [36], then we present the general fundamental relations between the L&R sides of (ω, φ) -fractional integrals and derivatives. Accordingly, we derive the R.S definitions of our proposed operators and establish their properties due to the duality of the L.S. Lastly, as an application, we explore the EaU of the solution to (ω, φ)-Caputo fractional IDEs involving these new derivatives.

This paper is divided into several sections. In Sect. 2, we introduce some important definitions and lemmas. In Sect. 3 and Sect. 4, we extended the new (ω, φ)-fractional derivatives and integrals involving the modified GMLF to higher orders in the sense of RL and Caputo types. In Sect. 5, we introduced the R.S definitions of our proposed operators by using the general reflection operator Q and its reciprocal. In Sect. 6, we use the new operators to explore the solution of the (ω, φ)-Caputo fractional IDEs. Finally, in Sect. 7, we present some illustrative examples to validate our new results.

2 Preliminaries

Here, we offer some fundamental definitions and lemmas of the fractional calculus, which are closely related to our study analysis.

Definition 2.1 [35] Let $\varphi : [a, b] \longrightarrow \mathbb{R}$ is a non-decreasing function on $[a, b]$, such that $\varphi'(r) \neq 0, \forall r \in [a, b]$. Assume that $Re(\mu) > 0, \gamma, \beta > 0, \alpha \in (0, 1), \sigma \in \mathbb{R}$ and

$\mathfrak{f}(r)$ is continuous on $[a,b]$. The L.S (ω,φ)-fractional derivative of order α in the setting of RL involving the modified GMLF is presented by

$$\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)=\frac{\delta(\alpha)}{1-\alpha}\frac{1}{\omega(r)}\frac{1}{\varphi'(r)}\frac{d}{dr}\int_a^r\varphi'(s)\mathrm{E}^{\alpha,\sigma}_{\alpha,\mu}\big(\lambda_\alpha,\big(\varphi(r)-\varphi(s)\big)\big)(\omega\mathfrak{f})(s)ds, \quad (2.1)$$

where $\lambda_\alpha=\frac{-\alpha}{1-\alpha}$, $\delta(\alpha)$ is normalization function with $\delta(0)=\delta(1)=1$, ω is a weighted function, such that $\omega>0$ on $[a,b]$, and $\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}$ is the modified version of GMLF defined as:

$$\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\varphi(r)-\varphi(s)\big)\big)=\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_\alpha(\varphi(r)-\varphi(s)^{i\gamma+\mu-1}}{(i)!\,\Gamma(\beta i+\mu)}. \quad (2.2)$$

Furthermore, the α^{th} L.S (ω,φ)-fractional derivative in the setting of Caputo involving the modified GMLF is presented by:

$$\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)=\frac{\delta(\alpha)}{1-\alpha}\frac{1}{\omega(r)}\int_a^r\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\varphi(r)-\varphi(s)\big)\big)\varphi'(s)\omega(s)\mathrm{D}^1_{a,\omega,\varphi}\mathfrak{f}(s)ds. \quad (2.3)$$

And the corresponding α^{th} L.S generalized (ω,φ)-fractional integral of $\mathfrak{f}$ is given as below:

$$\left(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)=\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha)^i}{(1-\alpha)^{i-1}\,\delta(\alpha)}\left({}^{RL}\mathrm{I}^{i\beta-\mu+1}_{a,\omega,\varphi}\mathfrak{f}\right)(r), \quad (2.4)$$

where ${}^{RL}\mathrm{I}^{i\beta-\mu+1}_{a,\omega,\varphi}$ is the L.S–RL integral operator [23], which defined as

$$\left({}^{RL}\mathrm{I}^{\mu}_{a,\omega,\varphi}\mathfrak{f}\right)(r)=\frac{1}{\omega(r)\Gamma(\mu)}\int_a^r\big(\varphi(r)-\varphi(s)\big)^{\mu-1}\varphi'(s)(\omega\mathfrak{f})(s)ds, \quad (2.5)$$

and the R.S–RL integral operator [23], defined as

$$\left({}^{RL}\mathrm{I}^{\mu}_{b,\omega,\varphi}\mathfrak{f}\right)(r)=\frac{\omega(r)}{\Gamma(\mu)}\int_r^b\frac{\big(\varphi(s)-\varphi(r)\big)^{\mu-1}}{\omega(s)}\varphi'(s)\mathfrak{f}(s)ds, \quad (2.6)$$

such that $\Gamma(\mu)=\int_0^\infty e^{-r}r^{\mu-1}dr$, $\mu>0$.

Definition 2.2 [23, 24] Let $n\in\mathbb{N}$, ω is a non zero weighted function, and φ is strictly differentiable function, then the (ω,φ)-derivative of n^{th} order is given as

$$\big(\mathrm{D}^{n}_{a,\omega,\varphi}\mathfrak{f}\big)(r) = \mathrm{D}^{1}_{a,\omega,\varphi}(\mathrm{D}^{n-1}_{a,\omega,\varphi}\mathfrak{f})(r) = \omega^{-1}(r)\Big(\frac{\mathrm{D_r}}{\varphi'(r)}\Big)^{n}(\omega\mathfrak{f})(r), \tag{2.7}$$

$$\big(\mathrm{D}^{n}_{b,\omega,\varphi}\mathfrak{f}\big)(r) = \mathrm{D}^{1}_{b,\omega,\varphi}(\mathrm{D}^{n-1}_{b,\omega,\varphi}\mathfrak{f})(r) = \omega(r)\Big(\frac{-\mathrm{D_r}}{\varphi'(r)}\Big)^{n}(\omega^{-1}\mathfrak{f})(r),$$

where $\mathrm{D}_r = \frac{d}{dr}$, $\omega^{-1}(r) = \frac{1}{\omega(r)}$, and $(\omega\mathfrak{f})(r) = \omega(r)\mathfrak{f}(r)$.

Lemma 2.3 *([24]). For $n \in \mathbb{N}$ and $\mathfrak{f} \in AC^{n-1}_{\omega,\varphi}[a,b)$, one has*

(i) $(\mathrm{I}^{n}_{a,\omega,\varphi}\,\mathrm{D}^{n}_{\omega,\varphi})\mathfrak{f}(r) = \mathfrak{f}(r) - \omega^{-1}(r)\displaystyle\sum_{i=0}^{n-1}\frac{\big(\varphi(r)-\varphi(a)\big)^{i}}{i!}\Big(\frac{\mathrm{D}_r}{\varphi'(r)}\Big)^{i}(\omega\mathfrak{f})(a),$

(ii) $(\mathrm{D}^{n}_{a,\omega,\varphi}\,\mathrm{I}^{n}_{a,\omega,\varphi}\mathfrak{f})(r) = \mathfrak{f}(r)$;

(iii) $({}^{RL}\mathrm{D}^{\beta}_{a,\omega,\varphi}\;{}^{RL}\mathrm{I}^{\alpha}_{a,\omega,\varphi}\mathfrak{f})(r) = ({}^{RL}\mathrm{I}^{\alpha-\beta}_{a,\omega,\varphi}\mathfrak{f})(r), \quad \alpha,\beta \in \mathbb{N}$;

(iv) $\mathcal{L}_{\omega,\varphi}\{({}^{RL}\mathrm{I}^{\mu}_{a,\omega,\varphi}\mathfrak{f})(r)\}(s) = s^{-\mu}\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s)$;

(v) $\mathcal{L}_{\omega,\varphi}\{(\mathrm{D}^{n}_{a,\omega,\varphi}\mathfrak{f})(r)\}(s) = s^{n}\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s) - \sum_{i=0}^{n-1} s^{n-1-i}\Big(\frac{\mathrm{D}_r}{\varphi'(r)}\Big)^{i}(\omega\mathfrak{f})(a)$;

(vi) $\mathcal{L}_{\omega,\varphi}\{\omega^{-1}(r)(\varphi(r)-\varphi(a))^{\vartheta-1}\}(s) = \dfrac{\Gamma(\vartheta)}{s^{\vartheta}}, \quad \vartheta > -1, s > 0.$

Theorem 2.4 *(Banach's* FPT*) [37]: Consider $\Pi : G \longrightarrow G$ be a contraction operator, such that G is a Banach space. Then, there is only one fixed point for Π in G.*

Theorem 2.5 *(Schaefer's* FPT*) [38]: Let a Banach space denoted by G, and $\Pi : G \longrightarrow G$ is a completely continuous mapping. For some $\varrho \in (0,1)$, there is a bounded set $U = \{\varsigma \in G : \ \varsigma = \varrho\ \Pi\ \varsigma\}$. So the map Π owns a fixed point.*

Theorem 2.6 *(Ascoli-Arzelá Theorem) [39] Consider $Y \subset G$, where G is a Banach space, then Y is called a relatively compact set if Y is uniformly bounded and equicontinuous set.*

3 Higher-order fractional derivatives

Definition 3.1 Suppose that $\varphi : [a,b] \to \mathbb{R}$ is a non-decreasing function on $[a,b]$, such that $\varphi'(r) \neq 0, \forall r \in [a,b]$. Assume that $Re(\mu) > 0, \gamma, \beta > 0, \alpha \in (k, k+1), \sigma \in \mathbb{R}$ and $\mathfrak{f}(r)$ is continuous on $[a,b]$. The higher order α^{th} L.S (ω,φ)-fractional derivative in the setting of RL involving the modified GMLF is presented by

$$\begin{aligned}
\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \mathrm{D}^{k}_{a,\omega,\varphi}\left({}^{R}\mathrm{D}^{\nu,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\,\mathfrak{f}\right)(r)\\
&= \frac{1}{\omega(r)}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k}\left(\omega(r)\,{}^{R}\mathrm{D}^{\nu,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\,\mathfrak{f}\right)(r)\\
&= \frac{1}{\omega(r)}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k}\left(\frac{\omega(r)\delta(\nu)}{(1-\nu)}\frac{1}{\omega(r)\varphi'(r)}\frac{d}{dr}\right)\int_a^r \varphi'(s)\mathtt{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_\nu,\left(\varphi(r)-\varphi(s)\right)\right)(\omega\mathfrak{f})(s)ds\\
&= \frac{1}{\omega(r)}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k+1}\frac{\delta(\nu)}{(1-\nu)}\int_a^r \varphi'(s)\,\mathtt{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_\nu,\left(\varphi(r)-\varphi(s)\right)\right)(\omega\mathfrak{f})(s)ds\\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\frac{1}{\omega(r)}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k+1}\int_a^r \varphi'(s)\mathtt{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k},\left(\varphi(r)-\varphi(s)\right)\right)(\omega\mathfrak{f})(s)ds,
\end{aligned}$$

where $\nu=\alpha-k$, $\lambda_{\alpha-k}=\frac{-(\alpha-k)}{(k+1-\alpha)}$, $k=0,1,2,\ldots$, and $\delta(\nu)$ is normalization function with $\delta(0)=\delta(1)=1$, and $\mathtt{E}^{\gamma,\sigma}_{\beta,\mu}$ is the modified version of GMLF given in Eq. (2.2).

Remark 3.2

(i) If $k=0$, then Definition 3.1 reduces to the L.S (ω,φ)-fractional derivative in the RL sense involving the modified GMLF with $\alpha\in(0,1)$ [35].
(ii) By putting $\alpha-k=\gamma=\beta$, $\sigma=\mu=1$ and $\omega(r)=1$ into Definition 3.1, we find the φ-ABR derivative of fractional higher orders [22].
(iii) By putting $\alpha-k=\gamma=\beta$, $\sigma=\mu=1$, $\omega(r)=1$, and $\varphi(r)=r$ into Definition 3.1, we have the ABR derivative of fractional higher orders [40].
(iv) By putting $\gamma=\beta=\sigma=\mu=1$, into Definition 3.1, one finds the (ω,φ)-CFR derivative of fractional higher orders [26].
(v) By putting $\gamma=\beta=\sigma=\mu=1$ and $\omega(r)=1$, into Definition 3.1, one obtains the φ-CFR derivative of fractional higher orders [26].
(vi) By putting $\gamma=\beta=\sigma=\mu=1$, $\varphi(r)=r$ and $\omega(r)=1$, into Definition 3.1, we have the CFR derivative of fractional higher order s [41].
(vii) Definition 3.1 returns to the integer derivative of order $k+1$, if we take $\gamma,\beta,\mu,\sigma\to 1,\varphi(r)=r$, $\omega(r)=1$, and $\alpha\to k+1$. Also, for $\mu\in(0,1)$ the Definition 3.1 owns singular kernel.

Definition 3.3 Suppose that $\varphi:[a,b]\to\mathbb{R}$ is a non-decreasing function on $[a,b]$, such that $\varphi'(r)\neq 0,\forall r\in[a,b]$. Assume that $Re(\mu)>0,\gamma,\beta>0,\alpha\in(k,k+1),\sigma\in\mathbb{R}$, and $\mathfrak{f}\in H^1(a,b)$. The higher order α^{th} L.S (ω,φ)-fractional derivative in the setting of Caputo involving the modified GMLF is presented by

$$\begin{aligned}
\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \left({}^{C}\mathrm{D}^{\nu,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\,\mathrm{D}^{k}_{a,\omega,\varphi}\mathfrak{f}\right)(r)\\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\frac{1}{\omega(r)}\int_a^r \mathtt{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k},\left(\varphi(r)-\varphi(s)\right)\right)\varphi'(s)\omega(s)\,\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}(s)ds.
\end{aligned}$$

Remark 3.4

(i) If $k=0$, then Definition 3.3 reduces to the L.S (ω,φ)-fractional derivative in the setting of Caputo involving the modified GMLF with $\alpha\in(0,1)$ [35].

(ii) By putting $\alpha - k = \gamma = \beta$, $\sigma = \mu = 1$ and $\omega(r) = 1$ into Definition 3.3, we find the $\varphi -$ ABC derivative of fractional higher orders [22].
(iii) By putting $\alpha - k = \gamma = \beta$, $\sigma = \mu = 1$, $\omega(r) = 1$, and $\varphi(r) = r$ into Definition 3.3, we get the ABC derivative of fractional higher orders [40].
(iv) By putting $\gamma = \beta = \sigma = \mu = 1$, into Definition 3.3, one finds the $(\omega, \varphi) -$ CFC derivative of fractional higher orders [26].
(v) By putting $\gamma = \beta = \sigma = \mu = 1$ and $\omega(r) = 1$, into Definition 3.3, one obtains the $\varphi -$ CFC derivative of fractional higher orders [26].
(vi) By putting $\gamma = \beta = \sigma = \mu = 1$, $\varphi(r) = r$, and $\omega(r) = 1$, into Definition 3.3, we have the CFC derivative of fractional higher orders [42].
(vii) Definition 3.3 returns to the integer derivative of order $k+1$, if we take $\gamma, \beta, \mu, \sigma \to 1, \varphi(r) = r$, $\omega(r) = 1$, and $\alpha \to k+1$. Also, for $\mu \in (0,1)$ the Definition 3.3 owns singular kernel.

Proposition 3.5 *The series form of the higher order α^{th} L.S (ω, φ)-fractional derivative involving the modified GMLF which are given in Definition 3.1 and Definition 3.3 can be represented, respectively, as follows:*

$$\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) = \frac{\delta(\alpha-k)}{(k+1-\alpha)} \sum_{i=0}^{\infty} (\sigma)_i \, \lambda^i_{\alpha-k} \frac{\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)} \, {}^{RL}\mathrm{I}^{\gamma i+\mu-k-1}_{a,\omega,\varphi}\mathfrak{f}(r),$$

$$\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) = \frac{\delta(\alpha-k)}{(k+1-\alpha)} \sum_{i=0}^{\infty} (\sigma)_i \frac{\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)} \, {}^{RL}\mathrm{I}^{\gamma i+\mu}_{a,\omega,\varphi}\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}(r),$$

respectively, where ${}^{RL}\mathrm{I}^{(\gamma i+\mu-k-1)}_{a,\omega,\varphi}$, and ${}^{RL}\mathrm{I}^{\gamma i+\mu}_{a,\omega,\varphi}$ are RL-fractional integrals defined in Eq.(3.1)

Proof Since the series of the modified version of GMLF which is given in Eq. (2.2) is locally uniformly convergent for all complex domains. Thus, the Definition 3.1 can be represented as below:

$$\begin{aligned}
\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)} \frac{1}{\omega(r)} \left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k+1} \int_a^r \varphi'(s) \mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k}, \left(\varphi(r)-\varphi(s)\right)\right)(\omega\mathfrak{f})(s)ds \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)} \frac{1}{\omega(r)} \left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k+1} \\
&\quad \times \int_a^r \varphi'(s) \sum_{i=0}^{\infty} (\sigma)_i \frac{\lambda^i_{\alpha-k}\left(\varphi(r)-\varphi(s)\right)^{i\gamma+\mu-1}}{(i)!\Gamma(\beta i+\mu)} (\omega\mathfrak{f})(s)ds \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)} \frac{1}{\omega(r)} \left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{(k+1)} \sum_{i=0}^{\infty} (\sigma)_i \frac{\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)} \frac{1}{\Gamma(\gamma i+\mu)} \\
&\quad \times \int_a^r \varphi'(s)\left(\varphi(r)-\varphi(s)\right)^{i\gamma+\mu-1}(\omega\mathfrak{f})(s)ds \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)} \sum_{i=0}^{\infty} (\sigma)_i \frac{\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)} \frac{1}{\omega(r)} \left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{(k+1)} \left(\omega(r)\, {}^{RL}\mathrm{I}^{\gamma i+\mu}_{a,\omega,\varphi}\mathfrak{f}\right)(r) \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)} \sum_{i=0}^{\infty} (\sigma)_i \, \lambda^i_{\alpha-k} \frac{\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)} \, {}^{RL}\mathrm{I}^{\gamma i+\mu-k-1}_{a,\omega,\varphi}\mathfrak{f}(r).
\end{aligned}$$

Next, the Definition 3.3 can be represented as follows:

$$
\begin{aligned}
\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\frac{1}{\omega(r)}\int_a^r \varphi'(s)\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k},\left(\varphi(r)-\varphi(s)\right)\right)\omega(s)\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}(s)ds \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\frac{1}{\omega(r)}\int_a^r \varphi'(s)\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\left(\varphi(r)-\varphi(s)\right)^{i\gamma+\mu-1}}{(i)!\Gamma(\beta i+\mu)}\omega(s)\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}(s)ds \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)}\frac{1}{\omega(r)\Gamma(\gamma i+\mu)} \\
&\quad\times\int_a^r \varphi'(s)\left(\varphi(r)-\varphi(s)\right)^{i\gamma+\mu-1}\omega(s)\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}(s)ds \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)}\,{}^{RL}\mathrm{I}^{\gamma i+\mu}_{a,\omega,\varphi}\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}(r).
\end{aligned}
$$

Hence, finished the proof. □

Proposition 3.6 *The (ω,φ)-Laplace transform for $\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)$ and $\left({}^{c}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)$ are represented as*

$$
\mathcal{L}_{\omega,\varphi}\{\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s) = \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu-k-1}}\,\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s).
$$

$$
\begin{aligned}
\mathcal{L}_{\omega,\varphi}\{\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu}} \\
&\quad\times\left[s^{k+1}\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s)-\omega(a)\sum_{j=0}^{k}s^{k-j}(\mathrm{D}^{j}_{a,\omega,\varphi}\,\mathfrak{f})(a)\right].
\end{aligned}
$$

Proof In view of Proposition 3.5, and Lemma 2.3, we get

$$
\begin{aligned}
\mathcal{L}_{\omega,\varphi}\{\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)}\,\mathcal{L}_{\omega,\varphi}\{\left({}^{RL}\mathrm{I}^{\gamma i+\mu-k-1}_{a,\omega,\varphi}\mathfrak{f}(r)\right)\}(s) \\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu-k-1}}\,\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s).
\end{aligned}
$$

Next,

$$
\begin{aligned}
\mathcal{L}_{\omega,\varphi}\{\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\,\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)}\,\mathcal{L}_{\omega,\varphi}\{\left({}^{RL}\mathrm{I}^{\gamma i+\mu}_{a,\omega,\varphi}\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}\right)(r)\}(s)\\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\,\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu}}\,\mathcal{L}_{\omega,\varphi}\{\left(\mathrm{D}^{k+1}_{a,\omega,\varphi}\mathfrak{f}\right)(r)\}(s)\\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\,\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu}}\\
&\quad\times\left[s^{k+1}\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s)-\omega(a)\sum_{j=0}^{k}s^{k-j}(\mathrm{D}^{j}_{a,\omega,\varphi}\,\mathfrak{f})(a)\right].
\end{aligned}
$$

Thus, the required result is satisfied. □

Lemma 3.7 *For $\alpha\in(k,k+1)$, the relationship between the* RL *and Caputo versions of the* L.S *(ω,φ)-fractional derivatives involving the modified* GMLF *for a function $\mathfrak{f}$ is given as*

$$
\begin{aligned}
\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)-\omega(a)\frac{\delta(\alpha-k)}{(k+1-\alpha)}\\
&\quad\times\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)\left(\varphi(r)-\varphi(a)\right)^{\gamma i+\mu-k+j-1}}{(i)!\,\omega(r)\,\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}^{j}_{a,\omega,\varphi}\mathfrak{f})(a).
\end{aligned}
\tag{3.1}
$$

Proof According to Lemma 2.3 (vi), and Proposition 3.6, one has

$$
\begin{aligned}
&\mathcal{L}_{\omega,\varphi}\{\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s)\\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\,\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu}}\left[s^{k+1}\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s)-\omega(a)\sum_{j=0}^{k}s^{k-j}(\mathrm{D}^{j}_{a,\omega,\varphi}\,\mathfrak{f})(a)\right]\\
&= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\,\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu-k-1}}\mathcal{L}_{\omega,\varphi}\{\mathfrak{f}(r)\}(s)\\
&\quad-\omega(a)\frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}(\sigma)_i\,\frac{\lambda^i_{\alpha-k}\,\Gamma(\gamma i+\mu)}{(i)!\Gamma(\beta i+\mu)s^{\gamma i+\mu}}\sum_{j=0}^{k}s^{k-j}(\mathrm{D}^{j}_{a,\omega,\varphi}\,\mathfrak{f})(a)\\
&= \mathcal{L}_{\omega,\varphi}\{\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s)\\
&\quad-\omega(a)\frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{k}(\sigma)_i\frac{\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)\Gamma(\gamma i+\mu-k+j)}{(i)!\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)s^{\gamma i+\mu-k+j}}(\mathrm{D}^{j}_{a,\omega,\varphi}\mathfrak{f})(a)\\
&= \mathcal{L}_{\omega,\varphi}\{\left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\}(s)\\
&\quad-\omega(a)\frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)\mathcal{L}_{\omega,\varphi}\{\omega^{-1}(r)\left(\varphi(r)-\varphi(a)\right)^{\gamma i+\mu-k+j-1}\}(s)}{(i)!\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}^{j}_{a,\omega,\varphi}\mathfrak{f})(a).
\end{aligned}
$$

In view of the Laplace inverse, the proof has done. □

In what follows, we infer for the first time the relationship between higher order ABC and ABR fractional derivatives; also the relationship between higher order CFC and CFR fractional derivatives.

Corollary 3.8 *If $\alpha-k=\gamma=\beta$, $\sigma=\mu=\omega(r)=1$, and $\varphi(r)=r$, we obtain the relationship between higher order* ABC *and* ABR *fractional derivatives as follows:*

$$\left({}^{ABC}\mathrm{D}_a^{\alpha}\,\mathfrak{f}\right)(r) = \left({}^{ABR}\mathrm{D}_a^{\alpha}\,\mathfrak{f}\right)(r) - \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{j=0}^{k}\mathrm{E}_{\alpha-k,j-k+1}\left(\lambda_{\alpha-k},(r-a)^{\alpha-k}\right)(r-a)^{j-k}(\mathrm{D}_a^{j}\mathfrak{f})(a).$$

Corollary 3.9 *If $\gamma=\beta=\sigma=\mu=\omega(r)=1$, and $\varphi(r)=r$, we find the relationship between higher order* CFC *and* CFR *fractional derivatives as below:*

$$\left({}^{CFC}\mathrm{D}_a^{\alpha}\,\mathfrak{f}\right)(r) = \left({}^{CFR}\mathrm{D}_a^{\alpha}\,\mathfrak{f}\right)(r) - \frac{\delta(\alpha-k)}{(k+1-\alpha)}\sum_{j=0}^{k}\mathrm{E}_{1,j-k+1}\left(\lambda_{\alpha-k},(r-a)\right)(r-a)^{j-k}(\mathrm{D}_a^{j}\mathfrak{f})(a).$$

Remark 3.10 We note the following:

(i) If $k=0$, then $\alpha\in(0,1)$, and Corollary 3.8 coincides with the relationship between ABC and ABR fractional derivatives which is given [8].
(ii) If $k=0$, then $\alpha\in(0,1)$, and Corollary 3.9 coincides with the relationship between CFC and CFR fractional derivatives which is given in [26].

4 Higher-order fractional integral

This section presents a generalized (ω,φ)-integral of fractional higher orders which corresponds to the Caputo and RL versions of the (ω,φ)-fractional derivatives involving the modified GMLF, and investigate some properties related to new fractional derivative and integral definitions. In the rest of study, for simplicity, if $\gamma=\beta$, then ${}^{C}\mathrm{D}^{\alpha,\beta,\beta,\mu}_{a,\sigma,\omega,\varphi}$ can be written as ${}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}$.

Definition 4.1 Consider a function $\mathfrak{f}$ to be continuous on $[a,b]$. Let $\alpha\in(k,k+1)$, $\sigma\in\mathbb{R}$, $\beta>0$, and $Re(\mu)>0$. The higher order α^{th} L.S generalized (ω,φ)-fractional integral is given as follows:

$$\begin{aligned}
\left(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \left(\mathrm{I}^{k}_{a,\omega,\varphi}\,\mathrm{I}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\\
&= \mathrm{I}^{k}_{a,\omega,\varphi}\left[\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{\nu^{i}}{(1-\nu)^{i-1}\,\delta(\nu)}\left({}^{RL}\mathrm{I}^{i\beta-\mu+1}_{a,\omega,\varphi}\mathfrak{f}\right)(r)\right]\\
&= \sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{\nu^{i}}{(1-\nu)^{i-1}\delta(\nu)}\left(\mathrm{I}^{k}_{a,\omega,\varphi}\,{}^{R}\mathrm{I}^{i\beta-\mu+1}_{a,\omega,\varphi}\mathfrak{f}\right)(r)\\
&= \sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-k)^{i}}{(k+1-\alpha)^{i-1}\,\delta(\alpha-k)}\left({}^{RL}\mathrm{I}^{i\beta-\mu+k+1}_{a,\omega,\varphi}\mathfrak{f}\right)(r).
\end{aligned}$$

where $\mathbf{C}(\sigma,i)=\binom{\sigma}{i}$, $\nu=\alpha-k$, and ${}^{RL}\mathrm{I}^{i\beta-\mu+k+1}_{a,\omega,\varphi}$ is defined in Eq. (2.5).

Remark 4.2 For specifically parameters $\beta, \alpha, \sigma, \mu, \omega, \varphi$, we observe the existing definitions in the literature as follows:

(i) By putting $k = 0$ into Definition 4.1, we obtain the α^{th} `L.S` generalized (ω, φ) -fractional integral, where $\alpha \in (0, 1)$ [35].
(ii) By putting $\beta = \alpha$, $\sigma = \mu = 1$, $\omega(r) = 1$, and $\varphi(r) = r$ into Definition 4.1, we get the `AB` integral of fractional higher orders [40].
(iii) By putting $\beta = \sigma = \mu = 1$, $\varphi(r) = r$, and $\omega(r) = 1$, into Definition 4.1, we return to the `CF` integral of fractional higher orders [42].
(iv) By putting $\beta = \alpha$, $\varphi(r) = r$, and $\omega(r) = 1$, into Definition 4.1, one gets the `AB` integral of fractional higher orders with `GMLF` function [33].
(v) By putting $\beta = \alpha, \sigma = \mu = 1$, and $\omega(r) = 1$, into Definition 4.1, one finds the φ -`AB` integral of fractional higher orders [22].
(vi) Definition 4.1 returns to an integral of order $k + 1$, if we take $\beta, \mu, \sigma \to 1, \varphi(r) = r, \omega(r) = 1$, and $\alpha \to k + 1$.

Proposition 4.3 *The following relations hold:*

(i) $$\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\,\mathfrak{f})(r) = \mathfrak{f}(r) - \sum_{i=0}^{k} \frac{(\varphi(r) - \varphi(a))^i}{i!}(\mathrm{D}^i_{a,\omega,\varphi}\mathfrak{f})(a),$$

(ii) $${}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\big(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\big)(r) = \begin{cases} \mathfrak{f}(r) - \dfrac{\mathrm{E}^{\beta,\sigma}_{\beta,1}\big(\lambda_{\alpha-k}, (\varphi(r) - \varphi(a))\big)}{\omega(r)}(\omega\mathfrak{f})(a), & \text{if } \mu = 1; \\ \mathfrak{f}(r), \quad \text{if } \mu \neq 1, Re(1-\mu) > 0, Re(i\beta - \mu + 1) > 0. \end{cases}$$

Proof

(i) Let $\nu = \alpha - k, k = 0, 1, \ldots,$ we recall (Proposition 3.12, part (iii) [35]), Lemma 2.3 (i), and Defs. 3.3, 4.1, we find

$$\begin{aligned}
\left(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\ {}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\,\mathfrak{f}\right)(r) &= \left(\mathrm{I}^{k}_{a,\omega,\varphi}\mathrm{I}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\ {}^{C}\mathrm{D}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\ \mathrm{D}^{k}_{a,\omega,\varphi}\mathfrak{f}\right)(r) \\
&= \mathrm{I}^{k}_{a,\omega,\varphi}\left(\mathrm{D}^{k}_{a,\omega,\varphi}\mathfrak{f}(r) - \frac{\omega(a)}{\omega(r)}\mathrm{D}^{k}_{a,\omega,\varphi}\mathfrak{f}(a)\right) \\
&= \mathfrak{f}(r) - \omega^{-1}(r)\sum_{i=0}^{k-1}\frac{(\varphi(r) - \varphi(a))^i}{i!}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^i(\omega\mathfrak{f})(a) \\
&\quad - \frac{1}{\omega(r)}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^k \frac{(\omega\mathfrak{f})(a)\,(\varphi(r) - \varphi(a))^k}{k!} \\
&= \mathfrak{f}(r) - \omega^{-1}(r)\sum_{i=0}^{k}\frac{(\varphi(r) - \varphi(a))^i}{i!}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^i(\omega\mathfrak{f})(a) \\
&= \mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(r) - \varphi(a))^i}{i!}(\mathrm{D}^i_{a,\omega,\varphi}\mathfrak{f})(a).
\end{aligned}$$

(ii) Let $\nu=\alpha-k, k=0,1,\ldots$, we recall (Proposition 3.12, part(vi) [35]), Lemma 2.3 (ii), and Defs. 3.3, 4.1, we find

$$\begin{aligned}
\left({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\ \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) &= \left({}^{C}\mathrm{D}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\ \mathrm{D}^{k}_{a,\omega,\varphi}\ \mathrm{I}^{k}_{a,\omega,\varphi}\mathrm{I}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\\
&= \left({}^{C}\mathrm{D}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\ \mathrm{I}^{\nu,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)\\
&= \begin{cases}\mathfrak{f}(r)-\dfrac{\mathrm{E}^{\beta,\sigma}_{\beta,1}\left(\lambda_{\nu},(\varphi(r)-\varphi(a))\right)}{\omega(r)}(\omega\mathfrak{f})(a), & \text{if } \mu=1;\\ \mathfrak{f}(r), & \text{if}\mu\neq 1, Re(1-\mu)>0, Re(i\beta-\mu+1)>0.\end{cases}\\
&= \begin{cases}\mathfrak{f}(r)-\dfrac{\mathrm{E}^{\beta,\sigma}_{\beta,1}\left(\lambda_{\alpha-k},(\varphi(r)-\varphi(a))\right)}{\omega(r)}(\omega\mathfrak{f})(a), & \text{if } \mu=1;\\ \mathfrak{f}(r), & \text{if}\mu\neq 1, Re(1-\mu)>0, Re(i\beta-\mu+1)>0.\end{cases}
\end{aligned}$$

Hence, the proof is completed. □

5 General Q-operator

Samko et al. [18] introduced the reflection operator Q. This operator is used to convert L.S fractional differentiation/ integration to R.S differentiation/ integration, and it is given by $Q\mathfrak{f}(r)=\mathfrak{f}(a+b-r)$, for any function $\mathfrak{f}(r)$ defined on $[a,b]$. In classical fractional calculus, the following identities hold: $\mathrm{I}^{\alpha}_{a}Q\mathfrak{f}=Q\,\mathrm{I}^{\alpha}_{b}\mathfrak{f}$ and $\mathrm{D}^{\alpha}_{a}Q\mathfrak{f}=Q\,\mathrm{D}^{\alpha}_{b}\mathfrak{f}$. Very recently, Fan and Huang [36], derived the general Q-operator to convert the L.S to R.S in fractional calculus wrta function φ, which is defined as: For any $\mathfrak{f}$ be a function defined on $[a,b]$, and $\varphi:[a,b]\to\mathbb{R}$ is a non-decreasing function on $[a,b]$, such that $\varphi'(r)\neq 0$, $\forall r\in[a,b]$, the general Q-operator is given as

$$Q\mathfrak{f}(r)=\mathfrak{f}\left(\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(r)\right)\right),\quad \forall r\in[a,b], \tag{5.1}$$

and satisfies $Q^2\mathfrak{f}(r)=\mathfrak{f}(r)$ [36], as below:

$$\begin{aligned}
Q^2\mathfrak{f}(r)=QQ\mathfrak{f}(r)&=Q\mathfrak{f}\left(\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(r)\right)\right)\\
&=\mathfrak{f}\left(\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi\left(\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(r)\right)\right)\right)\right)=\mathfrak{f}(r).
\end{aligned}$$

Furthermore, the following identities regarding φ-fractional calculus are fulfilled:

$$\mathrm{I}^{\mu}_{a,\varphi}Q\mathfrak{f}=Q\mathrm{I}^{\mu}_{b,\varphi}\mathfrak{f},\ \text{ and } \mathrm{D}^{\mu}_{a,\varphi}Q\mathfrak{f}=Q\mathrm{D}^{\mu}_{b,\varphi}\mathfrak{f}.$$

In what follows, we define the reciprocal of the new general Q-operator, and we denote it by ${}_{R}Q$.

Definition 5.1 Reciprocal of the general Q-operator ${}_{R}Q$ for a function $\mathfrak{f}$ defined on $[a,b]$, wrta function φ, such that $\varphi'(r)\neq 0$, $\forall r\in[a,b]$ is defined as:

$$
{}_RQ\mathfrak{f}(r)=\frac{1}{Q\mathfrak{f}(r)},\quad \forall r\in[a,b], \tag{5.2}
$$

where $Q\mathfrak{f}$ is given in (5.1).

Remark 5.2. *We note the following:*

$$
{}_RQ^2\mathfrak{f}(r)={}_RQ_RQ\mathfrak{f}(r)={}_RQ\left(\frac{1}{Q\mathfrak{f}(r)}\right)=QQ\mathfrak{f}(r)=\mathfrak{f}(r),
$$

and

$$
{}_RQQ\mathfrak{f}(r)=\frac{1}{\mathfrak{f}(r)}=Q\ {}_RQ\mathfrak{f}(r).
$$

Now, inspired by the new general Q-operator and its reciprocal ${}_RQ$, which are defined in (5.1) and (5.2), respectively, we present the general fundamental relations between the L&R sides of (ω,φ)-fractional calculus as follows:

$$
\mathrm{I}^{\mu}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f}=Q\mathrm{I}^{\mu}_{b,\omega,\varphi}\mathfrak{f},\quad \text{and}\quad \mathrm{D}^{\mu}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f}=Q\mathrm{D}^{\mu}_{b,\omega,\varphi}\mathfrak{f},\quad \forall\mu>0, \tag{5.3}
$$

Lemma 5.3 *The relations between the* L&R *sides of* (ω,φ)-RL-*fractional integral can be address as follows:*

$$
\begin{aligned}
\left({}^{RL}\mathrm{I}^{\mu}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f}\right)(r)&=\left(Q\,{}^{RL}\mathrm{I}^{\mu}_{b,\omega,\varphi}\mathfrak{f}\right)(r),\quad \textit{and}\quad \left({}^{RL}\mathrm{I}^{\mu}_{b,{}_RQ\omega,\varphi}Q\mathfrak{f}\right)(r)\\
&=\left(Q\,{}^{RL}\mathrm{I}^{\mu}_{a,\omega,\varphi}\mathfrak{f}\right)(r).
\end{aligned}
$$

Proof In the light of formulations (2.5) and (2.6), we obtain

$$
\left({}^{RL}\mathrm{I}^{\mu}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f}\right)(r)=\frac{1}{{}_RQ\omega(r)\Gamma(\mu)}\int_a^r\left(\varphi(r)-\varphi(s)\right)^{\mu-1}\varphi'(s)(\omega^{-1}\ \mathfrak{f})\left(\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(s)\right)\right)ds.
$$

We introduce the change of variable $x=\varphi^{-1}\big(\varphi(a)+\varphi(b)-\varphi(s)\big)$, for $s\in[a,r]$, then $\varphi(x)=\varphi(a)+\varphi(b)-\varphi(s)$, so we find that $x\in[b,\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(r)\right)]$, and by taking the total differential of both sides, we have $\varphi'(x)dx=-\varphi'(s)ds$. It follows that

$$
\begin{aligned}
&\left({}^{RL}\mathrm{I}^{\mu}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f}\right)(r)\\
&=\frac{Q\omega(r)}{\Gamma(\mu)}\int_{\varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(r))}^{b}\left(\varphi(r)-\varphi(a)-\varphi(b)+\varphi(x)\right)^{\mu-1}\varphi'(x)(\omega^{-1}\ \mathfrak{f})\,(x)\,dx\\
&=\left(Q\,{}^{RL}\mathrm{I}^{\mu}_{b,\omega,\varphi}\mathfrak{f}\right)(r).
\end{aligned}
$$

Thus, the required result is satisfied. □

In what follows, we investigate the relations between the L&R sides of (ω,φ)-fractional operators involving the modified GMLF.

Lemma 5.4 *Assume that* $Re(\mu) > 0, \gamma, \beta > 0, \alpha \in (k, k+1), \sigma \in \mathbb{R}$, *and* $\mathfrak{f} \in H^1(a, b)$. *Then,*

1. $\left({}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \left(Q\,{}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\, \mathfrak{f}\right)(r)$, *and* $\left({}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \left(Q\,{}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\, \mathfrak{f}\right)(r)$.
2. $\left({}^C\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \left(Q\,{}^C\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\, \mathfrak{f}\right)(r)$, *and* $\left({}^C\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \left(Q\,{}^C\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,\omega,\varphi}\, \mathfrak{f}\right)(r)$.
3. $\left(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \left(Q\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)$, *and* $\left(\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \left(Q\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\right)(r)$.

Proof 1 In view of Definition 3.1, we have

$$\begin{aligned}\left({}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\frac{1}{{}_RQ\omega(r)}\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k+1}\\ &\times \int_a^r \varphi'(s)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k}, (\varphi(r)-\varphi(s))\right)(\omega^{-1}\,\mathfrak{f})\left(\varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(s))\right)ds.\end{aligned}$$

We introduce the change of variable $x = \varphi^{-1}\big(\varphi(a)+\varphi(b)-\varphi(s)\big)$, for $s \in [a, r]$, then $\varphi(x) = \varphi(a)+\varphi(b)-\varphi(s)$, so we find that $x \in [b, \varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(r))]$, and by taking the total differential of both sides, we have $\varphi'(x)dx = -\varphi'(s)ds$. It follows that

$$\begin{aligned}\left({}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\,Q\omega(r)\left(\frac{1}{\varphi'(r)}\frac{d}{dr}\right)^{k+1}\\ &\quad\times \int_{Q(r)}^b \varphi'(x)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k}, (\varphi(r)-\varphi(a)-\varphi(b)+\varphi(x))\right)(\omega^{-1}\,\mathfrak{f})(x)dx\\ &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\,(-1)^{k+1}Q\omega(r)\left(\frac{1}{-\varphi'(r)}\frac{d}{dr}\right)^{k+1}\\ &\quad\times \int_{Q(r)}^b \varphi'(x)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k}, (\varphi(r)-\varphi(a)-\varphi(b)+\varphi(x))\right)(\omega^{-1}\,\mathfrak{f})(x)dx\\ &= \frac{\delta(\alpha-k)}{(k+1-\alpha)}\,(-1)^{k+1}Q\omega(r)\left(\frac{1}{(Q\varphi(r))'}\frac{d}{dr}\right)^{k+1}\\ &\quad\times \int_{Q(r)}^b \varphi'(x)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k}, (\varphi(x)-Q\varphi(r))\right)(\omega^{-1}\,\mathfrak{f})(x)dx\\ &= \left(Q\,{}^R\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\,\mathfrak{f}\right)(r).\end{aligned}$$

2. According to Definition 3.3, one has

$$\begin{aligned}&\left({}^C\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f}\right)(r) = \frac{\delta(\alpha-k)}{(k+1-\alpha)}\frac{1}{{}_RQ\omega(r)}\\ &\times \int_a^r \varphi'(s)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_{\alpha-k}, (\varphi(r)-\varphi(s))\right)\left(\frac{1}{\varphi'(s)}\frac{d}{ds}\right)^{k+1}(\omega^{-1}\,\mathfrak{f})\left(\varphi^{-1}(\varphi(a)+\varphi(b)-\varphi(s))\right)ds.\end{aligned}$$

We introduce the change of variable $x=\varphi^{-1}\big(\varphi(a)+\varphi(b)-\varphi(s)\big)$, for $s\in[a,r]$, then $\varphi(x)=\varphi(a)+\varphi(b)-\varphi(s)$, so we find that $x\in[b,\varphi^{-1}\left(\varphi(a)+\varphi(b)-\varphi(r)\right)]$, and by taking the total differential of both sides, we have $\varphi'(x)dx=-\varphi'(s)ds$.It follows that

$$
\begin{aligned}
&\big({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\,Q\mathfrak{f}\big)(r)=\frac{\delta(\alpha-k)}{(k+1-\alpha)}\,Q\omega(r)\\
&\times\int_{Q(r)}^{b}\varphi'(x)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_{\alpha-k},(\varphi(r)-\varphi(a)-\varphi(b)+\varphi(x))\big)\Big(\frac{1}{-\varphi'(x)}\frac{d}{dx}\Big)^{k+1}(\omega^{-1}\,\mathfrak{f})(x)dx\\
&=\frac{\delta(\alpha-k)}{(k+1-\alpha)}\,(-1)^{k+1}Q\omega(r)\\
&\times\int_{Q(r)}^{b}\varphi'(x)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_{\alpha-k},(\varphi(x)-Q\varphi(r))\big)\Big(\frac{1}{\varphi'(x)}\frac{d}{dx}\Big)^{k+1}(\omega^{-1}\,\mathfrak{f})(x)dx\\
&=\big(Q\,{}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\,\mathfrak{f}\big)(r).
\end{aligned}
$$

3. Based on Definition 4.1 and Lemma 5.3, we find

$$
\begin{aligned}
\big(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}Q\mathfrak{f}\big)(r)&=\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-k)^{i}}{(k+1-\alpha)^{i-1}\,\delta(\alpha-k)}\big({}^{RL}\mathrm{I}^{i\beta-\mu+k+1}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f}\big)(r)\\
&=\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-k)^{i}}{(k+1-\alpha)^{i-1}\,\delta(\alpha-k)}\big(Q\,{}^{RL}\mathrm{I}^{i\beta-\mu+k+1}_{b,\omega,\varphi}\mathfrak{f}\big)(r)\\
&=\big(Q\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\big)(r).
\end{aligned}
$$

Hence, the proof is completed. □

According to the above Lemma 5.4, we define the higher order of R.S (ω,φ)-fractional operators as follows:

Definition 5.5 Suppose that $\varphi:[a,b]\to\mathbb{R}$ is a non-decreasing function on $[a,b]$, such that $\varphi'(r)\neq 0$, $\forall r\in[a,b]$. Assume that $Re(\mu)>0$, γ, $\beta>0$, $\alpha\in(k,k+1)$, $\sigma\in\mathbb{R}$ and $\mathfrak{f}(r)$ is continuous on $[a,b]$. The higher order α^{th} R.S (ω,φ)-fractional derivative in the setting of RL involving the modified GMLF is presented by

$$
\big({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\big)(r)=\frac{\delta(\alpha-k)}{(k+1-\alpha)}\omega(r)\Big(\frac{-1}{\varphi'(r)}\frac{d}{dr}\Big)^{k+1}\int_{r}^{b}\varphi'(s)\,\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_{\alpha-k},\big(\varphi(s)-\varphi(r)\big)\big)(\omega^{-1}\mathfrak{f})(s)ds.
$$

And the higher order α^{th} R.S (ω,φ)-fractional derivative in the setting of Caputo involving the modified GMLF is presented by

$$
\big({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\big)(r)=\frac{\delta(\alpha-k)}{(k+1-\alpha)}\,(-1)^{k+1}\omega(r)\int_{r}^{b}\mathrm{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_{\alpha-k},(\varphi(s)-\varphi(r))\big)\varphi'(s)\,\omega^{-1}(s)\,\mathrm{D}^{k+1}_{b,\omega,\varphi}\mathfrak{f}(s)ds.
$$

Also, the higher order α^{th} R.S generalized (ω,φ)-fractional integral is given as follows:

$$\left(\mathrm{I}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{f}\right)(r)=\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-k)^{i}}{(k+1-\alpha)^{i-1}\,\delta(\alpha-k)}\left({}^{RL}\mathrm{I}_{b,\omega,\varphi}^{i\beta-\mu+k+1}\mathfrak{f}\right)(r),$$

where ${}^{RL}\mathrm{I}_{b,\omega,\varphi}^{i\beta-\mu+k+1}$ is the R.S–RL fractional integral.

Lemma 5.6 *For $\alpha\in(k,k+1)$, the relationship between the* RL *and Caputo versions of the* R.S *(ω,φ)-fractional derivatives involving the modified* GMLF *for a function $\mathfrak{f}$ is given as*

$$\begin{aligned}\left({}^{C}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r)&=\left({}^{R}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r)-\frac{1}{\omega(b)}\frac{\delta(\alpha-k)}{(k+1-\alpha)}\\&\quad\times\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda_{\alpha-k}^{i}\Gamma(\gamma i+\mu)\omega(r)\big(\varphi(b)-\varphi(r)\big)^{\gamma i+\mu-k+j-1}}{(i)!\;\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}_{b,\omega,\varphi}^{j}\;\mathfrak{f})(b).\end{aligned}\tag{5.4}$$

Proof According to Lemma 5.4, one obtains that

$$\left({}^{C}\mathrm{D}_{a,\sigma,{}_RQ\omega,\varphi}^{\alpha,\beta,\gamma,\mu}Q\mathfrak{f}\right)(r)=\left(Q^{C}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r),\quad\text{and}\quad\left({}^{R}\mathrm{D}_{a,\sigma,{}_RQ\omega,\varphi}^{\alpha,\beta,\gamma,\mu}Q\mathfrak{f}\right)(r)=\left(Q^{R}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r).$$

And in view of Lemma 3.7, we have

$$\begin{aligned}\left({}^{C}\mathrm{D}_{a,\sigma,{}_RQ\omega,\varphi}^{\alpha,\beta,\gamma,\mu}Q\mathfrak{f}\right)(r)&=\left({}^{R}\mathrm{D}_{a,\sigma,{}_RQ\omega,\varphi}^{\alpha,\beta,\gamma,\mu}Q\mathfrak{f}\right)(r)-{}_RQ\omega(a)\frac{\delta(\alpha-k)}{(k+1-\alpha)}\\&\quad\times\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda_{\alpha-k}^{i}\Gamma(\gamma i+\mu)\big(\varphi(r)-\varphi(a)\big)^{\gamma i+\mu-k+j-1}}{(i)!\;{}_RQ\omega(r)\,\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}_{a,{}_RQ\omega,\varphi}^{j}Q\mathfrak{f})(a)\\&=\left(Q^{R}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r)-\frac{1}{\omega(b)}\frac{\delta(\alpha-k)}{(k+1-\alpha)}\\&\quad\times\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda_{\alpha-k}^{i}\Gamma(\gamma i+\mu)Q\omega(r)\big(\varphi(r)-\varphi(a)\big)^{\gamma i+\mu-k+j-1}}{(i)!\;\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(Q\mathrm{D}_{b,\omega,\varphi}^{j}\;\mathfrak{f})(a)\\&=\left(Q^{R}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r)-\frac{1}{\omega(b)}\frac{\delta(\alpha-k)}{(k+1-\alpha)}\\&\quad\times\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda_{\alpha-k}^{i}\Gamma(\gamma i+\mu)Q\omega(r)\big(\varphi(r)-\varphi(a)\big)^{\gamma i+\mu-k+j-1}}{(i)!\;\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}_{b,\omega,\varphi}^{j}\;\mathfrak{f})(b).\end{aligned}$$

Thus,

$$\begin{aligned}\left(Q^{C}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r)&=\left(Q^{R}\mathrm{D}_{b,\sigma,\omega,\varphi}^{\alpha,\beta,\gamma,\mu}\mathfrak{f}\right)(r)-\frac{1}{\omega(b)}\frac{\delta(\alpha-k)}{(k+1-\alpha)}\\&\quad\times\sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda_{\alpha-k}^{i}\Gamma(\gamma i+\mu)Q\omega(r)\big(\varphi(r)-\varphi(a)\big)^{\gamma i+\mu-k+j-1}}{(i)!\;\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}_{b,\omega,\varphi}^{j}\;\mathfrak{f})(b).\end{aligned}$$

By operating general Q-operator on both sides of the above equation, it implies that

$$\left({}^{C}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) = \left({}^{R}\mathrm{D}^{\alpha,\beta,\gamma,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\right)(r) - \frac{1}{\omega(b)}\frac{\delta(\alpha-k)}{(k+1-\alpha)} \times \sum_{i=0}^{\infty}\sum_{j=0}^{k}\frac{(\sigma)_i\lambda^i_{\alpha-k}\Gamma(\gamma i+\mu)\omega(r)\big(\varphi(b)-\varphi(r)\big)^{\gamma i+\mu-k+j-1}}{(i)!\,\Gamma(\beta i+\mu)\Gamma(\gamma i+\mu-k+j)}(\mathrm{D}^{j}_{b,\omega,\varphi}\,\mathfrak{f})(b).$$

□

Proposition 5.7 *The following relations hold:*

(i) $\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\,\mathfrak{f})(r) = \mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(b)-\varphi(r))^i}{i!}(\mathrm{D}^{i}_{b,\omega,\varphi}\mathfrak{f})(b),$

(ii) ${}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}(\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f})(r) = \begin{cases}\mathfrak{f}(r) - \omega(r)\mathrm{E}^{\beta,\sigma}_{\beta,1}\big(\lambda_{\alpha-k},(\varphi(b)-\varphi(r))\big)(\omega^{-1}\,\mathfrak{f})(b), & \text{if } \mu=1;\\ \mathfrak{f}(r), & \text{if } \mu\neq 1, Re(1-\mu)>0, Re(i\beta-\mu+1)>0.\end{cases}$

Proof

(i) According to Lemma 5.4, one finds that

$$\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\,Q\mathfrak{f})(r) = Q\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\,\mathfrak{f})(r).$$

Thus, in view of Proposition 4.3 part (i), one obtains

$$\begin{aligned}\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\,Q\mathfrak{f})(r) &= Q\mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(r)-\varphi(a))^i}{i!}(\mathrm{D}^{i}_{a,{}_RQ\omega,\varphi}Q\mathfrak{f})(a)\\ &= Q\mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(r)-\varphi(a))^i}{i!}(Q\mathrm{D}^{i}_{b,\omega,\varphi}\mathfrak{f})(a)\\ &= Q\mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(r)-\varphi(a))^i}{i!}(\mathrm{D}^{i}_{b,\omega,\varphi}\mathfrak{f})(b).\end{aligned}$$

Therefore,

$$Q\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\,\mathfrak{f})(r) = Q\mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(r)-\varphi(a))^i}{i!}(\mathrm{D}^{i}_{b,\omega,\varphi}\mathfrak{f})(b).$$

By operating general Q-operator on both sides of the above equation, it implies that

$$\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\,\mathfrak{f})(r) = \mathfrak{f}(r) - \sum_{i=0}^{k}\frac{(\varphi(b)-\varphi(r))^i}{i!}(\mathrm{D}^{i}_{b,\omega,\varphi}\mathfrak{f})(b),$$

and it yields the result.

(ii) According to Lemma 5.4, one finds that

$$ {}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f})(r) = Q\,{}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}(\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\, \mathfrak{f})(r). $$

Thus, in view of Proposition 4.3 part (ii), one gets

$$ \begin{aligned} {}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,{}_RQ\omega,\varphi}\, Q\mathfrak{f})(r) &= \begin{cases} Q\mathfrak{f}(r) - \dfrac{\mathrm{E}^{\beta,\sigma}_{\beta,1}\big(\lambda_{\alpha-k},(\varphi(r)-\varphi(a))\big)}{{}_RQ\omega(r)}\big(\dfrac{Q\mathfrak{f}}{Q\omega}\big)(a), & \text{if } \mu = 1; \\ Q\mathfrak{f}(r), \quad \text{if } \mu \neq 1, Re(1-\mu) > 0, Re(i\beta - \mu + 1) > 0 \end{cases} \\ &= \begin{cases} Q\mathfrak{f}(r) - Q\omega(r)\mathrm{E}^{\beta,\sigma}_{\beta,1}\big(\lambda_{\alpha-k},(\varphi(r)-\varphi(a))\big)(\omega^{-1}\,\mathfrak{f})(b), & \text{if } \mu = 1; \\ Q\mathfrak{f}(r), \quad \text{if } \mu \neq 1, Re(1-\mu) > 0, Re(i\beta - \mu + 1) > 0. \end{cases} \end{aligned} $$

Then,

$$ Q\,{}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}(\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\, \mathfrak{f})(r) = \begin{cases} Q\mathfrak{f}(r) - Q\omega(r)\mathrm{E}^{\beta,\sigma}_{\beta,1}\big(\lambda_{\alpha-k},(\varphi(r)-\varphi(a))\big)(\omega^{-1}\,\mathfrak{f})(b), & \text{if } \mu = 1; \\ Q\mathfrak{f}(r), \quad \text{if } \mu \neq 1, Re(1-\mu) > 0, Re(i\beta - \mu + 1) > 0. \end{cases} $$

By operating general Q-operator on both sides of the above equation, we find

$$ {}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\big(\mathrm{I}^{\alpha,\beta,\mu}_{b,\sigma,\omega,\varphi}\mathfrak{f}\big)(r) = \begin{cases} \mathfrak{f}(r) - \omega(r)\mathrm{E}^{\beta,\sigma}_{\beta,1}\big(\lambda_{\alpha-k},(\varphi(b)-\varphi(r))\big)(\omega^{-1}\,\mathfrak{f})(b), & \text{if } \mu = 1; \\ \mathfrak{f}(r), \quad \text{if } \mu \neq 1, Re(1-\mu) > 0, Re(i\beta - \mu + 1) > 0. \end{cases} $$

Thus, the result is fulfilled. □

6 Application to fractional integro-differential equations

This part applies the new derivative to exploring the solution of the following fractional IDEs involving (ω,φ)-Caputo derivative of the modified GMLF:

$$ \begin{cases} ({}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\, x)(r) = \mathfrak{f}\left(r, x(r), \displaystyle\int_a^r g(r,s,x(s))ds\right), \quad r \in [a,b], \\ c_1(\omega x)(a) + c_2(\omega x)(b) = c_3, \\ d_1(\omega x)'_{\varphi}(a) + d_2(\omega x)'_{\varphi}(b) = d_3, \quad c_i, d_i \in \mathbb{R}, i = 1,2,3, \end{cases} \tag{6.1} $$

where ${}^{C}\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}$ is the (ω,φ)-Caputo fractional derivative with modified GMLF, for $\alpha \in (1,2)$, $\beta > 0$, $Re(\mu) \neq 1$, $\sigma \in \mathbb{R}$, and the functions $\mathfrak{f} : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Furthermore, $c_1 + c_2 \neq 0$, and $d_1 + d_2 \neq 0$.

Let $\Lambda := \mathcal{C}(J = [a,b], \mathbb{R})$ symbolize the Banach space of all continuous functions x with the supremum norm $\|x\| = \sup_{r\in[a,b]} |x(r)|$, and the weighted Banach space is defined as:

$$\mathcal{C}_\omega(J=[a,b],\mathbb{R})=\{x\in\mathcal{C}(J,\mathbb{R})\big|(\omega x)(r)\in\mathcal{C}(J,\mathbb{R})\},$$

equipped with the norm

$$\|x\|_\omega=\sup_{r\in|a,b|}\big|(\omega x)(r)\big|,$$

where $(\omega x)(r)=\omega(r)x(r)$.

Lemma 6.1 *Let $\mathfrak{h}\in\mathcal{C}_\omega(J,\mathbb{R})$, and $\varphi\in\mathcal{C}^1(J,\mathbb{R})$, $\alpha\in(1,2)$, and $Re(\mu)\neq 1$. The solution of (ω,φ)-Caputo fractional* `IDEs`*:*

$$\begin{cases}({}^C\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\,x)(r)=\mathfrak{h}(r), \quad r\in[a,b],\\ c_1(\omega x)(a)+c_2(\omega x)(b)=c_3,\\ d_1(\omega x)'_\varphi(a)+d_2(\omega x)'_\varphi(b)=d_3, \quad c_i,d_i\in\mathbb{R}, i=1,2,3,\end{cases}\tag{6.2}$$

is given by the form:

$$\begin{aligned}x(r)&=\omega^{-1}(r)\Big[\psi_1-\psi_2\left(\varphi(b)-\varphi(a)\right)\Big(\psi_3-\psi_4\omega(b)\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h}(b)\Big)-\psi_2\omega(b)\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h}(b)\Big]\\&\quad+\omega^{-1}(r)(\varphi(r)-\varphi(a))\Big(\psi_3-\psi_4\omega(b)\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h}(b)\Big)+\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h}(r),\end{aligned}\tag{6.3}$$

where $\psi_1=\dfrac{c_3}{c_1+c_2}$, $\psi_2=\dfrac{c_2}{c_1+c_2}$, $\psi_3=\dfrac{d_3}{d_1+d_2}$, $\psi_4=\dfrac{d_2}{d_1+d_2}$.

Proof By operating $\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}$ on both sides of Eq.(6.2), we get

$$(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\ {}^C\mathrm{D}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\,x)(r)=(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h})(r).$$

Thus, by using Proposition 4.3, part (i), one has

$$x(r)=\omega^{-1}(r)\big[(\omega x)(a)+(\varphi(r)-\varphi(a))(\omega x)'_\varphi(a)\big]+(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h})(r),\tag{6.4}$$

that is

$$(\omega x)(r)=(\omega x)(a)+(\varphi(r)-\varphi(a))(\omega x)'_\varphi(a)+\omega(r)(\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\ \mathfrak{h})(r).\tag{6.5}$$

Using the boundary condition $c_1(\omega x)(a)+c_2(\omega x)(b)=c_3$, we find

$$c_1(\omega x)(a)+c_2(\omega x)(a)+c_2(\varphi(b)-\varphi(a))(\omega x)'_\varphi(a)+c_2\,\omega(b)\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h}(b)=c_3.$$

It implies that

$$(\omega x)(a)=\frac{1}{c_1+c_2}\Big[c_3-c_2\left(\varphi(b)-\varphi(a)\right)(\omega x)'_\varphi(a)-c_2\,\omega(b)\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{h}(b)\Big].\tag{6.6}$$

Also, by using the boundary condition $d_1(\omega x)'_{\varphi}(a) + d_2(\omega x)'_{\varphi}(b) = d_3$, we obtain

$$d_1(\omega x)'_{\varphi}(a) + d_2(\omega x)'_{\varphi}(a) + d_2 \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\delta(\alpha-1)} \left({}^{RL}\mathrm{I}_{a,\varphi}^{i\beta-\mu+1}\, \omega\mathfrak{h}\right)(b) = d_3,$$

which yields that

$$\begin{aligned}(\omega x)'_{\varphi}(a) &= \frac{1}{d_1+d_2}\Big[d_3 - d_2 \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\delta(\alpha-1)} \left({}^{RL}\mathrm{I}_{a,\varphi}^{i\beta-\mu+1}\, \omega\mathfrak{h}\right)(b)\Big] \\ &= \frac{1}{d_1+d_2}\Big[d_3 - d_2\omega(b)\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big].\end{aligned} \tag{6.7}$$

Now, substituting the values of $(\omega x)(a)$ and $(\omega x)'_{\varphi}(a)$ into Eq.(6.4), we infer that

$$\begin{aligned}x(r) &= \omega^{-1}(r)\Big[\frac{1}{c_1+c_2}\Big[c_3 - c_2\,(\varphi(b)-\varphi(a))\Big(\frac{d_3}{d_1+d_2} - \frac{d_2\omega(b)}{d_1+d_2}\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big) \\ &\quad - c_2\,\omega(b)\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big]\Big] + \omega^{-1}(r)(\varphi(r)-\varphi(a))\Big(\frac{d_3}{d_1+d_2} - \frac{d_2\omega(b)}{d_1+d_2}\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big) \\ &\quad + \mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(r) = \omega^{-1}(r)\Big[\psi_1 - \psi_2\,(\varphi(b)-\varphi(a))\Big(\psi_3 - \psi_4\omega(b)\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big) \\ &\quad - \psi_2\omega(b)\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big] + \omega^{-1}(r)(\varphi(r)-\varphi(a))\Big(\psi_3 - \psi_4\omega(b)\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(b)\Big) \\ &\quad + \mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{h}(r).\end{aligned}$$

Hence, the required result follows. □

Next, according to Lemma 6.1, the solution of (ω, φ)-Caputo fractional integro-differential (6.1) is given as:

$$\begin{aligned}x(r) &= \frac{\psi_1}{\omega(r)} - \frac{\psi_2\ \psi_3}{\omega(r)}\,(\varphi(b)-\varphi(a)) + \frac{\psi_3}{\omega(r)}(\varphi(r)-\varphi(a)) \\ &\quad + \frac{\psi_2\ \psi_4\ \omega(b)}{\omega(r)}\,(\varphi(b)-\varphi(a))\ \mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{f}\left(b, x(b), \int_a^b g(b,s,x(s))ds\right) \\ &\quad - \frac{\psi_2\ \omega(b)}{\omega(r)}\mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{f}\left(b, x(b), \int_a^b g(b,s,x(s))ds\right) \\ &\quad - \frac{\psi_4\ \omega(b)}{\omega(r)}\ (\varphi(r)-\varphi(a))\ \mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{f}\left(b, x(b), \int_a^b g(b,s,x(s))ds\right) \\ &\quad + \mathrm{I}_{a,\sigma,\omega,\varphi}^{\alpha,\beta,\mu}\mathfrak{f}\left(r, x(r), \int_a^r g(r,s,x(s))ds\right),\end{aligned} \tag{6.8}$$

where $\psi_1 = \dfrac{c_3}{c_1+c_2}$, $\psi_2 = \dfrac{c_2}{c_1+c_2}$, $\psi_3 = \dfrac{d_3}{d_1+d_2}$, $\psi_4 = \dfrac{d_2}{d_1+d_2}$.

6.1 Existence and uniqueness

This subsection investigates criteria of the uniqueness solution for the problem (6.1), by utilizing the Banach's FPT. So, we assume that the below assumption is fulfilled:

$(\mathbf{A_1})$ There exist the constants $q_i > 0$, $(i = 1, 2, 3)$, such that $\forall r \in [a, b]$ and $x_j, \bar{x}_j \in \mathbb{R}$, $(j = 1, 2)$ satisfy:

$$|\mathfrak{f}(r, x_1, x_2) - \mathfrak{f}(r, \bar{x}_1, \bar{x}_2)| \leqslant q_1|x_1 - \bar{x}_1| + q_2|x_2 - \bar{x}_2|, \; and \; |g(r, s, x_1) - g(r, s, \bar{x}_1)| \leqslant q_3|x_1 - \bar{x}_1|.$$

Theorem 6.2 *Let $\mathbf{A_1}$ fulfilled, and if*

$$\tau_1 := \Big(q_1 + q_2 q_3 (b-a)\Big)\Big[\Big(\big|\psi_2 \; \psi_4\big| + \big|\psi_4\big|\Big)(\varphi(b) - \varphi(a)) + \big|\psi_2\big| + 1\Big]$$
$$\times \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,\big|\delta(\alpha-1)\big|\,\Gamma(i\beta - \mu + 3)} \; (\varphi(b) - \varphi(a))^{i\beta-\mu+2} < 1.$$

such that $\mu \neq 1$. Then, there is one and only one solution for the (ω, φ)-Caputo fractional IDEs *(6.1).*

Proof Firstly, we will define the operator $\Pi : \mathcal{C}_\omega(J, \mathbb{R}) \to \mathcal{C}_\omega(J, \mathbb{R})$ as follows:

$$\begin{aligned}
(\Pi x(r)) = {} & \frac{\psi_1}{\omega(r)} - \frac{\psi_2 \; \psi_3}{\omega(r)} (\varphi(b) - \varphi(a)) + \frac{\psi_3}{\omega(r)} (\varphi(r) - \varphi(a)) \\
& + \frac{\psi_2 \; \psi_4 \; \omega(b)}{\omega(r)} (\varphi(b) - \varphi(a)) \; \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi} \mathfrak{f}\left(b, x(b), \int_a^b g(b, s, x(s)) ds\right) \\
& - \frac{\psi_2 \; \omega(b)}{\omega(r)} \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi} \mathfrak{f}\left(b, x(b), \int_a^b g(b, s, x(s)) ds\right) \\
& - \frac{\psi_4 \; \omega(b)}{\omega(r)} (\varphi(r) - \varphi(a)) \; \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi} \mathfrak{f}\left(b, x(b), \int_a^b g(b, s, x(s)) ds\right) \\
& + \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi} \mathfrak{f}\left(r, x(r), \int_a^r g(r, s, x(s)) ds\right).
\end{aligned} \tag{6.9}$$

Now, we will use the Banach's FPT to explore uniqueness of the solution to the (ω, φ) -Caputo fractional IDEs (6.1). Then, let $x, \bar{x} \in \mathcal{C}_\omega(J, \mathbb{R})$, $r \in [a, b]$, and by using $\mathbf{A_1}$, we have

$$\begin{aligned}
& \big|\omega(r)\Pi x(r) - \omega(r)\Pi \bar{x}(r)\big| \\
& \leq \Big(|\psi_2 \; \psi_4 \; \omega(b)| \; |(\varphi(b) - \varphi(a))| + |\psi_2 \; \omega(b)| + |\psi_4 \; \omega(b)| \; |(\varphi(r) - \varphi(a))|\Big) \\
& \quad \times \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\Big|\mathfrak{f}\Big(b, x(b), \int_a^b g(b, s, x(s)) ds\Big) - \mathfrak{f}\Big(b, \bar{x}(b), \int_a^b g(b, s, \bar{x}(s)) ds\Big)\Big| \\
& + \omega(r) \; \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\Big|\mathfrak{f}\Big(r, x(r), \int_a^r g(r, s, x(s)) ds\Big) - \mathfrak{f}\Big(r, \bar{x}(r), \int_a^r g(r, s, \bar{x}(s)) ds\Big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \Big(|\psi_2 \ \psi_4 \ \omega(b)| \ |(\varphi(b) - \varphi(a))| + |\psi_2 \ \omega(b)| + |\psi_4 \ \omega(b)| \ |(\varphi(r) - \varphi(a))| \Big) \\
&\quad \times \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} |\delta(\alpha-1)| \, \omega(b) \, \Gamma(i\beta - \mu + 2)} \int_a^b \varphi'(s)\omega(s)(\varphi(b) - \varphi(s))^{i\beta-\mu+1} \\
&\qquad \times \Big| \mathfrak{f}\Big(s, x(s), \int_a^s g(s,t,x(t))dt\Big) - \mathfrak{f}\Big(s, \bar{x}(s), \int_a^s g(s,t,\bar{x}(t))dt\Big) \Big| ds \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} |\delta(\alpha-1)| \, \Gamma(i\beta - \mu + 2)} \int_a^r \varphi'(s)\omega(s)(\varphi(r) - \varphi(s))^{i\beta-\mu+1} \\
&\quad \times \Big| \mathfrak{f}\Big(s, x(s), \int_a^s g(s,t,x(t))dt\Big) - \mathfrak{f}\Big(s, \bar{x}(s), \int_a^s g(s,t,\bar{x}(t))dt\Big) \Big| ds \\
&\le \Big(|\psi_2 \ \psi_4 \ \omega(b)| \ |(\varphi(b) - \varphi(a))| + |\psi_2 \ \omega(b)| + |\psi_4 \ \omega(b)| \ |(\varphi(r) - \varphi(a))| \Big) \\
&\quad \times \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} |\delta(\alpha-1)| \, \omega(b) \, \Gamma(i\beta - \mu + 2)} \int_a^b \varphi'(s)\omega(s)(\varphi(b) - \varphi(s))^{i\beta-\mu+1} \\
&\quad \times \Big(q_1 \ \Big|x(s) - \bar{x}(s)\Big| + q_2 \ \Big| \int_a^s g(s,t,x(t))dt - \int_a^s g(s,t,\bar{x}(t))dt \Big| \Big) ds \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} |\delta(\alpha-1)| \, \Gamma(i\beta - \mu + 2)} \int_a^r \varphi'(s)\omega(s)(\varphi(r) - \varphi(s))^{i\beta-\mu+1} \\
&\times \Big(q_1 \ \Big|x(s) - \bar{x}(s)\Big| + q_2 \ \Big| \int_a^r g(s,t,x(t))dt - \int_a^r g(s,t,\bar{x}(t))dt \Big| \Big) ds \\
&\le \Big(|\psi_2 \ \psi_4| \ |(\varphi(b) - \varphi(a))| + |\psi_2| + |\psi_4| \ |(\varphi(r) - \varphi(a))| \Big) \\
&\quad \times \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} \Big|\delta(\alpha-1)\Big| \, \Gamma(i\beta - \mu + 2)} \int_a^b \varphi'(s)(\varphi(b) - \varphi(s))^{i\beta-\mu+1} \\
&\quad \times \Big(q_1\omega(s)|x(s) - \bar{x}(s)| + q_2\omega(s) \int_a^s q_3|x(t) - \bar{x}(t)|dt \Big) ds \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} \Big|\delta(\alpha-1)\Big| \, \Gamma(i\beta - \mu + 2)} \int_a^r \varphi'(s)(\varphi(r) - \varphi(s))^{i\beta-\mu+1} \\
&\quad \times \Big(q_1\omega(s)|x(s) - \bar{x}(s)| + q_2\omega(s) \int_a^s q_3|x(t) - \bar{x}(t)|dt \Big) ds \\
&\le \sum_{i=0}^{\infty} \mathbf{C}(\sigma, i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1} \Big|\delta(\alpha-1)\Big| \, \Gamma(i\beta - \mu + 3)} \big(q_1 + q_2 q_3 (b-a)\big) \left(\varphi(b) - \varphi(a)\right)^{i\beta-\mu+2} \\
&\quad \times \Big[\Big(\Big|\psi_2 \ \psi_4\Big| + \Big|\psi_4\Big| \Big) \ (\varphi(b) - \varphi(a)) + \Big|\psi_2\Big| + 1 \Big] \|x - \bar{x}\|_{\omega}.
\end{aligned}
$$

Hence,

$$
\|\Pi x - \Pi \bar{x}\|_{\omega} \le \tau_1 \|x - \bar{x}\|_{\omega}.
$$

By condition (6.2), the Π is a contraction operator. Then, in view of the Banach's FPT 2.4, the (ω, φ)-Caputo fractional IDEs (6.1) owns one solution. □

Next, the existence of solution to the (ω, φ)-Caputo fractional IDEs (6.1) will discuss by using Schaefer's FPT. Regarding this, we propose the following hypotheses:

$(\mathbf{A_2})$ There is a constant $\omega^* > 0$, where $\omega^* = \sup_{s \in [a,b]} \omega(s)$.

($\mathbf{A_3}$) For all $r \in [a,b]$ and $x_i \in \mathbb{R}$, $(i=1,2)$, there are constants $\epsilon_j > 0\,(j=\overline{1,5})$, satisfy $|\mathfrak{f}(r,x_1,x_2)| \leq \epsilon_1 + \epsilon_2|x_1| + \epsilon_3|x_2|$, and $|g(r,s,x_1)| \leq \epsilon_4 + \epsilon_5|x_1|$.

Theorem 6.3 *Let $\mathbf{A_2}$ and $\mathbf{A_3}$ hold. Then, there exist solutions for the system (6.1), provided that*

$$\begin{aligned}\tau_2 &:= \big(\epsilon_2 + \epsilon_3\epsilon_5(b-a)\big)\Big(|\psi_2| + (|\psi_2\ \psi_4| + |\psi_4|)\ (|\varphi(b)-\varphi(a)|) + 1\Big)\\ &\quad\times \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+3)}(\varphi(b)-\varphi(a))^{i\beta-\mu+2} < 1,\end{aligned} \tag{6.10}$$

such that $\mu \neq 1$.

Proof To examine constraints of the Schaefer's FPT, we introduce operator $\Pi : \mathcal{C}_\omega(J,\mathbb{R}) \longrightarrow \mathcal{C}_\omega(J,\mathbb{R})$ as defined in (6.2). Moreover, we take the closed bounded convex ball $D_\zeta = \{x \in \mathcal{C}_\omega(J,\mathbb{R}) : \|x\|_\omega \leq \zeta, \zeta > 0\}$, with radius $\zeta \geq \dfrac{\tau_3}{1-\tau_2}$, and $\tau_2 < 1$ such that

$$\begin{aligned}\tau_3 &:= \Big(|\psi_1| + (|\psi_2\ \psi_3| + |\psi_3|)\ (|\varphi(b)-\varphi(a)|)\Big) + \big(\omega^*(\epsilon_1 + \epsilon_3\epsilon_4(b-a))\big)(\varphi(b)-\varphi(a))^{i\beta-\mu+2}\\ &\quad\times \Big(|\psi_2| + (|\psi_2\ \psi_4| + |\psi_4|)\ (|\varphi(b)-\varphi(a)|) + 1\Big)\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+3)}.\end{aligned}$$

This theorem will be rigorously verified in two sequential steps.

First step: We prove the operator $\Pi : \mathcal{C}_\omega(J,\mathbb{R}) \longrightarrow \mathcal{C}_\omega(J,\mathbb{R})$ is completely continuous. So, consider a sequence $\{x_n\}_{n\in\mathbb{N}} \longrightarrow x$ in the ball D_ζ as $n \longrightarrow \infty$. Thus, according to the continuity of functions $\mathfrak{f}$ and g, as well as using the Lebesgue dominated convergence result, one gets

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\Pi x_n)(r) &= \frac{\psi_1}{\omega(r)} - \frac{\psi_2\ \psi_3}{\omega(r)}\,(\varphi(b)-\varphi(a)) + \frac{\psi_3}{\omega(r)}(\varphi(r)-\varphi(a))\\ &\quad+ \frac{\psi_2\ \psi_4\ \omega(b)}{\omega(r)}\,(\varphi(b)-\varphi(a))\ \lim_{n\longrightarrow\infty}\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\left(b, x_n(b), \int_a^b g(b,s,x_n(s))ds\right)\\ &\quad- \frac{\psi_2\ \omega(b)}{\omega(r)}\ \lim_{n\longrightarrow\infty}\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\left(b, x_n(b), \int_a^b g(b,s,x_n(s))ds\right)\\ &\quad- \frac{\psi_4\ \omega(b)}{\omega(r)}\ (\varphi(r)-\varphi(a))\ \lim_{n\longrightarrow\infty}\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\left(b, x_n(b), \int_a^b g(b,s,x_n(s))ds\right)\\ &\quad+ \lim_{n\longrightarrow\infty}\mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\left(r, x_n(r), \int_a^r g(r,s,x_n(s))ds\right)\\ &= (\Pi x)(r).\end{aligned}$$

Hence the operator Π is continuous.

Now, we show that $\Pi D_\zeta \subset D_\zeta$. Accordingly to $\mathbf{A_2}$ and $\mathbf{A_3}$, for $x \in D_\zeta$, and $r \in [a,b]$, we acquire

$$
\begin{aligned}
&\big|\omega(r)(\Pi x)(r)\big| \\
&\leq \Big(|\psi_1| + |\psi_2\ \psi_3|\ |\varphi(b)-\varphi(a)| + |\psi_3|\ |\varphi(r)-\varphi(a)|\Big) \\
&+ \Big(|\psi_2| + \big(|\psi_2\ \psi_4|\ |\varphi(b)-\varphi(a)| + |\psi_4|\ \big|\varphi(r)-\varphi(a)\big|\Big) \\
&\quad \times \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+2)} \\
&\quad \times \int_a^b \varphi'(s)\omega(s)(\varphi(b)-\varphi(s))^{i\beta-\mu+1}\Big|\mathfrak{f}\Big(s,x(s),\int_a^s g(s,t,x(t))dt\Big)\Big|ds \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+2)} \\
&\quad \times \int_a^r \varphi'(s)\omega(s)(\varphi(r)-\varphi(s))^{i\beta-\mu+1}\Big|\mathfrak{f}\Big(s,x(s),\int_a^s g(s,t,x(t))dt\Big)\Big|ds \\
&\leq \Big(|\psi_1| + |\psi_2\ \psi_3|\ |\varphi(b)-\varphi(a)| + |\psi_3|\ |\varphi(r)-\varphi(a)|\Big) \\
&+ \Big(|\psi_2| + (|\psi_2\ \psi_4|\ |\varphi(b)-\varphi(a)| + |\psi_4|\ |\varphi(r)-\varphi(a)|\Big) \\
&\times \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+2)} \\
&\times \int_a^b \varphi'(s)(\varphi(b)-\varphi(s))^{i\beta-\mu+1}\Big(\omega(s)\big(\ (\epsilon_1+\epsilon_2|x(s)|) + \epsilon_3(\epsilon_4+\epsilon_5(|x(t)|))(s-a)\ \big)\Big)ds \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,\big|\delta(\alpha-1)\big|\,\Gamma(i\beta-\mu+2)} \\
&\times \int_a^r \varphi'(s)(\varphi(r)-\varphi(s))^{i\beta-\mu+1}\Big(\omega(s)\big(\ (\epsilon_1+\epsilon_2|x(s)|) + \epsilon_3(\epsilon_4+\epsilon_5(|x(t)|))(s-a)\ \big)\Big)ds \\
&\leq \Big(|\psi_1| + (|\psi_2\ \psi_3| + |\psi_3|)\ (|\varphi(b)-\varphi(a)|)\Big) \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+3)} \\
&\times \big(\omega^*(\epsilon_1+\epsilon_3\epsilon_4(b-a))\big)(\varphi(b)-\varphi(a))^{i\beta-\mu+2}\Big(|\psi_2| + (|\psi_2\ \psi_4| + |\psi_4|)\ (|\varphi(b)-\varphi(a)|) + 1\Big) \\
&+ \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+3)} \big(\epsilon_2+\epsilon_3\epsilon_5(b-a)\big)(\varphi(b)-\varphi(a))^{i\beta-\mu+2} \\
&\times \Big(|\psi_2| + (|\psi_2\ \psi_4| + |\psi_4|)\ (|\varphi(b)-\varphi(a)|) + 1\Big)\ \|x\|_\omega \\
&\leq \tau_3 + \tau_2\,\|x\|_\omega \\
&\leq \tau_3 + \tau_2\,\zeta \leq \zeta.
\end{aligned}
\tag{6.11}
$$

Hence, we prove that $\Pi D_\zeta \subset D_\zeta$, that is $\Pi : D_\zeta \longrightarrow D_\zeta$.

Next, we want to prove that the operator $\Pi : D_\zeta \longrightarrow D_\zeta$ is equicontinuous on D_ζ. So, we obtain $\forall x \in D_\zeta$, and $a < r_1 < r_2 < b$, the following:

$$
\begin{aligned}
&|\omega(r_2)(\Pi x)(r_2) - \omega(r_1)(\Pi x)(r_1)| \\
&\leq |\psi_3|\ |\varphi(r_2)-\varphi(r_1)| + \omega(b)|\psi_4|\ |\varphi(r_2)-\varphi(r_1)| \times \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\Big|\mathfrak{f}\big(b,x(b),\int_a^b g(b,s,x(s))ds\big)\Big| \\
&+ \Big|\omega(r_2)\ \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\big(r_2,x(r_2),\int_a^{r_2} g(r_2,s,x(s))ds\big) - \omega(r_1)\ \mathrm{I}^{\alpha,\beta,\mu}_{a,\sigma,\omega,\varphi}\mathfrak{f}\big(r_1,x(r_1),\int_a^{r_1} g(r_1,s,x(s))ds\big)\Big| \\
&\leq |\psi_3|\ |\varphi(r_2)-\varphi(r_1)| + |\psi_4|\ |\varphi(r_2)-\varphi(r_1)| \times \sum_{i=0}^{\infty} \mathbf{C}(\sigma,i) \frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+2)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_a^b \varphi'(s)\omega(s)(\varphi(b)-\varphi(s))^{i\beta-\mu+1}\Big|\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)\Big|ds\\
&+\Bigg|\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\left|\delta(\alpha-1)\right|\Gamma(i\beta-\mu+2)}\\
&\times \int_a^{r_1} \varphi'(s)\omega(s)(\varphi(r_2)-\varphi(s))^{i\beta-\mu+1}\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)ds\\
&+\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\left|\delta(\alpha-1)\right|\Gamma(i\beta-\mu+2)}\\
&\times \int_{r_1}^{r_2} \varphi'(s)\omega(s)(\varphi(r_2)-\varphi(s))^{i\beta-\mu+1}\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)ds\\
&-\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\left|\delta(\alpha-1)\right|\Gamma(i\beta-\mu+2)}\\
&\times \int_a^{r_1} \varphi'(s)\omega(s)(\varphi(r_1)-\varphi(s))^{i\beta-\mu+1}\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)ds\Bigg|\\
&\le \left|\psi_3\right|\left|\varphi(r_2)-\varphi(r_1)\right| + \left|\psi_4\right|\left|\varphi(r_2)-\varphi(r_1)\right| \times \sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\left|\delta(\alpha-1)\right|\Gamma(i\beta-\mu+2)}\\
&\times \int_a^b \varphi'(s)\omega(s)(\varphi(b)-\varphi(s))^{i\beta-\mu+1}\Big|\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)\Big|ds\\
&+\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\left\|\delta(\alpha-1)\right\|\Gamma(i\beta-\mu+2)}\\
&\times \int_a^{r_1} \varphi'(s)\left[(\varphi(r_2)-\varphi(s))^{i\beta-\mu+1}-(\varphi(r_1)-\varphi(s))^{i\beta-\mu+1}\right]\Big|\omega(s)\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)\Big|ds\\
&+\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\left|\delta(\alpha-1)\right|\Gamma(i\beta-\mu+2)}\\
&\times \int_{r_1}^{r_2} \varphi'(s)(\varphi(r_2)-\varphi(s))^{i\beta-\mu+1}\Big|\omega(s)\mathfrak{f}\big(s,x(s),\int_a^s g(s,t,x(t))dt\big)\Big|ds\\
&\le \big|\psi_3\big|\,\big|\varphi(r_2)-\varphi(r_1)\big| + \big|\psi_4\big|\,\big|\varphi(r_2)-\varphi(r_1)\big|\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\big|\delta(\alpha-1)\big|\Gamma(i\beta-\mu+3)}\\
&\times (\varphi(b)-\varphi(a))^{i\beta-\mu+2}\big(\omega^*(\epsilon_1+\epsilon_3\epsilon_4(b-a)\big)+\big(\epsilon_2+\epsilon_3\epsilon_5(b-a)\big)\|x\|_\omega\big)\\
&-\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\big|\delta(\alpha-1)\big|\Gamma(i\beta-\mu+3)}\\
&\times \left[(\varphi(r_2)-\varphi(r_1))^{i\beta-\mu+2}-(\varphi(r_2)-\varphi(a))^{i\beta-\mu+2}+(\varphi(r_1)-\varphi(a))^{i\beta-\mu+2}\right]\\
&\times \big(\omega^*(\epsilon_1+\epsilon_3\epsilon_4(b-a)\big)+\big(\epsilon_2+\epsilon_3\epsilon_5(b-a)\big)\|x\|_\omega\big)\\
&+\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\big|\delta(\alpha-1)\big|\Gamma(i\beta-\mu+3)}\\
&\times (\varphi(r_2)-\varphi(r_1))^{i\beta-\mu+2}\big(\omega^*(\epsilon_1+\epsilon_3\epsilon_4(b-a)\big)+\big(\epsilon_2+\epsilon_3\epsilon_5(b-a)\big)\|x\|_\omega\big).
\end{aligned}
$$

It is clear that, $|\omega(r_2)(\Pi x)(r_2) - \omega(r_1)(\Pi x)(r_1)| \longrightarrow 0$ as $r_1 \longrightarrow r_2$. Then, Π is equicontinuous, and we conclude that Π is completely continuous due to the Ascoli-Arzelá theorem 2.6.

Second step: We want to demonstrate that the set $\Omega = \{x \in \mathcal{C}_\omega (J, \mathbb{R}) : x = \varrho \Pi x,\ 0 < \varrho < 1\}$ is bounded. So, for $x \in \Omega$, $r \in [a, b]$, and by Eq. (6.11), we find that

$$\|x\|_\omega = \sup_{r \in [a,b]} |\varrho\, \omega(r)\, \Pi\, x(r)| \leq \tau_3 + \tau_2 \|x\|_\omega,$$

which implies that

$$\|x\|_\omega = \frac{\tau_3}{1 - \tau_2}.$$

Therefore, Ω is bounded set. Hence, based on Theorem 2.5, there exists at least one solution for the (ω, φ)-Caputo fractional `IDEs` (6.1). □

7 Examples

This section demonstrates the efficacy of our findings by providing the next two mathematical applications:

Example 7.1 Contemplate the below (ω, φ)-Caputo fractional `IDEs`:

$$\begin{cases} ({}^C\mathrm{D}^{1.6,5,0.5}_{1,2,r,r^2}\, x)(r) = \dfrac{sin(r^2)}{1+r} + \dfrac{|x(r)|}{\frac{1}{20} + |x(r)|} + \dfrac{1}{12} \displaystyle\int_1^r \Big(r + s^2 + \dfrac{|x(s)|}{\frac{1}{216} + |x(s)|}\Big) ds, r \in [1, 2], \\ 8(r\, x)(1) + 4(r\, x)(2) = 3, \\ 9(r\, x)'_{r^2}(1) + 4(r\, x)'_{r^2}(2) = 5. \end{cases} \tag{7.1}$$

Here, $\alpha = 1.6$, $\beta = 5$, $\mu = 0.5$, $a = 1$, $b = 2$, $\sigma = 2$, $\omega(r) = r$, $\varphi(r) = r^2$, $\varphi(b) - \varphi(a) = 3$, and

$$\mathfrak{f}\left(r, x(r), \int_a^r g(r, s, x(s))ds\right) = \frac{sin(r^2)}{1+r} + \frac{|x(r)|}{\frac{1}{20} + |x(r)|} + \frac{1}{12} \int_1^r \Big(r + s^2 + \frac{|x(s)|}{\frac{1}{216} + |x(s)|}\Big) ds,$$

$$g(r, s, x(s)) = \Big(r + s^2 + \frac{|x(s)|}{\frac{1}{216} + |x(s)|}\Big).$$

Obviously, $\psi_1 = 0.25$, $\psi_2 = 0.33$, $\psi_3 = 0.38$, and $\psi_4 = 0.31$.

Also,

$$\begin{aligned} \big|\mathfrak{f}(r, x_1, x_2) - \mathfrak{f}(r, \bar{x}_1, \bar{x}_2)\big| &\leq \left|\frac{x_1}{\frac{1}{20} + x_1} - \frac{\bar{x}_1}{\frac{1}{20} + \bar{x}_1}\right| + \left|\frac{1}{12}(x_2 - \bar{x}_2)\right| \\ &\leq \frac{1}{20}|x_1 - \bar{x}_1| + \frac{1}{12}|x_2 - \bar{x}_2|, \end{aligned}$$

and

$$|g(r,s,x)-g(r,s,\bar{x})| \leq \left|\frac{x}{\frac{1}{216}+x}-\frac{\bar{x}}{\frac{1}{216}+\bar{x}}\right| \\ \leq \frac{1}{216}\ |x-\bar{x}|.$$

Thus, $q_1=\frac{1}{20}$, $q_2=\frac{1}{12}$, and $q_3=\frac{1}{216}$. Moreover, for $|\delta(\alpha-1)|=1$, so we compute that

$$\tau_1 := \Big(q_1+q_2q_3(b-a)\Big)\Big[\Big(|\psi_2\ \psi_4|+|\psi_4|\Big)\ \Big|(\varphi(b)-\varphi(a))\Big|+|\psi_2|+1\Big] \\ \times \sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,\big|\delta(\alpha-1)\big|\,\Gamma(i\beta-\mu+3)}\ (\varphi(b)-\varphi(a))^{i\beta-\mu+2} \approx\ 0.307206<1.$$

Additionally, Fig. 1 shows the values surface of τ_1 are less than one for vary fractional order $\alpha\in(1,2)$ on the interval $r\in[1,2]$.

So, the conditions of Theorem 6.2 hold. Hence, the (ω,φ)-Caputo fractional IDEs (7.1) possesses one solution.

Example 7.2 Let us take the following (ω,φ)-Caputo fractional IDEs:

$$\begin{cases} ({}^{C}\mathrm{D}^{1.2,7,1.5}_{1,2,r,\ln(r)}\ x)(r)=\ln\big(\frac{r}{2}\big)+\dfrac{2\sin(|x(r)|)}{3r}+\dfrac{1}{9}\displaystyle\int_1^r\Big(\frac{r-\sqrt{s}+1}{12}+\frac{\tan^{-1}(5|x(s)|)}{125}\Big)ds, \\ \frac{1}{5}(r\,x)(1)+\frac{2}{9}(r\,x)(e)=\frac{1}{13}, \\ \frac{1}{2}(r\,x)'_{\ln(r)}(1)+\frac{1}{4}(r\,x)'_{\ln(r)}(e)=\frac{1}{12}. \end{cases} \tag{7.2}$$

where $r\in[1,e]$, and we have $\alpha=1.2$, $\beta=7$, $\mu=1.5$, $a=1$, $b=e$, $\sigma=2$, $\omega(r)=r$, $\varphi(r)=\ln(r)$, and

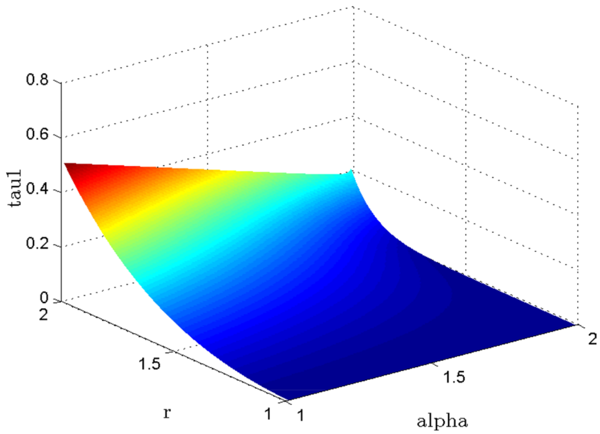
$$\mathfrak{f}\left(r,x(r),\int_a^r g(r,s,x(s))ds\right)=\ln\big(\frac{r}{2}\big)+\frac{2\sin(|x(r)|)}{30\,r}+\frac{1}{9}\int_1^r\Big(\frac{r-\sqrt{s}+1}{12}+\frac{\tan^{-1}(5|x(s)|)}{125}\Big)ds, \\ g(r,s,x(s))=\Big(\frac{r-s^2+1}{12}+\frac{\tan^{-1}(5|x(s)|)}{125}\Big).$$

Next, we can calculated that, $\psi_1=0.18219$, $\psi_2=0.52632$, $\psi_3=0.11111$, $\psi_4=0.33333$, and we find

$$\big|\mathfrak{f}(r,x_1,x_2)\big| \leq \ln(\frac{e}{2})+\frac{2}{30}|x_1|+\frac{1}{9}|x_2|, \\ \big|g(r,s,x_1)\big| \leq \frac{e}{12}+\frac{1}{25}|x_1|.$$

Hence, $\epsilon_1=\ln(\frac{e}{2})$, $\epsilon_2=\frac{2}{30}$, $\epsilon_3=\frac{1}{9}$, $\epsilon_4=\frac{e}{12}$, and $\epsilon_5=\frac{1}{25}$.

Fig. 1 3D plot of τ_1



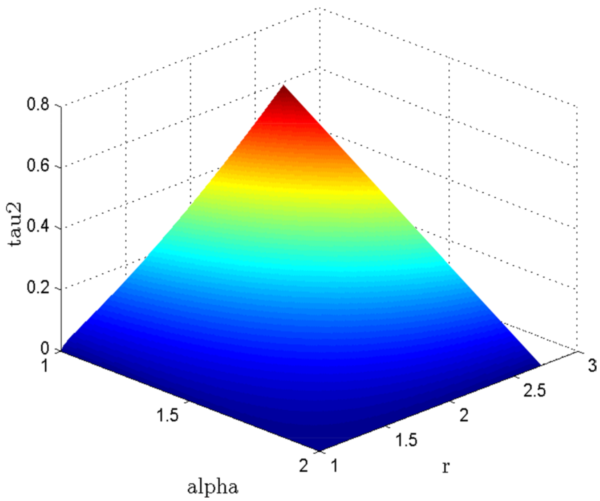
Then, by putting $|\delta(\alpha-1)|=1$, we can compute the following:

$$\tau_2 := \big(\epsilon_2+\epsilon_3\epsilon_5(b-a)\big)\Big(|\psi_2|+(|\psi_2\ \psi_4|+|\psi_4|)\ (|\varphi(b)-\varphi(a)|)+1\Big)$$
$$\times\sum_{i=0}^{\infty}\mathbf{C}(\sigma,i)\frac{(\alpha-1)^i}{(2-\alpha)^{i-1}\,|\delta(\alpha-1)|\,\Gamma(i\beta-\mu+3)}(\varphi(b)-\varphi(a))^{i\beta-\mu+2}\approx\ 0.142229<1.$$

Additionally, Fig. 2 shows the values surface of τ_2 are less than one for vary fractional order $\alpha\in(1,2)$ on the interval $r\in[1,e]$.

Hence, based on the hypotheses of Theorem 6.3, there are at least one solution of (ω,φ)-Caputo fractional `IDEs` (7.2).

Fig. 2 3D plot of τ_2



8 Conclusions

In this study, we extended the $\mathtt{L.S}$ $(\omega,\varphi)-$fractional derivatives and integrals involving the modified $\mathtt{GMLF}$ to higher orders in the setting of the $\mathtt{RL}$ and Caputo operators, and investigated some of their essential properties, such as their series versions, the relationship between Caputo and $\mathtt{RL}$ versions, the fundamental calculus theorem for these operators, and their (ω,φ)-Laplace transformation. Furthermore, we investigated some particular cases of the derived operators that exist in the literature, such as higher order fractional operators in the setting of the $\varphi-\mathtt{ABC}$, $\mathtt{ABC}$, $\varphi-\mathtt{ABR}$, $\mathtt{ABR}$, $(\omega,\varphi)-\mathtt{CFC}$, $\varphi-\mathtt{CFC}$, $\mathtt{CFC}$, $(\omega,\varphi)-\mathtt{CFR}$, $\varphi-\mathtt{CFR}$, and $\mathtt{CFR}$. Based on the results, we concluded for the first time the relationship between higher order $\mathtt{ABC}$ and $\mathtt{ABR}$ fractional derivatives, as well as the relationship between higher order $\mathtt{CFC}$ and $\mathtt{CFR}$ fractional derivatives.

Additionally, by utilizing the general Q-operator, we introduced its reciprocal operator (5.2), and then we presented the general fundamental identities between the $\mathtt{L\&R}$ sides of any (ω,φ)-fractional calculus, which is defined in (5.3). Accordingly, we concluded the higher orders $\mathtt{R.S}$ $(\omega,\varphi)-$fractional operators involving the modified $\mathtt{GMLF}$. As a result, we examined certain properties by utilizing the duality of the $\mathtt{L.S}$ approach. We demonstrated the duality of the $\mathtt{L\&R}$ sides of our proposed definitions by applying the fundamental identities for (ω,φ)-fractional calculus, utilizing the general Q-operator and its reciprocal. These identities (5.3) applicable for any (ω,φ)-fractional calculus, such as (ω,φ)-$\mathtt{RL}$, (ω,φ)-Caputo operators, (ω,φ)-Hilfer, (ω,φ)-Hadamard, (ω,φ)-Katugampola, (ω,φ)-$\mathtt{CF}$, and (ω,φ)-$\mathtt{AB}$ operators.

This research has addressed the $\mathtt{EaU}$ of solutions for the (ω,φ)-Caputo fractional $\mathtt{IDEs}$ (6.1) by utilizing Banach's and Schaefer's $\mathtt{FPTs}$. Finally, two examples were demonstrated to show the applicability of the findings.

As a future direction, this study opens new research areas. For instance, the researchers can solve $\mathtt{R.S}$ fractional differential problems by utilizing the duality of their $\mathtt{L.S}$ and the general identities of (ω,φ)-fractional calculus (5.3).

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Declarations

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