Commun. Korean Math. Soc. **41** (2026), No. 2, pp. 609–626
https://doi.org/10.4134/CKMS.c250242
pISSN: 1225-1763 / eISSN: 2234-3024

# INCLUSION RESULTS IN HARMONIC FUNCTION CLASSES USING POWER SERIES DISTRIBUTION

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Abstract. In this paper, we study inclusion relations for a previously introduced class of harmonic close-to-convex functions. Using operators defined via the Pascal and Poisson distribution series and applying suitable coefficient bounds, we obtain new inclusion results among the classes of harmonic convex functions, harmonic starlike functions, and the defined class. Our findings highlight the effectiveness of distribution series operators in geometric function theory.

## 1. Introduction

Let $\mathbb{C}$ denote the complex plane, and let

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

be the open unit disk. In [8], it was shown that a harmonic function

$$f = u + \overline{v}$$

is locally univalent and sense-preserving in $\mathbb{U}$ if and only if

$$|v'(z)| < |u'(z)|, \quad z \in \mathbb{U},$$

where $u$ and $v$ are analytic in $\mathbb{U}$.

A harmonic, univalent, and sense-preserving function in $\mathbb{U}$ is said to be normalized if it satisfies $f(0) = 0$ and $f_z(0) = 1$. The class of such functions is denoted by $\mathcal{SH}$. The subclass $\mathcal{SH}^0$ consists of functions in $\mathcal{SH}$ with $v'(0) = b_1 = 0$. The analytic and co-analytic parts of $f$ can be written as

$$u(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu, \quad v(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu. \tag{1}$$

If $v(z) \equiv 0$, then $\mathcal{SH}$ reduces to an analytic function in the class $\mathcal{S}$ of normalized univalent analytic functions on $\mathbb{U}$. It is clear that

$$\mathcal{S} \subset \mathcal{SH}^0 \subset \mathcal{SH}.$$

Received August 13, 2025; Revised December 18, 2025; Accepted January 23, 2026.
2020 *Mathematics Subject Classification.* 30C20, 30C45, 30C55.
*Key words and phrases.* Harmonic close-to-convex functions, Pascal distribution series, Poisson distribution series, inclusion relations.

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The subclasses $\mathcal{K}$ and $\mathcal{S}^*$ of $\mathcal{S}$ correspond to convex and starlike functions, respectively. Analogously, $\mathcal{KH}^0$ and $\mathcal{SH}^{*,0}$ denote the subclasses of $\mathcal{SH}^0$ consisting of convex and starlike harmonic mappings, respectively (see [3, 19]).

We now present some lemmas that will play a key role in the proofs of the subsequent results.

**Lemma 1.1** ([8]). *Let $f = u + \overline{v} \in \mathcal{KH}^0$, where $u$ and $v$ are given by* (1) *with $b_1 = 0$. Then, for $\nu \geq 2$,*

$$|a_\nu| \leq \frac{\nu+1}{2}, \quad |b_\nu| \leq \frac{\nu-1}{2}.$$

**Lemma 1.2** ([13]). *If $f = u + \overline{v}$ is given by* (1) *and*

$$\sum_{\nu=2}^{\infty}(\nu-\alpha)|a_\nu| + \sum_{\nu=2}^{\infty}(\nu+\alpha)|b_\nu| \leq 1-\alpha \tag{2}$$

*for some $\alpha$ with $0 \leq \alpha < 1$, then $f$ is harmonic, univalent, sense-preserving in $\mathbb{U}$, and belongs to the class $\mathcal{SH}^{*,0}(\alpha)$.*

**Lemma 1.3** ([13]). *If $f = u + \overline{v}$ is given by* (1) *and*

$$\sum_{\nu=2}^{\infty}\nu(\nu-\alpha)|a_\nu| + \sum_{\nu=2}^{\infty}\nu(\nu+\alpha)|b_\nu| \leq 1-\alpha \tag{3}$$

*for some $\alpha$ with $0 \leq \alpha < 1$, then $f \in \mathcal{KH}^0(\alpha)$ and is harmonic, univalent, and sense-preserving in $\mathbb{U}$.*

**Lemma 1.4** ([8]). *Let $u$ and $v$ be given by* (1) *with $b_1 = 0$. If $f = u+\overline{v} \in \mathcal{SH}^{*,0}$, then for $\nu \geq 2$,*

$$|a_\nu| \leq \frac{(2\nu+1)(\nu+1)}{6}, \quad |b_\nu| \leq \frac{(2\nu-1)(\nu-1)}{6}.$$

In this paper, we consider the harmonic close-to-convex function class introduced by Çakmak et al. [6]. We investigate inclusion relations for this class using operators defined through the Pascal and Poisson distribution series.

## 2. The class $\mathcal{RH}^0(\sigma, \tau, \xi)$

In this section, we recall the class $\mathcal{RH}^0(\sigma, \tau, \xi)$ of harmonic close-to-convex functions, which was introduced in [6]. We summarize its definition and list some known results that will be used in the following sections.

**Definition** ([6]). A function $f = u + \overline{v} \in \mathcal{H}^0$ is said to belong to the class $\mathcal{RH}^0(\sigma, \tau, \xi)$ if it satisfies

$$\begin{aligned} &\mathrm{Re}\Big[\sigma u'(z) + \tau z u''(z) + \frac{\tau-\sigma}{2}z^2 u'''(z) - \xi\Big] \\ &> \left|\sigma v'(z) + \tau z v''(z) + \frac{\tau-\sigma}{2}z^2 v'''(z)\right|, \end{aligned} \tag{4}$$

for $0 \leq \xi < \sigma \leq \tau$.

The first class defined by a third-order derivative operator was introduced by Yaşar and Yalçın [22], where it was shown that the inclusion of a third-order derivative preserves the close-to-convexity of the class. Subsequently, in [6], the class $\mathcal{R}_H^0(\sigma,\tau,\xi)$ was introduced via a third-order differential inequality, and within the same study it was proved that the newly defined class consists of close-to-convex harmonic functions.

In both of these works, the incorporation of third-order derivatives plays a crucial role. Despite increasing the order of differentiation, the fundamental geometric property of close-to-convexity is retained. Consequently, classes defined by third-order derivatives naturally generalize those constructed using first- and second-order derivatives, while preserving their essential geometric characteristics.

The strength and unifying nature of this approach are clearly illustrated by the following equivalences, which are obtained by selecting appropriate parameter values:

- $\mathcal{A}_H^0(\sigma,\sigma,\xi) \equiv \mathcal{R}_H^0(\sigma,\sigma,\xi)$ [7],
- $W_H^0 \equiv W_H^0(1) \equiv \mathcal{R}_H^0(1,1,0)$ [12,14],
- $W_H^0(1,\lambda) \equiv \mathcal{R}_H^0(1,1,\lambda)$ [18],
- $R_H^0\big(\delta,\frac{\delta-1}{2}\big) \equiv \mathcal{R}_H^0(1,\delta,0)$ [22].

Motivated by the fact that the class $\mathcal{R}_H^0(\sigma,\tau,\xi)$ is one of the most general close-to-convex classes defined via higher-order derivatives, this study focuses on the investigation of this class. By employing operators defined through the Pascal and Poisson distribution series, we examine its inclusion properties and thereby revisit and unify several known mapping results for its subclasses.

**Lemma 2.1** (Theorem 2.5, [6])**.** *If $f \in \mathcal{RH}^0(\sigma,\tau,\xi)$, then $f$ is close-to-convex in $\mathbb{U}$.*

**Lemma 2.2** (Theorem 2.6, [6])**.** *Let $f = u + \overline{v} \in \mathcal{RH}^0(\sigma,\tau,\xi)$. For $\nu \geq 2$,*

$$|b_\nu| \leq \frac{2(\sigma-\xi)}{\nu^2[2\sigma+(\tau-\sigma)(\nu-1)]}.$$

*Equality holds for*

$$f(z) = z + \frac{2(\sigma-\xi)}{\nu^2[2\sigma+(\tau-\sigma)(\nu-1)]}\overline{z}^\nu.$$

**Lemma 2.3** (Theorem 2.7, [6])**.** *Let $f = u + \overline{v} \in \mathcal{RH}^0(\sigma,\tau,\xi)$. For $\nu \geq 2$,*

$$\text{(i)}\quad |a_\nu| + |b_\nu| \leq \frac{4(\sigma-\xi)}{\nu^2[2\sigma+(\tau-\sigma)(\nu-1)]},$$

$$\text{(ii)}\quad \big||a_\nu| - |b_\nu|\big| \leq \frac{4(\sigma-\xi)}{\nu^2[2\sigma+(\tau-\sigma)(\nu-1)]},$$

$$\text{(iii)}\quad |a_\nu| \leq \frac{4(\sigma-\xi)}{\nu^2[2\sigma+(\tau-\sigma)(\nu-1)]}.$$

*Equality holds when*

$$f(z) = z + \sum_{\nu=2}^{\infty} \frac{4(\sigma - \xi)}{\nu^2[2\sigma + (\tau - \sigma)(\nu - 1)]} z^{\nu}.$$

**Lemma 2.4** ([6]). *If $f = u + \overline{v} \in \mathcal{H}^0$ satisfies*

$$\sum_{\nu=2}^{\infty} \nu^2[2\sigma + (\tau - \sigma)(\nu - 1)]\big(|a_\nu| + |b_\nu|\big) \leq 2(\sigma - \xi), \tag{5}$$

*then $f \in \mathcal{RH}^0(\sigma, \tau, \xi)$.*

## 3. Applications to Poisson distribution series

Recently, the Poisson distribution series has become a useful tool in Geometric Function Theory for studying various subclasses of analytic and harmonic univalent functions [1, 2, 15–17]. Inspired by previous research that used convolution operators involving hypergeometric and special functions to explore inclusion properties, this paper focuses on the class $\mathcal{RH}^0(\sigma, \tau, \xi)$ through an operator connected to the Poisson distribution series. We present new inclusion results relating the classes $\mathcal{KH}^0$, $\mathcal{SH}^{*,0}$, and $\mathcal{RH}^0(\sigma, \tau, \xi)$ by applying coefficient bounds introduced earlier.

For a positive parameter $\alpha > 0$, the Poisson distribution defines the power series

$$\Pi_\alpha(z) = z + \sum_{\nu=2}^{\infty} \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu - 1)!} z^{\nu},$$

which was introduced by Porwal [15]. Building on this, Porwal and Srivastava [17] proposed the operator

$$\Pi_{\alpha,\beta}(f)(z) = \Pi_\alpha(z) * u(z) + \overline{\Pi_\beta(z) * v(z)} = \mathfrak{U}(z) + \overline{\mathfrak{V}(z)},$$

where the analytic components $\mathfrak{U}$ and $\mathfrak{V}$ are given by

$$\begin{aligned} \mathfrak{U}(z) &= z + \sum_{\nu=2}^{\infty} \frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu - 1)!} a_\nu z^{\nu}, \\ \mathfrak{V}(z) &= b_1 z + \sum_{\nu=2}^{\infty} \frac{e^{-\beta}\beta^{\nu-1}}{(\nu - 1)!} b_\nu z^{\nu}. \end{aligned} \tag{6}$$

The following theorem establishes sufficient conditions under which the Poisson distribution operator maps the class $\mathcal{KH}^0$ into the more general class $\mathcal{RH}^0(\sigma, \tau, \xi)$.

**Theorem 3.1.** *Let $0 \leq \alpha, \beta < 1$ and $0 \leq \xi < \sigma \leq \tau$. If the inequality*

$$\begin{aligned} &(\tau - \sigma)\alpha^4 + (10\tau - 8\sigma)\alpha^3 + (24\tau - 10\sigma)\alpha^2 + (12\tau + 8\sigma)\alpha \\ &+ 4\sigma(1 - e^{-\alpha}) + (\tau - \sigma)\beta^4 + (8\tau - 6\sigma)\beta^3 \\ &+ (14\tau - 4\sigma)\beta^2 + 4(\tau + \sigma)\beta \leq 4(\sigma - \xi) \end{aligned} \tag{7}$$

*holds, then*

$$\Pi_{\alpha,\beta}\big(\mathcal{KH}^0\big) \subset \mathcal{RH}^0(\sigma,\tau,\xi).$$

*Proof.* Let $f = u + \overline{v} \in \mathcal{KH}^0$, where $u$ and $v$ have the form (1) with $b_1 = 0$. We aim to show that

$$\Pi_{\alpha,\beta}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{RH}^0(\sigma,\tau,\xi),$$

where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions given by (6) with $b_1 = 0$.

By Lemma 2.4, it suffices to verify that $\Phi_1 \le 2(\sigma - \xi)$, where

$$\Phi_1 := \sum_{\nu=2}^{\infty} \nu^2\big[2\sigma + (\tau-\sigma)(\nu-1)\big]\left(\left|\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!}a_\nu\right| + \left|\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}b_\nu\right|\right). \tag{8}$$

Using the coefficient bounds from Lemma 1.1, namely

$$|a_\nu| \le \frac{\nu+1}{2} \quad \text{and} \quad |b_\nu| \le \frac{\nu-1}{2} \quad \text{for } \nu \ge 2,$$

we obtain

$$\Phi_1 \le \frac{1}{2}\sum_{\nu=2}^{\infty}\big[2\sigma + (\tau-\sigma)(\nu-1)\big]\nu^2\left((\nu+1)\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} + (\nu-1)\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\right).$$

Expanding and rearranging terms, this can be written as

$$\begin{aligned}\Phi_1 \le{}& \frac{1}{2}\sum_{\nu=2}^{\infty}\big[(\tau-\sigma)\nu^4 + 2\sigma\nu^3 + (3\sigma-\tau)\nu^2\big]\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!}\\ &+ \frac{1}{2}\sum_{\nu=2}^{\infty}\big[(\tau-\sigma)\nu^4 + (4\sigma-2\tau)\nu^3 + (\tau-3\sigma)\nu^2\big]\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}.\end{aligned}$$

Using expansions that express powers of $\nu$ as combinations of descending products $(\nu-1)(\nu-2)\cdots$ with fixed coefficients, we rewrite the sums as

$$\begin{aligned}\Phi_1 \le \frac{1}{2}\Bigg\{&\sum_{\nu=2}^{\infty}\Big[(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)\\ &+ (10\tau-8\sigma)(\nu-1)(\nu-2)(\nu-3) + (24\tau-10\sigma)(\nu-1)(\nu-2)\\ &+ (12\tau+8\sigma)(\nu-1) + 4\sigma\Big]\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!}\\ &+ \sum_{\nu=2}^{\infty}\Big[(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)\\ &+ (8\tau-6\sigma)(\nu-1)(\nu-2)(\nu-3)\\ &+ (14\tau-4\sigma)(\nu-1)(\nu-2) + 4(\tau+\sigma)(\nu-1)\Big]\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}.\end{aligned}$$

By applying the necessary simplifications and rearranging the terms, the expression can be rewritten as

$$\begin{aligned}\Phi_1 \le \frac{1}{2}\Bigg\{&(\tau-\sigma)e^{-\alpha}\alpha^4\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}+(10\tau-8\sigma)e^{-\alpha}\alpha^3\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}\\&+(24\tau-10\sigma)e^{-\alpha}\alpha^2\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}+(12\tau+8\sigma)e^{-\alpha}\alpha\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}\\&+4\sigma e^{-\alpha}\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}+(\tau-\sigma)e^{-\beta}\beta^4\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}+(8\tau-6\sigma)e^{-\beta}\beta^3\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}\\&+(14\tau-4\sigma)e^{-\beta}\beta^2\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}+4(\tau+\sigma)e^{-\beta}\beta\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}\Bigg\}.\end{aligned}$$

Since $\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}=e^\alpha$, these sums simplify to

$$\begin{aligned}\Phi_1 \le \frac{1}{2}\Big\{&(\tau-\sigma)\alpha^4+(10\tau-8\sigma)\alpha^3+(24\tau-10\sigma)\alpha^2+(12\tau+8\sigma)\alpha\\&+4\sigma(1-e^{-\alpha})+(\tau-\sigma)\beta^4+(8\tau-6\sigma)\beta^3+(14\tau-4\sigma)\beta^2\\&+4(\tau+\sigma)\beta\Big\}.\end{aligned}$$

By the assumption (7), the right-hand side is at most $2(\sigma-\xi)$. ☐

**Theorem 3.2.** *Let $0\le\alpha,\beta<1$ and $0\le\xi<\sigma\le\tau$. Suppose the following inequality holds*:

$$\begin{aligned}&2(\tau-\sigma)\alpha^5+(31\tau-27\sigma)\alpha^4+(138\tau-92\sigma)\alpha^3+(192\tau-54\sigma)\alpha^2\\&+(60\tau+48\sigma)\alpha+128\sigma(1-e^{-\alpha})+2(\tau-\sigma)\beta^5+(25\tau-21\sigma)\beta^4\\&+(84\tau-50\sigma)\beta^3+(78\tau-12\sigma)\beta^2+(12\sigma+12\tau)\beta\le 12(\sigma-\xi).\end{aligned}\tag{9}$$

*Then, the inclusion*

$$\Pi_{\alpha,\beta}\big(\mathcal{SH}^{*,0}\big)\subset\mathcal{RH}^0(\sigma,\tau,\xi)$$

*holds.*

*Proof.* Assume $u$ and $v$ are of the form (1) with $b_1=0$ and $f=u+\overline{v}\in\mathcal{SH}^{*,0}$. It suffices to show that

$$\Pi_{\alpha,\beta}(f)=\mathfrak{U}+\overline{\mathfrak{V}}\in\mathcal{RH}^0(\sigma,\tau,\xi),$$

where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic in $\mathbb{U}$ defined as in (6) with $b_1=0$.

By Lemma 2.4, it is enough to verify that $\Phi_1\le 2(\sigma-\xi)$, where $\Phi_1$ is given by (8).

$$\Phi_1\le\frac{1}{6}\Bigg\{\sum_{\nu=2}^{\infty}\big[2\sigma+(\tau-\sigma)(\nu-1)\big]\nu^2(2\nu+1)(\nu+1)\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!}$$

$$+\sum_{\nu=2}^{\infty}\big[2\sigma+(\tau-\sigma)(\nu-1)\big]\nu^2(2\nu-1)(\nu-1)\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}$$
$$=\frac{1}{6}\Bigg\{\sum_{\nu=2}^{\infty}\Big[2(\tau-\sigma)\nu^5+(\tau+3\sigma)\nu^4+(8\sigma-2\tau)\nu^3+(3\sigma-\tau)\nu^2\Big]\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!}$$
$$+\sum_{\nu=2}^{\infty}\Big[2(\tau-\sigma)\nu^5+(9\sigma-5\tau)\nu^4+(4\tau-10\sigma)\nu^3+(3\sigma-\tau)\nu^2\Big]\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}.$$

Using Lemma 1.4, these polynomials in $\nu$ can be expressed in descending factorial form as

$$\begin{aligned}\Phi_1\le\frac{1}{6}\Bigg\{&\sum_{\nu=2}^{\infty}\Big[2(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)\\
&+(31\tau-27\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)\\
&+(138\tau-92\sigma)(\nu-1)(\nu-2)(\nu-3)\\
&+(192\tau-54\sigma)(\nu-1)(\nu-2)+(60\tau+48\sigma)(\nu-1)+12\sigma\Big]\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!}\\
&+\sum_{\nu=2}^{\infty}\Big[2(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)\\
&+(25\tau-21\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)\\
&+(84\tau-50\sigma)(\nu-1)(\nu-2)(\nu-3)\\
&+(78\tau-12\sigma)(\nu-1)(\nu-2)+(12\sigma+12\tau)(\nu-1)\Big]\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\Bigg\}.\end{aligned}$$

After simplification and using the known power series expansion of the exponential function, the expression reduces to

$$\begin{aligned}\Phi_1\le\frac{1}{6}\Bigg\{&e^{-\alpha}\sum_{\nu=0}^{\infty}\frac{\alpha^\nu}{\nu!}\Big[2(\tau-\sigma)\alpha^5+(31\tau-27\sigma)\alpha^4+(138\tau-92\sigma)\alpha^3\\
&+(192\tau-54\sigma)\alpha^2+(60\tau+48\sigma)\alpha+12\sigma\Big]\\
&+e^{-\beta}\sum_{\nu=0}^{\infty}\frac{\beta^\nu}{\nu!}\Big[2(\tau-\sigma)\beta^5+(25\tau-21\sigma)\beta^4+(84\tau-50\sigma)\beta^3\\
&+(78\tau-12\sigma)\beta^2+(12\sigma+12\tau)\beta\Big]\Bigg\}\\
=\frac{1}{6}\Big\{&2(\tau-\sigma)\alpha^5+(31\tau-27\sigma)\alpha^4+(138\tau-92\sigma)\alpha^3\\
&+(192\tau-54\sigma)\alpha^2+(60\tau+48\sigma)\alpha+128\sigma(1-e^{-\alpha})\\
&+2(\tau-\sigma)\beta^5+(25\tau-21\sigma)\beta^4+(84\tau-50\sigma)\beta^3\end{aligned}$$

$$+ (78\tau - 12\sigma)\beta^2 + (12\sigma + 12\tau)\beta\Big\}.$$

The proof is concluded by noting that the above expression is bounded by $2(\sigma - \xi)$ under the assumption given in (9). $\square$

**Theorem 3.3.** *Let $0 \le \alpha, \beta < 1$ and $0 \le \xi < \sigma \le \tau$. Suppose that the inequality*

$$2e^{-\alpha} - e^{-\beta} \ge 2 \tag{10}$$

*holds. Then the operator $\Pi_{\alpha,\beta}$ maps the class $\mathcal{RH}^0(\sigma,\tau,\xi)$ into itself; that is,*

$$\Pi_{\alpha,\beta}\big(\mathcal{RH}^0(\sigma,\tau,\xi)\big) \subseteq \mathcal{RH}^0(\sigma,\tau,\xi).$$

*Proof.* Assume that $u$ and $v$ are of the form (1) with $b_1 = 0$, and let $f = u + \overline{v} \in \mathcal{RH}^0(\sigma,\tau,\xi)$. It suffices to show that

$$\Pi_{\alpha,\beta}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{RH}^0(\sigma,\tau,\xi),$$

where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions in $\mathbb{U}$ defined as in (6) with $B_1 = 0$.

By Lemma 2.4, it is enough to verify that $\Phi_1 \le 2(\sigma - \xi)$, where $\Phi_1$ is defined as in Equation (8).

Applying Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}\Phi_1 &\le 4(\sigma - \xi)\sum_{\nu=2}^{\infty}\frac{e^{-\alpha}\alpha^{\nu-1}}{(\nu-1)!} + 2(\sigma - \xi)\sum_{\nu=2}^{\infty}\frac{e^{-\beta}\beta^{\nu-1}}{(\nu-1)!}\\ &= 4(\sigma - \xi)\big(1 - e^{-\alpha}\big) + 2(\sigma - \xi)\big(1 - e^{-\beta}\big).\end{aligned}$$

By the assumption (10), the above expression is bounded above by $2(\sigma - \xi)$. Hence, the result follows. $\square$

## 4. Applications to Pascal distribution series

The Pascal distribution series has recently gained attention in Geometric Function Theory for analyzing certain subclasses of analytic and harmonic univalent functions [4, 5, 9–11, 20, 21]. Motivated by earlier studies involving special function operators, this work explores the class $\mathcal{RH}^0(\sigma,\tau,\xi)$ using an operator based on the Pascal distribution series. We establish new inclusion relations among the classes $\mathcal{KH}^0$, $\mathcal{SH}^{*,0}$, and $\mathcal{RH}^0(\sigma,\tau,\xi)$ by applying previously derived coefficient estimates.

Consider the Pascal distribution with parameters $\kappa \ge 1$ and $0 \le \theta \le 1$ is defined by the probability mass function

$$\mathbb{P}(\mathcal{X} = \nu) = \binom{\nu + \kappa - 1}{\kappa - 1}\theta^{\nu}(1-\theta)^{\kappa}, \quad \nu = 0, 1, 2, \ldots. \tag{11}$$

This distribution induces the power series

$$\mathbb{P}_{\theta}^{\kappa}(z) = z + \sum_{\nu=2}^{\infty}\binom{\nu + \kappa - 2}{\kappa - 1}\theta^{\nu-1}(1-\theta)^{\kappa}z^{\nu},$$

which has an infinite radius of convergence. Using this series, Yaşar et al. [21] introduced the convolution operator

$$\mathbb{P}^{\kappa,\mu}_{\theta,\phi}(f)(z) = \mathbb{P}^{\kappa}_{\theta}(z) * u(z) + \overline{\mathbb{P}^{\mu}_{\phi}(z) * v(z)} = \mathbb{U}(z) + \overline{\mathbb{V}(z)}, \tag{12}$$

where the analytic functions $\mathbb{U}$ and $\mathbb{V}$ are defined by

$$\begin{aligned} \mathfrak{U}(z) &= z + \sum_{\nu=2}^{\infty} \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^{\kappa} a_\nu z^\nu, \\ \mathfrak{V}(z) &= b_1 z + \sum_{\nu=2}^{\infty} \binom{\nu+\mu-2}{\mu-1} \phi^{\nu-1}(1-\phi)^{\mu} b_\nu z^\nu, \end{aligned} \tag{13}$$

and the symbol "$*$" denotes the Hadamard (convolution) product of power series.

The following lemma collects several fundamental generating function identities involving binomial coefficients, which will be instrumental in the subsequent analysis:

**Lemma 4.1.** *Let $\kappa \geq 1$ be an integer and $0 \leq \theta < 1$ a real number. Then, the following identities hold*:

$$\begin{aligned} &\sum_{\nu=0}^{\infty} \binom{\nu+\kappa-1}{\kappa-1} \theta^\nu = \frac{1}{(1-\theta)^{\kappa}}, &&\sum_{\nu=0}^{\infty} \binom{\nu+\kappa+1}{\kappa+1} \theta^\nu = \frac{1}{(1-\theta)^{\kappa+2}}, \\ &\sum_{\nu=0}^{\infty} \binom{\nu+\kappa-2}{\kappa-2} \theta^\nu = \frac{1}{(1-\theta)^{\kappa-1}}, &&\sum_{\nu=0}^{\infty} \binom{\nu+\kappa+2}{\kappa+2} \theta^\nu = \frac{1}{(1-\theta)^{\kappa+3}}, \\ &\sum_{\nu=0}^{\infty} \binom{\nu+\kappa}{\kappa} \theta^\nu = \frac{1}{(1-\theta)^{\kappa+1}}, &&\sum_{\nu=0}^{\infty} \binom{\nu+\kappa+3}{\kappa+3} \theta^\nu = \frac{1}{(1-\theta)^{\kappa+4}}. \end{aligned}$$

**Theorem 4.2.** *Let the parameters satisfy $0 \leq \theta, \phi < 1$, $\sigma > \xi \geq 0$, and $\tau \geq 0$. Suppose that the inequality*

$$\begin{aligned} &\frac{(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)\theta^4}{(1-\theta)^4} + \frac{(10\tau-8\sigma)\kappa(\kappa+1)(\kappa+2)\theta^3}{(1-\theta)^3} \\ &+ \frac{(24\tau-10\sigma)\kappa(\kappa+1)\theta^2}{(1-\theta)^2} + \frac{(12\tau+8\sigma)\kappa\theta}{(1-\theta)} + 4\sigma - 4\sigma(1-\theta)^{\kappa} \\ &+ \frac{(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)\phi^4}{(1-\phi)^4} + \frac{(8\tau-6\sigma)\mu(\mu+1)(\mu+2)\phi^3}{(1-\phi)^3} \\ &+ \frac{(14\tau-4\sigma)\mu(\mu+1)\phi^2}{(1-\phi)^2} + \frac{4(\tau+\sigma)\mu\phi}{(1-\phi)} \leq 4(\sigma-\xi) \end{aligned} \tag{14}$$

*holds, then the Pascal distribution convolution operator $\mathbb{P}^{\kappa,\mu}_{\theta,\phi}$ maps the class of convex harmonic functions $K^0_{\mathcal{H}}$ into the class $\mathcal{RH}^0(\sigma,\tau,\xi)$, that is,*

$$\mathbb{P}^{\kappa,\mu}_{\theta,\phi}\big(K^0_{\mathcal{H}}\big) \subset \mathcal{RH}^0(\sigma,\tau,\xi).$$

*Proof.* Assume that $u$ and $v$ have the form given in (1) with $b_1 = 0$, and let $f = u + \overline{v} \in K^0_{\mathcal{H}}$. To prove the theorem, it suffices to show that the operator $\mathbb{P}^{\kappa,\mu}_{\theta,\phi}$ maps $f$ into $\mathcal{RH}^0(\sigma,\tau,\xi)$, that is,

$$\mathbb{P}^{\kappa,\mu}_{\theta,\phi}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{RH}^0(\sigma,\tau,\xi),$$

where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions in $\mathbb{U}$ defined by (6) when $B_1 = 0$.

By Lemma 2.4, it suffices to verify that

$$\Phi_2 \leq 2(\sigma - \xi),$$

where

$$\begin{aligned} \Phi_2 = &\sum_{\nu=2}^{\infty} \nu^2\big[2\sigma + (\tau-\sigma)(\nu-1)\big] \left| \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^{\kappa} a_\nu \right| \\ &+ \sum_{\nu=2}^{\infty} \nu^2\big[2\sigma + (\tau-\sigma)(\nu-1)\big] \left| \binom{\nu+\mu-2}{\mu-1} \phi^{\nu-1}(1-\phi)^{\mu} b_\nu \right|. \end{aligned} \tag{15}$$

Applying Lemma 1.1, we obtain

$$\begin{aligned} \Phi_2 \leq \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \big[2\sigma + (\tau-\sigma)(\nu-1)\big]\nu^2(\nu+1) \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^{\kappa} \\ &+ \sum_{\nu=2}^{\infty} \big[2\sigma + (\tau-\sigma)(\nu-1)\big]\nu^2(\nu-1) \binom{\nu+\mu-2}{\mu-1} \phi^{\nu-1}(1-\phi)^{\mu} \Bigg\}. \end{aligned}$$

Grouping the terms according to the powers of $\nu$, we have

$$\begin{aligned} \Phi_2 \leq \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \big[(\tau-\sigma)\nu^4 + 2\sigma\nu^3 + (3\sigma-\tau)\nu^2\big] \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^{\kappa} \\ &+ \sum_{\nu=2}^{\infty} \big[(\tau-\sigma)\nu^4 + (4\sigma-2\tau)\nu^3 \\ &+ (\tau-3\sigma)\nu^2\big] \binom{\nu+\mu-2}{\mu-1} \phi^{\nu-1}(1-\phi)^{\mu} \Bigg\}. \end{aligned}$$

Next, by expressing the powers of $\nu$ as combinations of descending factors $(\nu-1)(\nu-2)\cdots$, the expression is simplified and applying Lemma 4.1 then yields

$$\begin{aligned} \Phi_2 \leq \frac{1}{2}\Bigg\{ &\sum_{\nu=2}^{\infty} \big[(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4) \\ &+ (10\tau-8\sigma)(\nu-1)(\nu-2)(\nu-3) + (24\tau-10\sigma)(\nu-1)(\nu-2) \\ &+ (12\tau+8\sigma)(\nu-1) + 4\sigma\big] \binom{\nu+\kappa-2}{\kappa-1} \theta^{\nu-1}(1-\theta)^{\kappa} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=2}^{\infty}\big[(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)\\
&+(8\tau-6\sigma)(\nu-1)(\nu-2)(\nu-3)+(14\tau-4\sigma)(\nu-1)(\nu-2)\\
&+4(\tau+\sigma)(\nu-1)\big]\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\mu}\Bigg\}.\\
=&\ \frac{1}{2}\Bigg\{(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(1-\theta)^{\kappa}\theta^{4}\sum_{\nu=5}^{\infty}\binom{\nu+\kappa-2}{\kappa+3}\theta^{\nu-5}\\
&+(10\tau-8\sigma)\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3}\sum_{\nu=4}^{\infty}\binom{\nu+\kappa-2}{\kappa+2}\theta^{\nu-4}\\
&+(24\tau-10\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2}\sum_{\nu=3}^{\infty}\binom{\nu+\kappa-2}{\kappa+1}\theta^{\nu-3}\\
&+(12\tau+8\sigma)\kappa(1-\theta)^{\kappa}\theta\sum_{\nu=2}^{\infty}\binom{\nu+\kappa-2}{\kappa}\theta^{\nu-2}\\
&+4\sigma(1-\theta)^{\kappa}\sum_{\nu=1}^{\infty}\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}\Bigg\}\\
&+\frac{1}{2}\Bigg\{(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(1-\phi)^{\mu}\phi^{4}\sum_{\nu=5}^{\infty}\binom{\nu+\mu-2}{\mu+3}\phi^{\nu-5}\\
&+(8\tau-6\sigma)\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3}\sum_{\nu=4}^{\infty}\binom{\nu+\mu-2}{\mu+2}\phi^{\nu-4}\\
&+(14\tau-4\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2}\sum_{\nu=3}^{\infty}\binom{\nu+\mu-2}{\mu+1}\phi^{\nu-3}\\
&+4(\tau+\sigma)\mu(1-\phi)^{\mu}\phi\sum_{\nu=2}^{\infty}\binom{\nu+\mu-2}{\mu+2}\phi^{\nu-2}\Bigg\}\\
=&\ \frac{1}{2}\Bigg\{(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(1-\theta)^{\kappa}\theta^{4}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+3}{\kappa+3}\theta^{\nu}\\
&+(10\tau-8\sigma)\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+2}{\kappa+2}\theta^{\nu}\\
&+(24\tau-10\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+1}{\kappa+1}\theta^{\nu}\\
&+(12\tau+8\sigma)\kappa(1-\theta)^{\kappa}\theta\sum_{\nu=0}^{\infty}\binom{\nu+\kappa}{\kappa}\theta^{\nu}
\end{aligned}
$$

$$
\begin{aligned}
&+4\sigma(1-\theta)^{\kappa}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa-1}{\kappa-1}\theta^{\nu}-4\sigma(1-\theta)^{\kappa}\Bigg\}\\
&+\frac{1}{2}\Bigg\{(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(1-\phi)^{\mu}\phi^{4}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+3}{\mu+3}\phi^{\nu}\\
&+(8\tau-6\sigma)\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+2}{\mu+2}\phi^{\nu}\\
&+(14\tau-4\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+1}{\mu+1}\phi^{\nu}\\
&+4(\tau+\sigma)\mu(1-\phi)^{\mu}\phi\sum_{\nu=0}^{\infty}\binom{\nu+\mu}{\mu}\phi^{\nu}\Bigg\}\\
=&\ \frac{1}{2}\Bigg\{\frac{(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)\theta^{4}}{(1-\theta)^{4}}+\frac{(10\tau-8\sigma)\kappa(\kappa+1)(\kappa+2)\theta^{3}}{(1-\theta)^{3}}\\
&+\frac{(24\tau-10\sigma)\kappa(\kappa+1)\theta^{2}}{(1-\theta)^{2}}+\frac{(12\tau+8\sigma)\kappa\theta}{(1-\theta)}+4\sigma-4\sigma(1-\theta)^{\kappa}\\
&+\frac{(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)\phi^{4}}{(1-\phi)^{4}}+\frac{(8\tau-6\sigma)\mu(\mu+1)(\mu+2)\phi^{3}}{(1-\phi)^{3}}\\
&+\frac{(14\tau-4\sigma)\mu(\mu+1)\phi^{2}}{(1-\phi)^{2}}+\frac{4(\tau+\sigma)\mu\phi}{(1-\phi)}\Bigg\}.
\end{aligned}
$$

The upper bound of the last algebraic equation is $2(\sigma-\xi)$ by (7). We so have the intended outcome. □

**Theorem 4.3.** *Let $0\le\theta,\phi<1$ and $0\le\xi<\sigma\le\tau$. If the inequality*

$$
\begin{aligned}
&\frac{2(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(\kappa+4)\theta^{5}}{(1-\theta)^{5}}\\
&+\frac{(31\tau-27\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)\theta^{4}}{(1-\theta)^{4}}\\
&+\frac{(138\tau-92\sigma)\kappa(\kappa+1)(\kappa+2)\theta^{3}}{(1-\theta)^{3}}+\frac{(192\tau-54\sigma)\kappa(\kappa+1)\theta^{2}}{(1-\theta)^{2}}\\
&+\frac{(60\tau+48\sigma)\kappa\theta}{1-\theta}+12\sigma\big(1-(1-\theta)^{\kappa}\big)\\
&+\frac{2(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)\phi^{5}}{(1-\phi)^{5}}\\
&+\frac{(25\tau-21\sigma)\mu(\mu+1)(\mu+2)(\mu+3)\phi^{4}}{(1-\phi)^{4}}\\
&+\frac{(84\tau-50\sigma)\mu(\mu+1)(\mu+2)\phi^{3}}{(1-\phi)^{3}}+\frac{(78\tau-12\sigma)\mu(\mu+1)\phi^{2}}{(1-\phi)^{2}}
\end{aligned}
\tag{16}
$$

$$+\frac{(12\tau+12\sigma)\mu\phi}{1-\phi}\leq 12(\tau-\xi)$$

*holds, then*

$$\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(\mathcal{SH}^{*,0})\subset\mathcal{RH}^0(\gamma,\delta,\lambda).$$

*Proof.* Let $f=u+\overline{v}\in\mathcal{SH}^{*,0}$, where $u$ and $v$ are of the form 1 with $b_1=0$. We aim to show that

$$\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(f)=\mathfrak{U}+\overline{\mathfrak{V}}\in\mathcal{RH}^0(\sigma,\tau,\xi),$$

where $\mathfrak{U}$ and $\mathfrak{V}$, defined by (6) with $B_1=0$, are analytic functions in $\mathbb{U}$.

By Lemma 2.4, it suffices to verify that

$$\Phi_2\leq 2(\sigma-\xi),$$

where $\Phi_2$ is as defined in Equation (15).

Using Lemmas 1.4 and 4.1, we obtain

$$\begin{aligned}
\Phi_2\leq{}&\frac{1}{6}\Bigg\{\sum_{\nu=2}^{\infty}\big[2\sigma+(\tau-\sigma)(\nu-1)\big]\nu^2(2\nu+1)(\nu+1)\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa}\\
&+\sum_{\nu=2}^{\infty}\big[2\sigma+(\tau-\sigma)(\nu-1)\big]\nu^2(2\nu-1)(\nu-1)\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\kappa}\Bigg\}\\
={}&\frac{1}{6}\Bigg\{\sum_{\nu=2}^{\infty}\big[2(\tau-\sigma)\nu^5+(\tau+3\sigma)\nu^4+(8\sigma-2\tau)\nu^3\\
&+(3\sigma-\tau)\nu^2\big]\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa}\\
&+\sum_{\nu=2}^{\infty}\big[2(\tau-\sigma)\nu^5+(9\sigma-5\tau)\nu^4+(4\tau-10\sigma)\nu^3\\
&+(3\sigma-\tau)\nu^2\big]\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\kappa}\Bigg\}\\
={}&\frac{1}{6}\Bigg\{\sum_{\nu=2}^{\infty}\big[2(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)\\
&+(31\tau-27\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)\\
&+(138\tau-92\sigma)(\nu-1)(\nu-2)(\nu-3)\\
&+(192\tau-54\sigma)(\nu-1)(\nu-2)+(60\tau+48\sigma)(\nu-1)\\
&+12\sigma\big]\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa}\\
&+\sum_{\nu=2}^{\infty}\big[2(\tau-\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)\\
&+(25\tau-21\sigma)(\nu-1)(\nu-2)(\nu-3)(\nu-4)
\end{aligned}$$

$$
\begin{aligned}
&+(84\tau-50\sigma)(\nu-1)(\nu-2)(\nu-3)\\
&+(78\tau-12\sigma)(\nu-1)(\nu-2)\\
&+(12\tau+12\sigma)(\nu-1)\big]\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\kappa}\Bigg\}\\
=\frac{1}{6}\Bigg\{&2(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(\kappa+4)(1-\theta)^{\kappa}\theta^{5}\sum_{\nu=6}^{\infty}\binom{\nu+\kappa-2}{\kappa+4}\theta^{\nu-6}\\
&+(31\tau-27\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(1-\theta)^{\kappa}\theta^{4}\sum_{\nu=5}^{\infty}\binom{\nu+\kappa-2}{\kappa+3}\theta^{\nu-5}\\
&+(138\tau-92\sigma)\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3}\sum_{\nu=4}^{\infty}\binom{\nu+\kappa-2}{\kappa+2}\theta^{\nu-4}\\
&+(192\tau-54\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2}\sum_{\nu=3}^{\infty}\binom{\nu+\kappa-2}{\kappa+1}\theta^{\nu-3}\\
&+(60\tau+48\sigma)\kappa(1-\theta)^{\kappa}\theta\sum_{\nu=2}^{\infty}\binom{\nu+\kappa-2}{\kappa}\theta^{\nu-2}\\
&+12\sigma(1-\theta)^{\kappa}\sum_{\nu=1}^{\infty}\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}\Bigg\}\\
+\frac{1}{6}\Bigg\{&2(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)(1-\phi)^{\mu}\phi^{5}\sum_{\nu=6}^{\infty}\binom{\nu+\mu-2}{\mu+4}\phi^{\nu-6}\\
&+(25\tau-21\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(1-\phi)^{\mu}\phi^{4}\sum_{\nu=5}^{\infty}\binom{\nu+\mu-2}{\mu+3}\phi^{\nu-5}\\
&+(84\tau-50\sigma)\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3}\sum_{\nu=4}^{\infty}\binom{\nu+\mu-2}{\mu+2}\phi^{\nu-4}\\
&+(78\tau-12\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2}\sum_{\nu=3}^{\infty}\binom{\nu+\mu-2}{\mu+1}\phi^{\nu-3}\\
&+(12\tau+12\sigma)\mu(1-\phi)^{\mu}\phi\sum_{\nu=2}^{\infty}\binom{\nu+\mu-2}{\mu}\phi^{\nu-2}\Bigg\}\\
=\frac{1}{6}\Bigg\{&2(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(\kappa+4)(1-\theta)^{\kappa}\theta^{5}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+4}{\kappa+4}\theta^{\nu}\\
&+(31\tau-27\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(1-\theta)^{\kappa}\theta^{4}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+3}{\kappa+3}\theta^{\nu}
\end{aligned}
$$

$$
\begin{aligned}
&+ (138\tau - 92\sigma)\kappa(\kappa+1)(\kappa+2)(1-\theta)^{\kappa}\theta^{3}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+2}{\kappa+2}\theta^{\nu} \\
&+ (192\tau - 54\sigma)\kappa(\kappa+1)(1-\theta)^{\kappa}\theta^{2}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa+1}{\kappa+1}\theta^{\nu} \\
&+ (60\tau + 48\sigma)\kappa(1-\theta)^{\kappa}\theta\sum_{\nu=0}^{\infty}\binom{\nu+\kappa}{\kappa}\theta^{\nu} \\
&+ 12\sigma(1-\theta)^{\kappa}\sum_{\nu=0}^{\infty}\binom{\nu+\kappa-1}{\kappa-1}\theta^{\nu} - 12\sigma(1-\theta)^{\kappa}\Bigg\} \\
&+ \frac{1}{6}\Bigg\{2(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)(1-\phi)^{\mu}\phi^{5}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+4}{\mu+4}\phi^{\nu} \\
&+ (25\tau - 21\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(1-\phi)^{\mu}\phi^{4}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+3}{\mu+3}\phi^{\nu} \\
&+ (84\tau - 50\sigma)\mu(\mu+1)(\mu+2)(1-\phi)^{\mu}\phi^{3}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+2}{\mu+2}\phi^{\nu} \\
&+ (78\tau - 12\sigma)\mu(\mu+1)(1-\phi)^{\mu}\phi^{2}\sum_{\nu=0}^{\infty}\binom{\nu+\mu+1}{\mu+1}\phi^{\nu} \\
&+ (12\tau + 12\sigma)\mu(1-\phi)^{\mu}\phi\sum_{\nu=0}^{\infty}\binom{\nu+\mu}{\mu}\phi^{\nu}\Bigg\} \\
= {}& \frac{1}{6}\Bigg[\frac{2(\tau-\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)(\kappa+4)\theta^{5}}{(1-\theta)^{5}} \\
&+ \frac{(31\tau - 27\sigma)\kappa(\kappa+1)(\kappa+2)(\kappa+3)\theta^{4}}{(1-\theta)^{4}} \\
&+ \frac{(138\tau - 92\sigma)\kappa(\kappa+1)(\kappa+2)\theta^{3}}{(1-\theta)^{3}} \\
&+ \frac{(192\tau - 54\sigma)\kappa(\kappa+1)\theta^{2}}{(1-\theta)^{2}} + \frac{(60\tau + 48\sigma)\kappa\theta}{1-\theta} + 12\sigma - 12\sigma(1-\theta)^{\kappa}\Bigg] \\
&+ \frac{1}{6}\Bigg[\frac{2(\tau-\sigma)\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)\phi^{5}}{(1-\phi)^{5}} \\
&+ \frac{(25\tau - 21\sigma)\mu(\mu+1)(\mu+2)(\mu+3)\phi^{4}}{(1-\phi)^{4}} + \frac{(84\tau - 50\sigma)\mu(\mu+1)(\mu+2)\phi^{3}}{(1-\phi)^{3}} \\
&+ \frac{(78\tau - 12\sigma)\mu(\mu+1)\phi^{2}}{(1-\phi)^{2}} + \frac{(12\tau + 12\sigma)\mu\phi}{1-\phi}\Bigg]
\end{aligned}
$$

The last expression is bounded above by $2(\sigma-\xi)$ by the given condition (16). Thus, the proof of Theorem 4.3 is complete. □

**Theorem 4.4.** *Let $0 \le \theta, \phi < 1$ and $0 \le \xi < \sigma \le \tau$. If the inequality*

$$2(1-\theta)^{\kappa} + (1-\phi)^{\mu} \ge 2 \tag{17}$$

*holds, then*

$$\mathbb{P}_{\theta,\phi}^{\kappa,\mu}\big(\mathcal{RH}^0(\sigma,\tau,\xi)\big) \subset \mathcal{RH}^0(\sigma,\tau,\xi).$$

*Proof.* Assume that $u$ and $v$ are of the form 1 with $b_1 = 0$, and let $f = u + \overline{v} \in \mathcal{RH}^0(\sigma,\tau,\xi)$. It suffices to prove

$$\mathbb{P}_{\theta,\phi}^{\kappa,\mu}(f) = \mathfrak{U} + \overline{\mathfrak{V}} \in \mathcal{RH}^0(\sigma,\tau,\xi),$$

where $\mathfrak{U}$ and $\mathfrak{V}$ are analytic functions of $\mathbb{U}$ defined by (6) with $B_1 = 0$.

By Lemma 2.4, it is enough to show that $\Phi_2 \le 2(\sigma-\xi)$, where $\Phi_2$ is defined in Equation (15).

Using Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}
\Phi_2 &\le 4(\sigma-\xi)\sum_{\nu=2}^{\infty}\binom{\nu+\kappa-2}{\kappa-1}\theta^{\nu-1}(1-\theta)^{\kappa} \\
&\quad + 2(\sigma-\xi)\sum_{\nu=2}^{\infty}\binom{\nu+\mu-2}{\mu-1}\phi^{\nu-1}(1-\phi)^{\mu} \\
&= 4(\sigma-\xi)(1-\theta)^{\kappa}\left(\sum_{\nu=0}^{\infty}\binom{\nu+\kappa-1}{\kappa-1}\theta^{\nu} - 1\right) \\
&\quad + 2(\sigma-\xi)(1-\phi)^{\mu}\sum_{\nu=0}^{\infty}\binom{\nu+\mu-1}{\mu-1}\phi^{\nu} \\
&= 4(\sigma-\xi)\left[1-(1-\theta)^{\kappa}\right] + 2(\sigma-\xi)\left[1-(1-\phi)^{\mu}\right] \\
&= 2(\sigma-\xi)\left[3 - 2(1-\theta)^{\kappa} - (1-\phi)^{\mu}\right].
\end{aligned}$$

By assumption (17), the right-hand side is bounded above by $2(\sigma-\xi)$. Hence, the result follows. □

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