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NOVEL APPROACHES FOR NONLINEAR TIME-FRACTIONAL TELEGRAPH EQUATION THROUGH STABILITY ANALYSIS

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Abstract

In recent years, many researchers have explored the use of Fractional Partial Differential Equations (FPDEs) to model real-world phenomena through transport of porous media in engineering and applied sciences. Investigating nonlinear time-fractional Telegraph equation with quadratic nonlinearity, we transform the Partial Differential Equation (PDE) into Ordinary Differential Equation (ODE) by using the stability analysis. The obtained solutions contain different ways of exponential functions. Our method uses stability analysis to solve the complex equations easily and their applications in physics, mathematics and engineering. This indicates how system behaves near an equilibrium point and whether the underlying model is stable or unstable using the concept of stability. Significant outcomes are obtained from the various solution families. To examine the impact of several gained parameters, the propagation behavior of the obtained data is shown in 2D, 3D, and contour representations. The significance of physical situations will be better understood by the researchers to these findings. The nonlinear time-fractional Telegraph equations play a central role in many areas of electrical engineering for transport in porous media. This paper reports that our research contributes to the existing literature, offering a fresh perspective on these equations and paving the way for future research for transport in porous media. Our approach provides stability analysis with a high accuracy for two-dimensional plane autonomous systems.

Keywords: 2-Dimensional Nonlinear Time-Fractional Telegraph Equation; Stability Analysis; Fractional Calculus; Traveling Wave Transformation; New Function Transformations.

1. INTRODUCTION

Numerous studies have been conducted on fluid transportation in porous media which is crucial in many areas such as soil science, petroleum engineering, polymer electrolyte membrane fuel cells, micro channel networks and more. However, the porous media's internal microstructure is extremely complex, making it challenging to measure its transport properties.[1] The theory of fractals, which was developed by Mandelbrot, has been widely used in the last 20 years since it allows for the analytical characterization of the complexities of porous media and accurate research of porous media with complex structures.[2]

The Boundary Element Method (BEM) is a popular method among the many numerical techniques developed to determine the opening displacement of cracks in crack problems. The BEM only discretizes the boundaries of a domain, reducing the number of elements in computation and allowing for a more efficient computation as a result. Additionally, the BEM utilizes analytical fundamental solutions, which may result in a more accurate computation. Therefore, identifying suitable analytical fundamental solutions becomes crucial for the analysis of crack problems for accurate construction of boundary element equations and effective subsequent implementation.[3] The governing equations of multilayered media are a system of Partial Differential Equations (PDEs) from the theory of elasticity. Thus, it is natural to conceive a method that determines the analytical fundamental solution by reducing the PDEs to Ordinary Differential Equations (ODEs) for easier computation.[4–7]

The theory of fractional derivative is widely chosen to investigate transport mechanisms in porous media, such as fluid flow in tree-like networks.[8–11] Fractional calculus theory is a logical progression of ordinary derivative, which has gained popularity as a research topic because of its use in many scientific and engineering domains.[12] In many different fields, models are crucial. The benefits of mandatory inequality have been addressed in economics,[13] continuity and Statistical mechanics,[14] solid mechanics,[15] electro-chemistry,[16] biology,[17] and acoustics.[18] Derivatives of the fractional order of a given function contain the whole history of the function where the following state of the fractional-order system depends not only on its current state but also on all its historical states.[19,20] The theory and applications of fractional calculus have advanced and developed in recent years, and this has become increasingly evident and significant.[21]

Telegraph, also known as telegrapher's equation[22]

$$\partial_x^2 u(x,t) = cku(x,t) + sc\partial_t^2 u(x,t). \tag{1}$$

It was discovered in 1876 by Heaviside, who found it to be "fundamental equations of telegraphy" while working on improving signaling with submarine cables at the time. Equation (1) is used to solve signal transmission problems. If Eq. (1) is used, then $u(x,t)$ indicates the voltage at a line point x while c, k, and s are capacitance, resistance and inductance, all of which are taken per unit Length, respectively. In Ref. 23, one of the most studied issues for more than 400 years is Telegraph's equation. It is typically used to simulate the vibrations of structures, such as buildings, beams, and machinery, and it serves as the foundation for basic atomic physics equations.

If compared to the diffusion equation proposed 20 years ago by Sir William Thompson to describe an electric transmission line, the main innovation introduced by Heaviside was that he took into account the self-structure, which Lord Calvin described as negligible in his model. From a mathematical point of view, the discovery of the Heaviside's side changed the type of required equation from first-order parabolic to second-order hyperbolic belonging to the class of variational wave equations. The latter was familiar to the community of physicists and successfully applied to quantitatively modeling processes. It was therefore natural that during the following years, telegraph equations found applications that had far greater utility than electrical engineering and provided a versatile tool for physicists to conduct a multi-type study. Some phenomena can be seen in the realm of physics, chemistry and life sciences.

In Ref. 24, the nonlinear telegraph equations present a number of difficult problems. For instance, generalized periodic solutions of the one-dimensional nonlinear telegraph equation of the form were explored by Fucik and Mawhin.[25]

$$u_{tt} - u_{xx} + \alpha u_t + \phi(t) = f(x,t), \tag{2}$$

where ϕ and f are functions of u and x and t, respectively, and $\alpha > 0$ is a constant. The presence of a weak solution to the periodic-Dirichlet problem was also examined using Eq. (2).[26] Using the Leray–Schauder degree theory, existence, nonexistence, and multiplicity for the periodic solutions of the nonlinear telegraph equation, Bereanu[27] later investigated Eq. (2) in more detail. A novel competitive numerical method was introduced by El-Azab and El-Gamel[28] to solve the two-dimensional nonlinear telegraph problem in the form

$$u_{tt} - (u_{xx} + u_{yy}) + \alpha u_t + R(x,y,t,u) = 0, \tag{3}$$

where $\alpha > 0$ denotes a constant, representing resistance, and inductance of line transmission, whereas R an appropriate real-valued function representing nonlinear effects (such as harmonic generation, frequency mixing and wave distortion).

Through the expression αu_t, the constant α that appears in Eqs. (2) and (3) physically suggests a damping parameter that is implicated in a (dissipative) damping effect of wave motion. As previously mentioned, the damping parameter α was taken to be constant in Eqs. (2) and (3).[25–28] Van Gorder and Vajravelu, however, recently considered a variable damping in the Nagumo telegraph equation.[29,30]

Therefore, two-dimensional nonlinear time-fractional Telegraph equation can be written more simply as

$$D_{2t}^{2\alpha} u - (u_{xx} + u_{yy}) + \alpha D_t^{\alpha} u = u^2 + u, \tag{4}$$

where $u^2 + u$ is quadratic nonlinearity. It describes the propagation of electrical signals in a transmission line. It's a nonlinear version of classical Telegraph equation, which includes additional terms to account of nonlinear effects. Propagation helps us to understand how information is transmitted, processed and received.

2. CHARACTERISTICS

Mathematical properties of the model give a hyperbolic type equation. It exhibits wave-like behavior with solutions. Nonlinear term $R(x,y,t,u)$ can lead to formation of shock waves and solitons in transport of porous media. Some of the notable applications include signal propagation in optical fibers, electrical transmission line and biological systems such as nerve conduction, which may provide clearer theoretical guidance for the study on fluid transport in porous media. The role of fractional derivative tells us fractional order in circuits and systems such as capacitors and inductors exhibits unique properties like constant phase elements and fractal impedance (resistance to current flow such as solars, batteries) for porous media consisting of converging–diverging capillaries with fractal roughened surfaces.

The physical terms include nonlinear resistance arising from semi-conductor devices, nonlinear inductance due to magnetic situation, and nonlinear capacitance due to voltage dependent capacitance, which reveal the physical mechanism of the flow in fractal porous media, providing a better theoretical basis for various practical applications, such as battery industry and fuel-cell industry.

3. PRELIMINARIES

In this section, we provide an overview of fundamental concepts, terminology, and principles related to the fractional derivative used in this study.

Definition 3.1 (Fractional derivative for conformable). The definition of the conformable fractional derivative with the limit operator is as follows:

$$D_t^{\alpha}(y(t)) = \lim_{\sigma\to 0}\frac{y(t+\sigma t^{1-\alpha}) - y(t)}{\sigma},$$

where $t > 0$, $\alpha \,\epsilon\, (0,1]$ for a purpose $y = y(t) : [0,\infty) \to R$.[31,32] This definition is extensively utilized in many different fields, including physics, engineering, and mathematics, and has been shown to be effective in modeling complex phenomena.

Remark 1. "The conformable fractional derivative is a powerful tool for modeling complex systems, offering a generalization of classical derivatives and enabling the accurate description of nonlocal and power-law behaviors. Its importance lies in its flexibility, simplification of complex problems and wide range of applications across various fields."

Definition 3.2 (Fractional derivative for exponential). The definition of fractional derivative for exponential is

$$D_t^{\alpha}(e^t) = t^{1-\alpha}e^t.$$

The result shows the solvability of this equation, we present the following argument.

According to Definition 3.1,

$$D_t^{\alpha}(e^t) = \lim_{\sigma\to 0}\frac{e^{t+\sigma t^{1-\alpha}} - e^t}{\sigma},$$

$$D_t^{\alpha}(e^t) = \lim_{\sigma\to 0}\frac{e^t[e^{\sigma t^{1-\alpha}} - 1]}{\sigma}.$$

When we apply $\lim_{\sigma\to 0}$, it becomes undetermined and using L'Hopital rule, we get the following form:

$$D_t^{\alpha}(e^t) = \lim_{\sigma\to 0}\frac{e^t[t^{1-\alpha}e^{\sigma t^{1-\alpha}}]}{1},$$

applying the limit again, we have

$$D_t^{\alpha}(e^t) = t^{1-\alpha}e^t.$$

Definition 3.3 (Fractional derivative for polynomial). The fractional derivatives deals with derivatives and integrals to an arbitrary order (real or even complex order). By using fractional derivatives of polynomial as

$$D_t^{\alpha}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)}t^{s-\alpha}.$$

Remark 2. "Other fractional derivatives like Caputo and Riemann-Liouville applied on a polynomial have the same result".

4. METHODOLOGY

The key steps of this method are summarized as in the following.

Step 1. First, we consider a general time-fractional differential equation form.

$$P(u, D_t^{\alpha}u, u_x, u_y, D_{2t}^{2\alpha}u, u_{xx}, u_{yy}, \ldots) = 0, \quad (5)$$

where P is the polynomial in the function $u(x,y,t)$ and its various partial derivatives, and $D_{2t}^{2\alpha}u$ indicates the two-times sequential conformable fractional derivative of the function $u(x,y,t)$.

Step 2. Applying the transformation

$$u(x.y.t) = f(\xi), \quad \xi = \xi(x,y,t). \quad (6)$$

Using (5), we can write (6) in the following integer-order nonlinear Ordinary Differential Equation (ODE):

$$Q(f', f'', f''', \ldots) = 0, \tag{7}$$

where $f' = \frac{df}{d\xi}$, $f'' = \frac{d^2 f}{d\xi^2}$, and Q is the polynomial in f and its derivatives.

Step 3. Now, we take a new independent variable $Y = Y(\xi)$ such that

$$X(\xi) = f(\xi), \quad Y(\xi) = \frac{df}{d\xi}.$$

Thus, (7) reduces to a new system of ODEs

$$\begin{aligned} \frac{dX(\xi)}{d\xi} &= Y(\xi), \\ \frac{dY(\xi)}{d\xi} &= H(X(\xi), Y(\xi)), \end{aligned} \tag{8}$$

where f'' can be expressed in the new function $H(X(\xi), Y(\xi))$.

Step 4. Stability analysis is a mathematical process used to determine the behavior of a dynamical system near its equilibrium points (fixed points). It helps to understand whether small disturbances near these points will decay (leading to stability) or grow (leading to instability). Here's a breakdown of the steps involved in stability analysis, including the use of the Jacobian matrix and eigenvectors.

(a) Identify the Fixed Points (Equilibrium Points) Set the system of differential equations $\dot{X} = g(X)$ to zero and solve for X.

(b) Linearize the system using the Jacobian matrix

$$J(X, Y) = \begin{bmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{bmatrix}.$$

For a system of equations $\dot{X} = f(X, Y)$ and $\dot{Y} = g(X, Y)$ Substitute the fixed point coordinates into the Jacobian matrix to obtain the specific matrix for that point.

(c) Calculate the Eigenvalues and Eigenvectors of J at fixed points

Solve the characteristic polynomial $\det(J - \lambda I) = 0$ to find the eigenvalues λ.

Solve $\det(J - \lambda I)v = 0$ for each eigenvalue λ to find the corresponding eigenvectors v. These eigenvectors indicate the directions of the trajectories in the phase space.

Step 5. Analyze the trajectories and behavior near fixed points

Plot or visualize the phase portrait to understand how trajectories behave near the fixed points.

Understanding these steps allows for a detailed analysis of the stability and behavior of dynamical systems.

5. APPLICATIONS

In this section, we would like to present different cases to obtain solutions of fluids and heat in fractal porous media of nonlinear time-fractional telegraph equations with quadratic nonlinearity. The fractal geometry in porous media has been utilized for nonlinear interactions between fluids and heat in various engineering applications, such as oil recovery, groundwater flow and heat transfer in porous media according to Refs. 4–11.

Case 1. For our purpose, we consider traveling wave transformation

$$u(x.y.t) = f(\xi), \quad \xi = nx + my - \frac{pt^\alpha}{\Gamma(1+\alpha)}, \tag{9}$$

where n, m and p are nonzero arbitrary constants. Substituting (9) into (4), we can know that (4) reduced into an ODE

$$p^2 f'' - n^2 f'' - m^2 f'' - \alpha p f' = f^2 + f, \tag{10}$$

$$(p^2 - n^2 - m^2) f'' - (\alpha p) f' = f^2 + f, \tag{11}$$

where $f' = \frac{df}{d\xi}$. Let us take $z = p^2 - n^2 - m^2$, $\beta = \alpha p$

After simplification, we get

$$f''(\xi) = \frac{1}{z}(f^2(\xi) + f(\xi) + \beta f'(\xi)). \tag{12}$$

New variables are introduced $X = f(\xi)$ and $Y = f'(\xi)$. Equation (12) is equivalent to 2D autonomous system

$$\frac{dX}{d\xi} = Y, \quad \frac{dY}{d\xi} = Y' = \frac{1}{z}(X^2 + X + \beta Y). \tag{13}$$

Then, we get the following form:

$$X' = f(X, Y) = Y, \tag{14}$$

$$Y' = g(X, Y) = \frac{1}{z}(X^2 + X + \beta Y). \tag{15}$$

To find the fixed points of the system, the smooth functions can be written in the following equations:

Fixed Points:

$$\begin{aligned} f(X,Y)=0, &\quad \Rightarrow Y=0,\\ g(X,Y)=0, &\quad \Rightarrow \frac{1}{z}(X^2+X+\beta Y)=0,\\ &\quad \Rightarrow X=0,-1. \end{aligned}$$

The fixed points for the system are

$$(X,Y)=(0,0),\quad (-1,0).$$

Jacobian Matrix:

To find the Jacobian matrix of the system, we have

$$\begin{aligned} J(X,Y) &= \begin{bmatrix} \dfrac{\partial f}{\partial X} & \dfrac{\partial f}{\partial Y}\\ \dfrac{\partial g}{\partial X} & \dfrac{\partial g}{\partial Y} \end{bmatrix}\\ &= \begin{bmatrix} 0 & 1\\ \dfrac{1}{z}(2X+1) & \dfrac{\beta}{z} \end{bmatrix},\\ J(0,0) &= \begin{bmatrix} 0 & 1\\ \dfrac{1}{z} & \dfrac{\beta}{z} \end{bmatrix},\\ J(-1,0) &= \begin{bmatrix} 0 & 1\\ -\dfrac{1}{z} & \dfrac{\beta}{z} \end{bmatrix}. \end{aligned}$$

Eigenvalues at Fixed Points:

Eigenvalues of Jacobian matrix J can be found by the following characteristic equation:

$$\det(J-\lambda I)=0.$$

- At $J(0,0)$

$$\begin{vmatrix} -\lambda & 1\\ \dfrac{1}{z} & \dfrac{\beta}{z}-\lambda \end{vmatrix}=0,\quad \lambda=\frac{\beta\pm\sqrt{\beta^2+4z}}{2z}$$

$$\Rightarrow \lambda_{11}=\frac{\beta+\sqrt{\beta^2+4z}}{2z},$$

$$\lambda_{12}=\frac{\beta-\sqrt{\beta^2+4z}}{2z}.$$

If we choose the values of parameters $\alpha=1$, $p=5$, $m=5$, $n=2$

$$\Rightarrow z=-4,\quad \beta=5$$

we get

$$\lambda_{11}=-1,\quad \lambda_{12}=-\frac{1}{4}.$$

The eigenvalues of λ_{11} and λ_{12} show that the fixed point $(0,0)$ is stable. Using these values of parameters, the system of ODEs reduces to

$$X'=Y,\quad Y'=-\frac{1}{4}(X^2+X+5Y). \tag{16}$$

Eigenvector at $\boldsymbol{\lambda=\frac{\beta\pm\sqrt{\beta^2+4z}}{2z}}$:

For each eigenvalue λ_{11} and λ_{12}, the eigenvectors are

$$v_{11}=\begin{bmatrix}1\\ \lambda_{11}\end{bmatrix},\quad v_{12}=\begin{bmatrix}1\\ \lambda_{12}\end{bmatrix}$$

$$\Rightarrow v_{11}=\begin{bmatrix}1\\ \dfrac{\beta+\sqrt{\beta^2+4z}}{2z}\end{bmatrix},$$

$$v_{12}=\begin{bmatrix}1\\ \dfrac{\beta-\sqrt{\beta^2+4z}}{2z}\end{bmatrix}.$$

- At $J(-1,0)$

$$\begin{vmatrix} -\lambda & 1\\ -\dfrac{1}{z} & \dfrac{\beta}{z}-\lambda \end{vmatrix}=0,\quad \lambda=\frac{\beta\pm\sqrt{\beta^2-4z}}{2z}$$

$$\Rightarrow \lambda_{13}=\frac{\beta+\sqrt{\beta^2-4z}}{2z},$$

$$\lambda_{14}=\frac{\beta-\sqrt{\beta^2-4z}}{2z}.$$

If we select the parametric values $\alpha=\frac{1}{2}$, $p=-12$, $m=10$, $n=6$

$$\Rightarrow z=8,\quad \beta=-6,$$

we get

$$\lambda_{13}=-\frac{1}{2},\quad \lambda_{14}=-\frac{1}{4},$$

The fixed points $(-1,0)$ are stable, as indicated by the eigenvalues of λ_{13} and λ_{14}. With these values of parameters, the system of ODEs reduces to

$$X'=Y,\quad Y'=\frac{1}{8}(X^2+X-6Y). \tag{17}$$

Eigenvector at $\boldsymbol{\lambda = \frac{\beta \pm \sqrt{\beta^2 - 4z}}{2z}}$:

For the eigenvalues λ_{13} and λ_{14}, the eigenvectors are

$$v_{13} = \begin{bmatrix} 1 \\ \lambda_{13} \end{bmatrix}, \quad v_{14} = \begin{bmatrix} 1 \\ \lambda_{14} \end{bmatrix} \Rightarrow v_{13} = \begin{bmatrix} 1 \\ \dfrac{\beta + \sqrt{\beta^2 - 4z}}{2z} \end{bmatrix}, \quad v_{14} = \begin{bmatrix} 1 \\ \dfrac{\beta - \sqrt{\beta^2 - 4z}}{2z} \end{bmatrix}.$$

Trajectory

The general solution for the system is

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = c_1 v_{11} e^{\lambda_{11}\xi} + c_2 v_{12} e^{\lambda_{12}\xi} = c_1 \begin{bmatrix} 1 \\ \dfrac{\beta + \sqrt{\beta^2 + 4z}}{2z} \end{bmatrix} e^{\lambda_{11}\xi} + c_2 \begin{bmatrix} 1 \\ \dfrac{\beta - \sqrt{\beta^2 + 4z}}{2z} \end{bmatrix} e^{\lambda_{12}\xi},$$

$$f(\xi) = c_1 \exp\left(\frac{\beta + \sqrt{\beta^2 + 4z}}{2z}\right)\xi + c_2 \exp\left(\frac{\beta - \sqrt{\beta^2 + 4z}}{2z}\right)\xi,$$

$$f(\xi) = c_1 e^{-\xi} + c_2 e^{-\frac{1}{4}\xi}, \tag{18}$$

$$\begin{aligned} u_{11}(x,y,t) = {} & c_1 \exp\left(\frac{\alpha p + \sqrt{(\alpha p)^2 + 4(p^2 - n^2 - m^2)}}{2(p^2 - n^2 - m^2)}\right)\left(nx + my - \frac{pt^\alpha}{\Gamma(1+\alpha)}\right) \\ & + c_2 \exp\left(\frac{\alpha p - \sqrt{(\alpha p)^2 + 4(p^2 - n^2 - m^2)}}{2(p^2 - n^2 - m^2)}\right)\left(nx + my - \frac{pt^\alpha}{\Gamma(1+\alpha)}\right), \end{aligned} \tag{19}$$

$$u_{11}(x,y,t) = c_1 e^{-(2x+5y-\frac{5t}{\Gamma(2)})} + c_2 e^{-\frac{1}{4}(2x+5y-\frac{5t}{\Gamma(2)})}. \tag{20}$$

On the other hand,

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = c_3 v_{13} e^{\lambda_{13}\xi} + c_4 v_{14} e^{\lambda_{14}\xi} = c_3 \begin{bmatrix} 1 \\ \dfrac{\beta + \sqrt{\beta^2 - 4z}}{2z} \end{bmatrix} e^{\lambda_{13}\xi} + c_4 \begin{bmatrix} 1 \\ \dfrac{\beta - \sqrt{\beta^2 - 4z}}{2z} \end{bmatrix} e^{\lambda_{14}\xi},$$

$$f(\xi) = c_3 exp\left(\frac{\beta + \sqrt{\beta^2 - 4z}}{2z}\right)\xi + c_4 exp\left(\frac{\beta - \sqrt{\beta^2 - 4z}}{2z}\right)\xi,$$

$$f(\xi) = c_3 e^{-\frac{1}{2}\xi} + c_4 e^{-\frac{1}{4}\xi}, \tag{21}$$

$$\begin{aligned} u_{12}(x,y,t) = {} & c_3 exp\left(\frac{\alpha p + \sqrt{(\alpha p)^2 - 4(p^2 - n^2 - m^2)}}{2(p^2 - n^2 - m^2)}\right)\left(nx + my - \frac{pt^\alpha}{\Gamma(1+\alpha)}\right) \\ & + c_4 \exp\left(\frac{\alpha p - \sqrt{(\alpha p)^2 - 4(p^2 - n^2 - m^2)}}{2(p^2 - n^2 - m^2)}\right)\left(nx + my - \frac{pt^\alpha}{\Gamma(1+\alpha)}\right), \end{aligned} \tag{22}$$

$$u_{12}(x,y,t) = c_3 e^{-\frac{1}{2}(6x+10y+\frac{12\sqrt{t}}{\Gamma(1.5)})} + c_4 e^{-\frac{1}{4}(6x+10y+\frac{12\sqrt{t}}{\Gamma(1.5)})}. \tag{23}$$

Case 2. We utilize the transformations as follows:

$$u(x.y.t) = f(\xi), \quad \xi = x^2 + y^2. \tag{24}$$

Using (24) into (4), we can know that (4) leads to an ODE as follows:

$$\begin{aligned} -(4\xi f''(\xi)+4f'(\xi)) &= \quad f^2(\xi)+f(\xi), \\ 4\xi f''(\xi)+4f'(\xi) &= \; -f^2(\xi)-f(\xi), \end{aligned} \tag{25}$$

where $f'(\xi)=\frac{df}{d\xi}$. After simplification, we get the following form:

$$f''(\xi)=-\frac{1}{4\xi}f^2(\xi)-\frac{1}{4\xi}(\xi)-\frac{1}{\xi}f'(\xi). \tag{26}$$

Introduce new variables $X=f(\xi)$ and $Y=f'(\xi)$. Equation (26) is identical to two-dimensional autonomous system

$$\frac{dX}{d\xi}=Y, \quad \frac{dY}{d\xi}=Y'=-\frac{1}{4\xi}(X^2+X)-\frac{1}{\xi}Y. \tag{27}$$

Then, we get the following form:

$$X'=f(X,Y)=Y, \tag{28}$$

$$Y'=g(X,Y)=-\frac{1}{4\xi}(X^2+X)-\frac{1}{\xi}Y. \tag{29}$$

The smooth function can be expressed in the following way to determine the fixed points of the system:

Fixed Points:

$$\begin{aligned} f(X,Y)&=0, \quad && \Rightarrow Y=0, \\ g(X,Y)&=0, && \Rightarrow -\frac{1}{4\xi}(X^2+X)-\frac{1}{\xi}Y=0, \\ & && \Rightarrow X=0,-1. \end{aligned}$$

The system's fixed points are

$$(X,Y)=(0,0), \quad (-1,0).$$

Jacobian Matrix:

To obtain the Jacobian matrix of the system, we have

$$J(X,Y)=\begin{bmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -\frac{1}{4\xi}(2X+1) & -\frac{1}{\xi} \end{bmatrix},$$

$$J(0,0)=\begin{bmatrix} 0 & 1 \\ -\frac{1}{4\xi} & -\frac{1}{\xi} \end{bmatrix}, \quad J(-1,0)=\begin{bmatrix} 0 & 1 \\ \frac{1}{4\xi} & -\frac{1}{\xi} \end{bmatrix}.$$

Eigenvalues at Fixed Points:

Eigenvalues of Jacobian matrix J can be obtained by the following characteristic equation:

$$\det(J-\lambda I)=0.$$

At $J(0,0)$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{1}{4\xi} & -\frac{1}{\xi}-\lambda \end{vmatrix}=0, \quad \lambda=\frac{-1\pm\sqrt{1-\xi}}{2\xi}$$

$$\Rightarrow \lambda_{21}=\frac{-1+\sqrt{1-\xi}}{2\xi},$$

$$\lambda_{22}=\frac{-1-\sqrt{1-\xi}}{2\xi}.$$

At $J(-1,0)$

$$\begin{vmatrix} -\lambda & 1 \\ \frac{1}{4\xi} & -\frac{1}{\xi}-\lambda \end{vmatrix}=0, \quad \lambda=\frac{-1\pm\sqrt{1+\xi}}{2\xi}$$

$$\Rightarrow \lambda_{23}=\frac{-1+\sqrt{1+\xi}}{2\xi},$$

$$\lambda_{24}=\frac{-1-\sqrt{1+\xi}}{2\xi}.$$

Eigenvector at $\boldsymbol{\lambda=\frac{-1\pm\sqrt{1-\xi}}{2\xi}}$:

For the eigenvalues λ_{21} and λ_{22}, the eigenvectors are

$$v_{21}=\begin{bmatrix} 1 \\ \lambda_{21} \end{bmatrix}, \quad v_{22}=\begin{bmatrix} 1 \\ \lambda_{22} \end{bmatrix},$$

$$\Rightarrow v_{21}=\begin{bmatrix} 1 \\ \frac{-1+\sqrt{1-\xi}}{2\xi} \end{bmatrix}, \quad v_{22}=\begin{bmatrix} 1 \\ \frac{-1-\sqrt{1-\xi}}{2\xi} \end{bmatrix}.$$

Eigenvector at $\lambda=\frac{-1\pm\sqrt{1+\xi}}{2\xi}$:

For the eigenvalues λ_{23} and λ_{24}, the eigenvectors are

$$v_{23}=\begin{bmatrix} 1 \\ \lambda_{23} \end{bmatrix}, \quad v_{24}=\begin{bmatrix} 1 \\ \lambda_{24} \end{bmatrix},$$

$$\Rightarrow v_{23}=\begin{bmatrix} 1 \\ \frac{-1+\sqrt{1+\xi}}{2\xi} \end{bmatrix}, \quad v_{24}=\begin{bmatrix} 1 \\ \frac{-1-\sqrt{1+\xi}}{2\xi} \end{bmatrix}.$$

Trajectory

The general solution for the system is

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = c_5 v_{21} e^{\lambda_{21}\xi} + c_6 v_{22} e^{\lambda_{22}\xi}$$

$$= c_5 \begin{bmatrix} 1 \\ \frac{-1+\sqrt{1-\xi}}{2\xi} \end{bmatrix} e^{\lambda_{21}\xi}$$

$$+ c_6 \begin{bmatrix} 1 \\ \frac{-1-\sqrt{1-\xi}}{2\xi} \end{bmatrix} e^{\lambda_{22}\xi},$$

$$f(\xi) = c_5 \exp\left(\frac{-1+\sqrt{1-\xi}}{2\xi}\right)\xi$$

$$+ c_6 \exp\left(\frac{-1-\sqrt{1-\xi}}{2\xi}\right)\xi,$$

$$u_{21}(x,y,t) = c_5 \exp\left(\frac{-1+\sqrt{1-(x^2+y^2)}}{2}\right)$$

$$+ c_6 \exp\left(\frac{-1-\sqrt{1-(x^2+y^2)}}{2}\right), \tag{30}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = c_7 v_{23} e^{\lambda_{23}\xi} + c_8 v_{24} e^{\lambda_{24}\xi}$$

$$= c_7 \begin{bmatrix} 1 \\ \frac{-1+\sqrt{1+\xi}}{2\xi} \end{bmatrix} e^{\lambda_{23}\xi}$$

$$+ c_8 \begin{bmatrix} 1 \\ \frac{-1-\sqrt{1+\xi}}{2\xi} \end{bmatrix} e^{\lambda_{24}\xi},$$

$$f(\xi) = c_7 \exp\left(\frac{-1+\sqrt{1+\xi}}{2\xi}\right)\xi$$

$$+ c_8 \exp\left(\frac{-1-\sqrt{1+\xi}}{2\xi}\right)\xi,$$

$$u_{22}(x,y,t) = c_7 \exp\left(\frac{-1+\sqrt{1+x^2+y^2}}{2}\right)$$

$$+ c_8 \exp\left(\frac{-1-\sqrt{1+x^2+y^2}}{2}\right). \tag{31}$$

Case 3.

Consider another new function transformation as follows:

$$u(x.y.t) = f(\xi), \quad \xi = e^{ax+bt}. \tag{32}$$

Under this transformation, chain rule and exponential definition, (4) takes the subsequent form of an ODE

$$(b^{2\alpha}-a^2)\xi^2 f'' + (b^{2\alpha}-a^2)\xi f' + \alpha b^{\alpha}\xi f'$$
$$- f^2 - f = 0,$$
$$(b^{2\alpha}-a^2)\xi^2 f'' + (b^{2\alpha}-a^2+\alpha b^{\alpha})\xi f' = f^2 + f, \tag{33}$$

where $f'(\xi) = \frac{df}{d\xi}$. After simplification, we get the following form:

$$f''(\xi) = \frac{1}{(b^{2\alpha}-a^2)\xi^2} f^2(\xi) + \frac{1}{(b^{2\alpha}-a^2)\xi^2} f(\xi)$$
$$- \frac{(b^{2\alpha}-a^2+\alpha b^{\alpha})}{(b^{2\alpha}-a^2)\xi} f'(\xi). \tag{34}$$

Now, if we let $t_1 = \frac{1}{(b^{2\alpha}-a^2)}$ and $t_2 = (b^{2\alpha}-a^2+\alpha b^{\alpha})t_1$, so (34) changes, we get

$$f''(\xi) = \frac{t_1}{\xi^2}(f^2(\xi)+f(\xi)) - \frac{t_2}{\xi} f'(\xi). \tag{35}$$

Introduce new independent variables $X = f(\xi)$ and $Y = f'(\xi)$, Eq. (35) is identical to a two-dimensional autonomous system

$$\frac{dX}{d\xi} = Y, \tag{36}$$

$$\frac{dY}{d\xi} = Y' = \frac{t_1}{\xi^2}(X^2+X) - \frac{t_2}{\xi} Y. \tag{37}$$

Then, we get the subsequent form

$$X' = f(X,Y) = Y, \tag{38}$$

$$Y' = g(X,Y) = \frac{t_1}{\xi^2}(X^2+X) - \frac{t_2}{\xi} Y. \tag{39}$$

The following term can be used to express the smooth functions in order to determine the system's fixed points

Fixed Points:

$$f(X,Y) = 0, \quad \Rightarrow Y = 0,$$
$$g(X,Y) = 0, \quad \Rightarrow \frac{t_1}{\xi^2}(X^2+X) - \frac{t_2}{\xi} Y = 0,$$
$$\Rightarrow X = 0, -1.$$

The fixed points for the system include

$$(X,Y) = (0,0), \quad (-1,0).$$

Jacobian Matrix:

To determine the Jacobian matrix of the system, we get

$$J(X,Y) = \begin{bmatrix} \dfrac{\partial f}{\partial X} & \dfrac{\partial f}{\partial Y} \\ \dfrac{\partial g}{\partial X} & \dfrac{\partial g}{\partial Y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{t_1}{\xi^2}(2X+1) & -\dfrac{t_2}{\xi} \end{bmatrix},$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ \dfrac{t_1}{\xi^2} & -\dfrac{t_2}{\xi} \end{bmatrix},$$

$$J(-1,0) = \begin{bmatrix} 0 & 1 \\ -\dfrac{t_1}{\xi^2} & -\dfrac{t_2}{\xi} \end{bmatrix}.$$

Eigenvalues at Fixed Points:

Eigenvalues of Jacobian matrix J can be obtained by the following characteristic equation:

$$\det(J - \lambda I) = 0.$$

- At $J(0,0)$

$$\begin{vmatrix} -\lambda & 1 \\ \dfrac{t_1}{\xi^2} & -\dfrac{t_2}{\xi} - \lambda \end{vmatrix} = 0,$$

$$\lambda = \frac{-t_2 \pm \sqrt{t_2^2 + 4t_1}}{2\xi}$$

$$\Rightarrow \lambda_{31} = \frac{-t_2 + \sqrt{t_2^2 + 4t_1}}{2\xi},$$

$$\lambda_{32} = \frac{-t_2 - \sqrt{t_2^2 + 4t_1}}{2\xi}.$$

If we select the parametric values t_1=3, $t_2 = 4$ we get

$$\lambda_{31} = \frac{-1}{\xi}, \quad \lambda_{32} = \frac{-3}{\xi}.$$

Both λ_{31} and λ_{32} are negative when $\lambda > 0$, the fixed point $(0,0)$ becomes stable. The ODE reduces to form

$$X' = Y, \quad Y' = -\frac{3}{\xi^2}(X^2 + X) - \frac{4}{\xi}Y. \tag{40}$$

Eigenvector at $\boldsymbol{\lambda = \frac{-t_2 \pm \sqrt{t_2^2+4t_1}}{2\xi}}$:

For the eigenvalues λ_{31} and λ_{32}, the eigenvectors are

$$v_{31} = \begin{bmatrix} 1 \\ \lambda_{31} \end{bmatrix}, \quad v_{32} = \begin{bmatrix} 1 \\ \lambda_{32} \end{bmatrix},$$

$$\Rightarrow v_{31} = \begin{bmatrix} 1 \\ \dfrac{-t_2 + \sqrt{t_2^2 + 4t_1}}{2\xi} \end{bmatrix},$$

$$v_{32} = \begin{bmatrix} 1 \\ \dfrac{-t_2 - \sqrt{t_2^2 + 4t_1}}{2\xi} \end{bmatrix}.$$

- At $J(-1,0)$

$$\begin{vmatrix} -\lambda & 1 \\ -\dfrac{t_1}{\xi^2} & -\dfrac{t_2}{\xi} - \lambda \end{vmatrix} = 0, \quad \lambda = \frac{-t_2 \pm \sqrt{t_2^2 - 4t_1}}{2\xi}$$

$$\Rightarrow \lambda_{33} = \frac{-t_2 + \sqrt{t_2^2 - 4t_1}}{2\xi},$$

$$\lambda_{34} = \frac{-t_2 - \sqrt{t_2^2 - 4t_1}}{2\xi}.$$

If we choose the values of parameters t_1=5, $t_2 = 6$, we get

$$\lambda_{33} = \frac{-1}{\xi}, \quad \lambda_{34} = \frac{-5}{\xi}.$$

Both λ_{33} and λ_{34} are negative when $\lambda > 0$, the fixed point $(-1,0)$ becomes stable. The ODE reduces to form

$$X' = Y, \quad Y' = \frac{5}{\xi^2}(X^2 + X) - \frac{6}{\xi}Y. \tag{41}$$

Eigenvector at $\boldsymbol{\lambda = \frac{-t_2 \pm \sqrt{t_2^2-4t_1}}{2\xi}}$:

For the eigenvalues λ_{33} and λ_{34}, the eigenvectors are

$$v_{33} = \begin{bmatrix} 1 \\ \lambda_{33} \end{bmatrix}, \quad v_{34} = \begin{bmatrix} 1 \\ \lambda_{34} \end{bmatrix},$$

$$\Rightarrow v_{33} = \begin{bmatrix} 1 \\ \dfrac{-t_2 + \sqrt{t_2^2 - 4t_1}}{2\xi} \end{bmatrix},$$

$$v_{34} = \begin{bmatrix} 1 \\ \dfrac{-t_2 - \sqrt{t_2^2 - 4t_1}}{2\xi} \end{bmatrix}.$$

Trajectory

The general solution for the system is

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = c_9 v_{31} e^{\lambda_{31}\xi} + c_{10} v_{32} e^{\lambda_{32}\xi} = c_9 \begin{bmatrix} 1 \\ \dfrac{-t_2}{2\xi} + \dfrac{\sqrt{\frac{t_2^2}{\xi^2} + \frac{4t_1}{\xi^2}}}{2} \end{bmatrix} e^{\lambda_{31}\xi} + c_{10} \begin{bmatrix} 1 \\ \dfrac{-t_2}{2\xi} - \dfrac{\sqrt{\frac{t_2^2}{\xi^2} + \frac{4t_1}{\xi^2}}}{2} \end{bmatrix} e^{\lambda_{32}\xi},$$

$$f(\xi) = c_9 \exp\left(\frac{-t_2}{2\xi} + \frac{\sqrt{\frac{t_2^2}{\xi^2} + \frac{4t_1}{\xi^2}}}{2}\right)\xi + c_{10} \exp\left(\frac{-t_2}{2\xi} - \frac{\sqrt{\frac{t_2^2}{\xi^2} + \frac{4t_1}{\xi^2}}}{2}\right)\xi,$$

$$\begin{aligned} u_{31}(x,y,t) &= c_9 \exp\left(\frac{-t_2}{2e^{ax+bt}} + \frac{\sqrt{\frac{t_2^2}{(e^{ax+bt})^2} + \frac{4t_1}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt} \\ &\quad + c_{10} \exp\left(\frac{-t_2}{2e^{ax+bt}} - \frac{\sqrt{\frac{t_2^2}{(e^{ax+bt})^2} + \frac{4t_1}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt}, \end{aligned} \tag{42}$$

$$u_{31}(x,y,t) = c_9 \exp\left(\frac{-2}{e^{ax+bt}} + \frac{\sqrt{\frac{4}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt} + c_{10} \exp\left(\frac{-2}{e^{ax+bt}} - \frac{\sqrt{\frac{4}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt}, \tag{43}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = c_{11} v_{33} e^{\lambda_{33}\xi} + c_{12} v_{34} e^{\lambda_{34}\xi} = c_{11} \begin{bmatrix} 1 \\ \dfrac{-t_2}{2\xi} + \dfrac{\sqrt{\frac{t_2^2}{\xi^2} - \frac{4t_1}{\xi^2}}}{2} \end{bmatrix} e^{\lambda_{33}\xi} + c_{12} \begin{bmatrix} 1 \\ \dfrac{-t_2}{2\xi} - \dfrac{\sqrt{\frac{t_2^2}{\xi^2} - \frac{4t_1}{\xi^2}}}{2} \end{bmatrix} e^{\lambda_{34}\xi},$$

$$f(\xi) = c_{11} \exp\left(\frac{-t_2}{2\xi} + \frac{\sqrt{\frac{t_2^2}{\xi^2} - \frac{4t_1}{\xi^2}}}{2}\right)\xi + c_{12} \exp\left(\frac{-t_2}{2\xi} - \frac{\sqrt{\frac{t_2^2}{\xi^2} - \frac{4t_1}{\xi^2}}}{2}\right)\xi,$$

$$\begin{aligned} u_{32}(x,y,t) &= c_{11} \exp\left(\frac{-t_2}{2e^{ax+bt}} + \frac{\sqrt{\frac{t_2^2}{(e^{ax+bt})^2} - \frac{4t_1}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt} \\ &\quad + c_{12} \exp\left(\frac{-t_2}{2e^{ax+bt}} - \frac{\sqrt{\frac{t_2^2}{(e^{ax+bt})^2} - \frac{4t_1}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt}, \end{aligned} \tag{44}$$

$$u_{32}(x,y,t) = c_{11} \exp\left(\frac{-3}{e^{ax+bt}} + \frac{\sqrt{\frac{16}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt} + c_{12} \exp\left(\frac{-3}{e^{ax+bt}} - \frac{\sqrt{\frac{16}{(e^{ax+bt})^2}}}{2}\right) e^{ax+bt}, \tag{45}$$

where $t_1 = \frac{1}{(b^{2\alpha} - a^2)}$ and $t_2 = (b^{2\alpha} - a^2 + \alpha b^{\alpha})t_1$.

6. GRAPHICAL REPRESENTATION

This section looks at how the solitary wave structures behave graphically for various values of the constant choices. In this context, "soliton solutions" refers to unique kind of solutions that are stable and unstable and retain their amplitude and shape throughout propagation of porous media.

Numerous disciplines such as physics, engineering and applied mathematics depend on solitons. The findings are highly helpful for experimentals and computational solutions and understanding nonlinear dynamics through fractal porous media. Plotting of the 2D, 3D and contours is done by choosing particular parameter values in order to validate the proposed model for determining the stability of transport in porous media. The researchers select different solutions to the distinct physical challenges in order to help them test the nonlinear dynamics precisely and also the seepage of particles in porous media has attracted considerable attention due to its extensive existence in nature. Our research will lead to the creation of a new global class.

Family of Case 1:

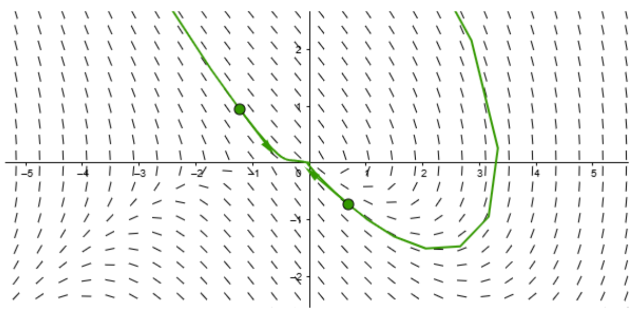

Fig. 1 Phase space for solution of (16) when $\alpha = 1$, $p = 5$, $m = 5$, $n = 2$.

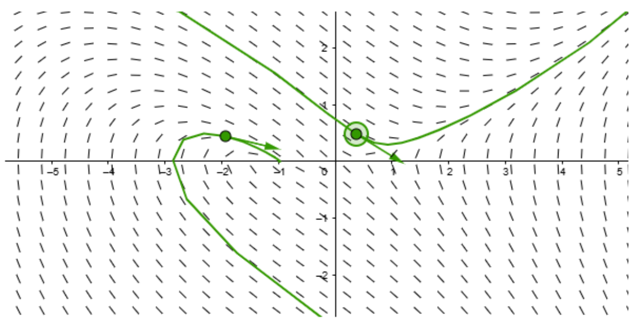

Fig. 2 Phase space for solution of (17) when $\alpha = \frac{1}{2}$, $p = -12$, $m = 10$, $n = 6$.

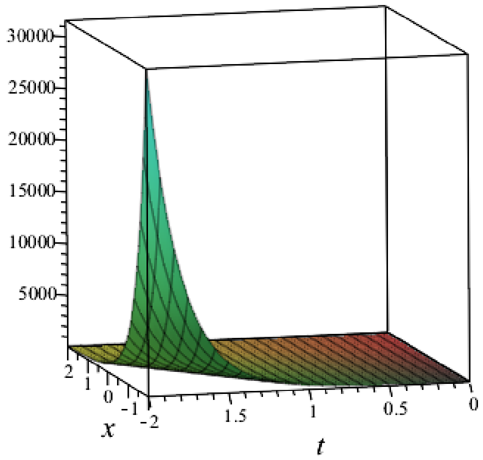

Fig. 3 3D surface for solution of (20).

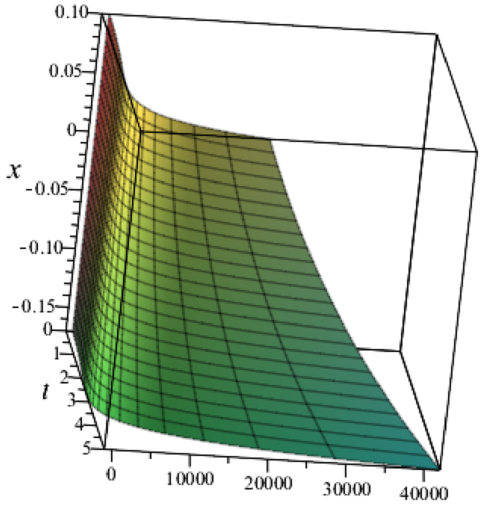

Fig. 4 3D surface for solution of (23).

Family of Case 2:

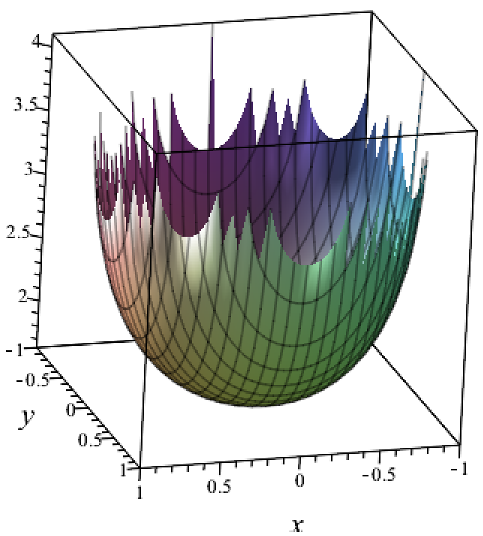

Fig. 5 3D plot for solution of (30).

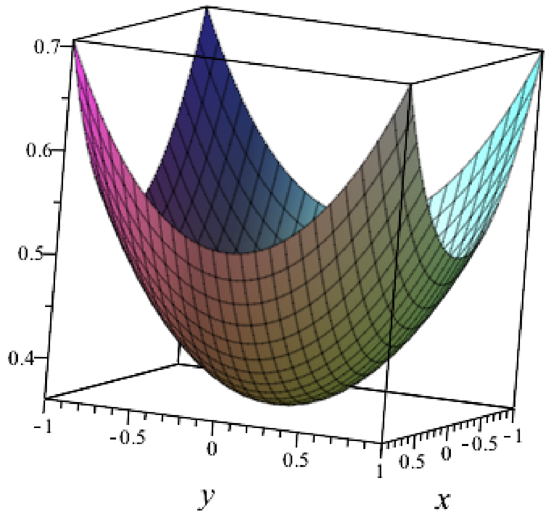

Fig. 6 3D plot for solution of (31).

7. RESULTS AND DISCUSSIONS

The discussion of this research lies in its pioneering application of the stability analysis to transform nonlinear time-fractional Telegraph equations from a PDE to an ODE, facilitating the derivation of stability analysis. The method used in this study provides exponential function of the obtained solutions. The analytic solutions through this research are original and substantial, making this study a valuable addition to existing literature and a significant advancement in the field of porous media.

The novel technique on stability analysis is applied to the above problems. The figures explain the behavior of nonlinear time-fractional Telegraph equations and show the method's accuracy is good compared to other literature methods. In this section, we note that our approaches can handle time-fractional Telegraph equations easily and perfectly by planning to conduct analysis of crack problems using our method in complex geometry. According to the solution graphs, this method can be applied to a wide range of nonlinear fractional derivatives in science and engineering. The result describes the better accuracy and broader applicability of our method compared to the Direct Method (DM) and conventional (DM) in Ref. 2, making it a convincing new method for crack problems in multilayered elastic media. To demonstrate the applicability of the proposed method, we present stability analysis of transport in porous media. The results show that our proposed method is accurately capturing the complex dynamics of transport in porous media.

8. CONCLUSION AND FUTURE WORK

In this paper, we successfully transformed nonlinear time-fractional Telegraph equations from a PDE to an ODE enabling the derivation of new function transformations as evidenced by the above figures in section of graphical representation. The stability analysis through this research offers valuable insights, empowering engineers to tackle complex problems with precision of roughened porous media composed of particles.

Our findings show that the solutions involve different ways of exponential functions. Various solitons and nonlinear wave solutions are displayed in the plotted graphs. Plotting of the 2D, 3D, and contours is done by choosing particular parameter values. Our method's physical meaning is that different solutions to the distinct physical challenges are selected to help the researchers test the nonlinear dynamics accurately.

The results obtained are found to be more accurate and efficient in determining stability for nonlinear time-fractional Telegraph equations, demonstrating the effectiveness of the proposed method. In this paper, all the results obtained are new and original. These results may be used in fluid dynamics, plasma theory, quantum field theory and electrical transmission line industry.

Future research directions may include extending this methodology to address more intricate problems, such as systems of nonlinear PDEs including coupled nonlinear time-fractional Telegraph equations, which are crucial in modeling complex phenomena in electrical engineering, signal processing and optical communications of roughened porous media. Also, the focus will be extending from this study to develop a new method for stability analysis with diverse behaviors particularly in neuroscience, machine learning for flow and power electronics, where nonlinear oscillators play an important role. Our research will change the world in a new way.

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