ORIGINAL RESEARCH

Existence and Approximate Controllability of Mild Solutions for Hadamard-Type Fractional Systems in Banach Spaces

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Abstract

This work establishes the approximate controllability of nonlinear Hadamard-type fractional systems with weakly singular logarithmic kernels in Banach spaces. Under suitable assumptions on the linearized control operator and a linear growth condition on the nonlinearity, we prove the existence of a mild solution steering the system from an arbitrary initial state arbitrarily close to a prescribed terminal state. The proof combines Krasnoselskii's fixed point theorem with refined estimates involving logarithmic convolutions.

Keywords Hadamard fractional derivative · Approximate controllability · Mild solution · Weakly singular kernel · Krasnoselskii's fixed point theorem · Logarithmic time scale · Banach space

Mathematics Subject Classification 34A08 · 34G20 · 47H10 · 93B05

1 Introduction

Fractional calculus has emerged as an effective mathematical framework for modeling dynamical processes with memory and hereditary effects. Its relevance has been demonstrated in a wide range of applications, including viscoelastic materials, anomalous diffusion, heat conduction with memory, and control theory. Classical references such as Podlubny [14], Samko et al. [15], and Kilbas et al. [9] laid the theoretical foundations of fractional integrals and derivatives, while further physical interpretations and applications were developed by Mainardi [11].

Beyond the classical Riemann-Liouville and Caputo derivatives, increasing attention has been devoted to fractional operators with nonstandard kernels. These operators allow for a finer description of nonlocal phenomena and complex memory structures. In particular, fractional derivatives involving logarithmic, exponential, or Mittag-Leffler type kernels have proven to be suitable for modeling scale-invariant and multiplicative memory effects. Significant contributions in this direction can be found in the works of Abdeljawad [1–4], where generalized fractional operators and related analytical properties are investigated.

At the same time, fixed point techniques and measures of noncompactness have played a central role in the qualitative analysis of fractional differential and integro-differential equations. Fundamental results due to Krasnoselskii [10] and Smart [16], together with the framework developed by Banaś and Goebel [5], provide powerful tools for studying nonlinear problems in infinite-dimensional spaces. More recent monographs by Benchohra et

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al. [6, 7] illustrate how these methods can be effectively combined with fractional operators to treat a broad class of nonlinear systems.

Motivated by these developments, and by applications arising in control theory, several authors have investigated the controllability of fractional evolution equations in Banach spaces. In particular, Wang and Zhou [18] studied complete controllability for fractional evolution systems, highlighting the role of fractional resolvent operators and suitable control mappings. More recently, controllability results for fractional integro-differential equations with delays and singular kernels have been obtained in more general settings, including Fréchet spaces; see, for example, [12].

However, most existing works focus on Caputo or Riemann-Liouville fractional derivatives, while the case of Hadamard-type fractional systems with weakly singular logarithmic kernels remains less explored. Furthermore, many studies assume exact controllability, which is known to be too restrictive in infinite-dimensional Banach spaces when the associated semigroup is compact. To the best of our knowledge, the approximate controllability of nonlinear Hadamard-type fractional integro-differential systems has not been fully investigated. This paper aims to fill this gap.

In this work, we focus on a class of controlled fractional integro-differential equations involving a weakly singular logarithmic kernel of Hadamard type. More precisely, we consider the following problem:

$$\begin{cases} \mathcal{H}^{\alpha}_{1+}u(t) + Au(t) = \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{-\beta} f\big(s, u(s)\big)\,\frac{ds}{s} + Bv(t), \quad t \in (1, T], \\ u(1) = u_0, \end{cases} \tag{1}$$

where $\mathcal{H}^{\alpha}_{1+}$ denotes the Caputo–Hadamard fractional derivative of order α, with $\frac{1}{2} < \alpha < 1$ and $0 < \beta < \alpha$. Here, $A : D(A) \subset F \to F$ is a densely defined closed linear operator on a Banach space F, $f : [1, T] \times F \to F$ is a nonlinear mapping, and $v(\cdot) \in L^2([1, T]; U)$ denotes the control function, with U being a Banach space. The bounded linear operator $B : U \to F$ represents the control action, while the weakly singular logarithmic kernel $\left(\log \frac{t}{s}\right)^{-\beta}$ captures the intrinsic multiplicative memory associated with Hadamard-type dynamics.

The main objective of this paper is to establish sufficient conditions for the approximate controllability of problem (1) on a finite time interval $[1, T]$. Working on a finite horizon allows us to develop the analysis within a Banach space framework while still accounting for the essential nonlocal effects induced by the logarithmic kernel. The approach combines properties of fractional resolvent operators with fixed point techniques based on measures of noncompactness, under suitable assumptions on the nonlinear term and the control operator.

The structure of the paper is as follows. Section 2 introduces the functional framework and preliminary results. Section 3 establishes the main existence and approximate controllability theorem. Section 4 provides an illustrative example demonstrating the applicability of our results. Finally, Section 5 concludes the paper with a summary of the main contributions.

2 Preliminaries

Let F be a separable Banach space and U a control Banach space. Fix $T > 1$ and consider the Banach space $X_T = C([1, T], F)$ equipped with the norm

$$\|u\|_T = \sup_{t \in [1,T]} \|u(t)\|_F.$$

Definition 2.1 (Caputo-Hadamard derivative [9]) For $\alpha \in (0,1)$ and a function $u \in C^1((1,T],F)$, the Caputo-Hadamard fractional derivative of order α is defined by

$$\mathcal{H}^{\alpha}_{1+}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{-\alpha} \left(s\frac{d}{ds}u(s)\right)\frac{ds}{s}, \quad t \in (1,T].$$

Theorem 2.2 (Krasnoselskii's Theorem [5, 10]) Let X be a Banach space and $\mathcal{M} \subset X$ nonempty, closed, bounded and convex. Suppose $\mathcal{A}, \mathcal{B} : \mathcal{M} \to X$ satisfy:

(i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ for all $x, y \in \mathcal{M}$;
(ii) $\mathcal{A}$ is a contraction;
(iii) $\mathcal{B}$ is continuous and compact on $\mathcal{M}$.

Then, $\mathcal{N} = \mathcal{A} + \mathcal{B}$ has a fixed point in $\mathcal{M}$.

3 Existence and Controllability Results

To analyze the controllability of problem (1), we first state the following structural assumptions

(H_1) The operator A generates an equicontinuous C_0-semigroup $(T(t))_{t\geq 0}$ on F. Consequently, the fractional resolvent operators T_α and S_α are uniformly bounded on $[0, \log T]$, i.e.,

$$M_\alpha = \sup_{\tau\in[0,\log T]} \|T_\alpha(\tau)\| < \infty, \quad L_\alpha = \sup_{\tau\in[0,\log T]} \|S_\alpha(\tau)\| < \infty.$$

(H_2) The nonlinearity $f : [1,T] \times F \to F$ is Carathéodory and satisfies the linear growth condition: there exist $m \in L^1([1,T];\mathbb{R}_+)$ and $c \geq 0$ such that

$$\|f(t,x)\|_F \leq m(t) + c\|x\|_F, \quad \text{for a.e. } t \in [1,T] \text{ and all } x \in F.$$

(H_3) The control operator $B : U \to F$ is bounded (i.e., $B \in \mathcal{L}(U,F)$), and the control function v belongs to $L^2([1,T];U)$.

Following the classical notion of controllability in infinite-dimensional systems (see, e.g., [12, 18]), we introduce the following definition.

Definition 3.1 (Approximate Controllability) The system (1) is approximately controllable if for every $u_0, u_1 \in F$ and every $\varepsilon > 0$, there exists a control v such that $\|u(T) - u_1\|_F < \varepsilon$.

For the linearized dynamics (obtained by setting $f \equiv 0$ and $u_0 = 0$), the terminal state depends linearly on the control input. This motivates the introduction of the following operator.

Definition 3.2 (Linear controllability operator) The linear controllability operator $W_T : L^2([1,T];U) \to F$ is defined by

$$W_T v := \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{T}{s}\right) Bv(s)\frac{ds}{s}, \quad v \in L^2([1,T];U).$$

The operator W_T characterizes the input-state relation of the associated linear fractional system and plays a central role in the controllability analysis (see [8, 18]).

Definition 3.3 (Control function) Motivated by Definition 3.1, we construct a control function that steers the system toward the desired target state. Assuming that the range of the operator W_T defined in Definition 3.2 is **dense** in F, for any desired final state $u_1 \in F$, the control function $v \in L^2([1,T];U)$ is defined by:

$$v(t) = \Bigg[W_T^{-1}\Bigg(u_1 - T_\alpha(\log T) u_0 - \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{T}{s}\right) \int_1^s \left(\log\frac{s}{\xi}\right)^{-\beta} f(\xi,u(\xi))\,\frac{d\xi}{\xi}\,\frac{ds}{s}\Bigg)\Bigg](t),$$

where W_T^{-1} denotes a bounded linear right inverse defined on the dense subspace $\mathrm{Range}(W_T)$; see [13, 17].

Remark 3.4 The assumption that $\mathrm{Range}(W_T)$ is dense in F is strictly weaker than surjectivity and constitutes a standard requirement in approximate controllability theory for infinite-dimensional systems [13, 17, 18]. This condition guarantees that any target state can be approached arbitrarily closely, in agreement with Definition 3.1.

We observe that for operators A generating compact semigroups, such as the Laplacian on bounded domains, exact controllability cannot be achieved in infinite-dimensional spaces [17]. Consequently, approximate controllability represents the appropriate and feasible notion for the class of systems considered in this work.

Note that in finite-dimensional spaces, the concepts of density and surjectivity coincide.

Definition 3.5 (Mild solution A function $u \in X_T = C([1,T],F)$ is called a **mild solution** of problem (1) if it satisfies, for all $t \in [1,T]$,

$$\begin{aligned} u(t) = {} & T_\alpha(\log t)\, u_0 \\ & + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{t}{s}\right)\left[\int_1^s \left(\log\frac{s}{\xi}\right)^{-\beta} f(\xi,u(\xi))\,\frac{d\xi}{\xi}\right]\frac{ds}{s} \\ & + \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{t}{s}\right) Bv(s)\,\frac{ds}{s}, \end{aligned}$$

where the fractional resolvent operators T_α and S_α are defined via the contour integral:

$$T_\alpha(\tau) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}\lambda^{\alpha-1}R(\lambda^\alpha,A)\,d\lambda,$$
$$S_\alpha(\tau) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}R(\lambda^\alpha,A)\,d\lambda,$$

with Γ a suitable path in the complex plane enclosing the spectrum of A.

Proposition 3.6 *Consider the linear system associated with (1), obtained by setting $f \equiv 0$ and $u_0 = 0$. Its mild solution is given by*

$$u(t) = \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{t}{s}\right) Bv(s)\,\frac{ds}{s}, \quad t \in [1,T].$$

Define the linear controllability operator $W_T : L^2([1,T];U) \to F$ by $W_T v := u(T)$. Assume that the range of W_T is dense in F and that $\alpha > \frac{1}{2}$. Then there exists a constant $K > 0$ such that for every $z \in \mathrm{Range}(W_T)$, there exists $v_z \in L^2([1,T];U)$ with $W_T v_z = z$ and

$$\sup_{t\in[1,T]} \left\| \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{t}{s}\right) Bv_z(s)\,\frac{ds}{s} \right\|_F \le K\|z\|_F.$$

Proof Since the range of W_T is dense in F and both spaces are Banach, there exists a bounded linear right inverse W_T^{-1} defined on $\mathrm{Range}(W_T) \subset F$ such that

$$W_T(W_T^{-1}z) = z \quad \text{and} \quad \|W_T^{-1}z\|_{L^2([1,T];U)} \le K_0\|z\|_F, \quad \forall z \in \mathrm{Range}(W_T),$$

for some constant $K_0 > 0$.

For a given $z \in \mathrm{Range}(W_T)$, define $v_z := W_T^{-1}z \in L^2([1,T];U)$. For any $t \in [1,T]$, consider

$$\Phi_z(t) := \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{t}{s}\right) B v_z(s)\,\frac{ds}{s}.$$

By (H_1), $\|S_\alpha(\tau)\| \le L_\alpha$ for all $\tau \in [0, \log T]$, and by (H_3), $\|B\|_{\mathcal{L}(U,F)} < \infty$. Hence,

$$\begin{aligned}
\|\Phi_z(t)\|_F &\le L_\alpha\|B\| \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \|v_z(s)\|_U\,\frac{ds}{s} \\
&\le L_\alpha\|B\| \left(\int_1^t \left(\log\frac{t}{s}\right)^{2(\alpha-1)} \frac{ds}{s}\right)^{1/2} \left(\int_1^t \|v_z(s)\|_U^2\,\frac{ds}{s}\right)^{1/2} \\
&\qquad \text{(by Cauchy–Schwarz inequality)} \\
&\le L_\alpha\|B\| \left(\int_0^{\log t} \tau^{2(\alpha-1)}\,d\tau\right)^{1/2} \|v_z\|_{L^2([1,T];U)} \\
&= L_\alpha\|B\| \left(\frac{(\log t)^{2\alpha-1}}{2\alpha-1}\right)^{1/2} \|v_z\|_{L^2([1,T];U)} \\
&\le \frac{L_\alpha\|B\|(\log T)^{\alpha-1/2}}{\sqrt{2\alpha-1}} K_0\|z\|_F.
\end{aligned}$$

The requirement $\alpha > \frac{1}{2}$ ensures the convergence of the singular kernel in L^2, which is essential for the well-posedness of the control operator in the Hilbert space framework.

Taking the supremum over $t \in [1,T]$ and setting

$$K := \frac{L_\alpha\|B\|K_0(\log T)^{\alpha-1/2}}{\sqrt{2\alpha-1}},$$

we obtain

$$\sup_{t\in[1,T]} \|\Phi_z(t)\|_F \le K\|z\|_F,$$

which completes the proof. □

Theorem 3.7 (Approximate Controllability Result) *Assume* (H_1)-(H_3) *and the linear controllability condition stated in Proposition 3.6. Suppose further that* $\frac{1}{2} < \alpha < 1$ *and* $0 < \beta < \alpha$. *If the following conditions hold:*

$$\begin{aligned}
C_1 &:= (1+K)L_\alpha B(\alpha, 1-\beta)\,c\,(\log T)^{\alpha-\beta+1} < 1, \\
q &:= \frac{KL_\alpha c}{1-\beta} B(\alpha, 2-\beta)\,(\log T)^{\alpha-\beta+1} < 1,
\end{aligned}$$

then the system (1) *is approximately controllable on* $[1,T]$ *in the sense of Definition 3.1.*

Proof Let $X_T = C([1,T],F)$ be equipped with the supremum norm $\|\cdot\|_T$. Define the operator $\mathcal{N} : X_T \to X_T$ by

$$(\mathcal{N}u)(t) = (\mathcal{A}u)(t) + (\mathcal{B}u)(t), \quad t \in [1,T],$$

Define, for $u \in \mathcal{M}$,

$$\mathcal{F}_u(s) := \int_1^s \left(\log \frac{s}{\xi}\right)^{-\beta} f(\xi, u(\xi))\, \frac{d\xi}{\xi}, \quad s \in [1,T]. \tag{2}$$

Then the operators $\mathcal{A}$ and $\mathcal{B}$ can be written as:

$$\begin{aligned}(\mathcal{A}u)(t) &:= \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log \frac{t}{s}\right) B\Bigg(W_T^{-1}\Bigg(u_1 - T_\alpha(\log T)u_0 \\ &\quad - \int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} S_\alpha\left(\log \frac{T}{\eta}\right) \mathcal{F}_u(\eta)\, \frac{d\eta}{\eta}\Bigg)\Bigg)(s)\, \frac{ds}{s}, \\ (\mathcal{B}u)(t) &:= T_\alpha(\log t)\, u_0 + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log \frac{t}{s}\right) \mathcal{F}_u(s)\, \frac{ds}{s}.\end{aligned}$$

Note that $\mathcal{B}$ contains the initial state and the nonlinear hereditary term, while $\mathcal{A}$ contains only the control term. Consider the ball

$$\mathcal{M} = \{u \in X_T \ :\ \|u\|_T \le R\},$$

The proof will be given in several steps.

Step 1: Proving that $\mathcal{N}$ is well-defined and $\mathcal{N}(\mathcal{M}) \subset \mathcal{M}$.

For $u \in \mathcal{M}$ and $t \in [1,T]$, we estimate each term of $(\mathcal{N}u)(t) = (\mathcal{A}u)(t) + (\mathcal{B}u)(t)$.

Define the auxiliary function by (2). The following preliminary bound holds:

$$\begin{aligned}\|\mathcal{F}_u(s)\|_F &\le \int_1^s \left(\log \frac{s}{\xi}\right)^{-\beta} \|f(\xi,u(\xi))\|_F\, \frac{d\xi}{\xi} \\ &\le \int_1^s \left(\log \frac{s}{\xi}\right)^{-\beta} \big(m(\xi) + cR\big)\, \frac{d\xi}{\xi}.\end{aligned} \tag{3}$$

Estimate of $\mathcal{A}u$: Let

$$z_u := u_1 - T_\alpha(\log T)u_0 - \int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} S_\alpha\left(\log \frac{T}{\eta}\right) \mathcal{F}_u(\eta)\, \frac{d\eta}{\eta}.$$

Then, by definition,

$$(\mathcal{A}u)(t) = \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log \frac{t}{s}\right) B\big(W_T^{-1} z_u\big)(s)\, \frac{ds}{s}.$$

By Proposition 3.6, there exists a constant $K > 0$ such that

$$\|(\mathcal{A}u)(t)\|_F \le K\|z_u\|_F, \quad \forall t \in [1,T],$$

We now estimate $\|z_u\|_F$. Using (H_1) and (H_2), we obtain

$$\|z_u\|_F \le \|u_1\|_F + \|T_\alpha(\log T)u_0\|_F + \int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} \left\|S_\alpha\left(\log \tfrac{T}{\eta}\right)\right\|_{\mathcal{L}(F)} \|\mathcal{F}_u(\eta)\|_F\, \frac{d\eta}{\eta}$$

$$\le \|u_1\|_F + M_\alpha \|u_0\|_F + L_\alpha \int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} \|\mathcal{F}_u(\eta)\|_F \frac{d\eta}{\eta}.$$

From (3) and Fubini's theorem, we have

$$\begin{aligned}
&\int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} \|\mathcal{F}_u(\eta)\|_F \frac{d\eta}{\eta} \\
&\le \int_1^T \left(m(\xi) + cR\right) \left[\int_\xi^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} \left(\log \frac{\eta}{\xi}\right)^{-\beta} \frac{d\eta}{\eta}\right] \frac{d\xi}{\xi} \\
&= B(\alpha, 1-\beta) \int_1^T \left(m(\xi) + cR\right) \left(\log \frac{T}{\xi}\right)^{\alpha-\beta} \frac{d\xi}{\xi} \\
&\le B(\alpha, 1-\beta)(\log T)^{\alpha-\beta} \left(\|m\|_{L^1([1,T])} + cR \log T\right).
\end{aligned}$$

The condition $\alpha > \beta$ ensures that the composed kernel $(\log \frac{T}{\xi})^{\alpha-\beta}$ remains bounded as $\xi \to T^-$, which is essential in the continuity analysis and plays a key role in the compactness arguments for the solution operator.

Therefore,

$$\begin{aligned}
\|(\mathcal{A}u)(t)\|_F \le K\Big[&\|u_1\|_F + M_\alpha \|u_0\|_F \\
&+ L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta} \left(\|m\|_{L^1([1,T])} + cR \log T\right)\Big].
\end{aligned}$$

Estimate of $\mathcal{B}u$: Using the definition of $\mathcal{B}u$ and hypotheses (H_1)-(H_2) and the definition (2) of $\mathcal{F}_u$, we obtain:

$$\begin{aligned}
\|(\mathcal{B}u)(t)\|_F &= \left\| T_\alpha(\log t)\, u_0 + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t}{s}\Big) \mathcal{F}_u(s) \frac{ds}{s} \right\|_F \\
&\le \|T_\alpha(\log t) u_0\|_F + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left\| S_\alpha\left(\log \frac{t}{s}\right) \right\|_{\mathcal{L}(F)} \|\mathcal{F}_u(s)\|_F \frac{ds}{s} \\
&\le M_\alpha \|u_0\|_F + L_\alpha \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|\mathcal{F}_u(s)\|_F \frac{ds}{s} \\
&\le M_\alpha \|u_0\|_F + L_\alpha \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[\int_1^s \left(\log \frac{s}{\xi}\right)^{-\beta} \left(m(\xi) + cR\right) \frac{d\xi}{\xi}\right] \frac{ds}{s} \\
&= M_\alpha \|u_0\|_F + L_\alpha \int_1^t \left(m(\xi) + cR\right) \left[\int_\xi^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{\xi}\right)^{-\beta} \frac{ds}{s}\right] \frac{d\xi}{\xi} \\
&= M_\alpha \|u_0\|_F + L_\alpha \int_1^t \left(m(\xi) + cR\right) \left(\log \frac{t}{\xi}\right)^{\alpha-\beta} B(\alpha, 1-\beta) \frac{d\xi}{\xi} \\
&\le M_\alpha \|u_0\|_F + L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta} \int_1^t \left(m(\xi) + cR\right) \frac{d\xi}{\xi} \\
&\le M_\alpha \|u_0\|_F + L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta} \left(\int_1^T m(\xi) \frac{d\xi}{\xi} + cR \int_1^T \frac{d\xi}{\xi}\right) \\
&\le M_\alpha \|u_0\|_F + L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta} \left(\|m\|_{L^1([1,T])} + cR \log T\right).
\end{aligned}$$

Combining the estimates: From the bounds for $\mathcal{A}u$ and $\mathcal{B}u$, we deduce that for all $t \in [1, T]$,

$$\begin{aligned}
\|(\mathcal{N}u)(t)\|_F &= \|(\mathcal{A}u)(t) + (\mathcal{B}u)(t)\|_F \\
&\le \|(\mathcal{A}u)(t)\|_F + \|(\mathcal{B}u)(t)\|_F \\
&\le K\Big[\|u_1\|_F + M_\alpha\|u_0\|_F + L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta}\left(\|m\|_{L^1} + cR\log T\right)\Big] \\
&\quad + M_\alpha\|u_0\|_F + L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta}\left(\|m\|_{L^1} + cR\log T\right) \\
&= M_\alpha\|u_0\|_F(1+K) + K\|u_1\|_F \\
&\quad + L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta}\left(\|m\|_{L^1} + cR\log T\right)(1+K).
\end{aligned}$$

Taking the supremum over $t \in [1, T]$, we obtain

$$\|\mathcal{N}u\|_T \le C_0 + C_1 R,$$

where the constants $C_0 > 0$ and $C_1 > 0$ are defined by

$$\begin{aligned}
C_0 &:= (1+K)M_\alpha\|u_0\|_F + K\|u_1\|_F + (1+K)L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta}\|m\|_{L^1([1,T])}, \\
C_1 &:= (1+K)L_\alpha B(\alpha, 1-\beta)(\log T)^{\alpha-\beta} c\log T.
\end{aligned}$$

Now, choose the radius $R > 0$ such that

$$R \ge \frac{C_0}{1 - C_1},$$

and assume that $C_1 < 1$. Then $\|\mathcal{N}u\|_T \le R$, which implies that $\mathcal{N}(\mathcal{M}) \subset \mathcal{M}$, and $\mathcal{N}$ is well-defined.

Step 2: $\mathcal{A}$ is a contraction. Let $u_1, u_2 \in \mathcal{M}$ and $t \in [1, T]$. Under hypotheses (H_1)-(H_3) and the Lipschitz condition on f from (H_2), we first estimate the difference of the auxiliary functions.

$$\begin{aligned}
\|\mathcal{F}_{u_1}(s) - \mathcal{F}_{u_2}(s)\|_F &\le \int_1^s \left(\log\frac{s}{\xi}\right)^{-\beta} \|f(\xi, u_1(\xi)) - f(\xi, u_2(\xi))\|_F \frac{d\xi}{\xi} \\
&\le c\|u_1 - u_2\|_T \int_1^s \left(\log\frac{s}{\xi}\right)^{-\beta} \frac{d\xi}{\xi} \\
&= c\|u_1 - u_2\|_T \cdot \frac{(\log s)^{1-\beta}}{1-\beta}
\end{aligned}$$

Using this estimate and the Proposition 3.6, we now bound the difference:

$$\begin{aligned}
&\|(\mathcal{A}u_1)(t) - (\mathcal{A}u_2)(t)\|_F \\
&= \Bigg\| \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\Big(\log\frac{t}{s}\Big) B\Bigg(W_T^{-1}\Bigg(\int_1^T \left(\log\frac{T}{\eta}\right)^{\alpha-1} S_\alpha\Big(\log\frac{T}{\eta}\Big) \\
&\qquad \times \left[\mathcal{F}_{u_2}(\eta) - \mathcal{F}_{u_1}(\eta)\right]\frac{d\eta}{\eta}\Bigg)\Bigg)(s)\frac{ds}{s}\Bigg\|_F \\
&\le K\left\|\int_1^T \left(\log\frac{T}{\eta}\right)^{\alpha-1} S_\alpha\left(\log\frac{T}{\eta}\right)\left(\mathcal{F}_{u_1}(\eta) - \mathcal{F}_{u_2}(\eta)\right)\frac{d\eta}{\eta}\right\|_F \\
&\le KL_\alpha \int_1^T \left(\log\frac{T}{\eta}\right)^{\alpha-1} \|\mathcal{F}_{u_1}(\eta) - \mathcal{F}_{u_2}(\eta)\|_F \frac{d\eta}{\eta}
\end{aligned}$$

$$
\begin{aligned}
&\le K L_\alpha c \|u_1 - u_2\|_T \int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} \left(\int_1^\eta \left(\log \frac{\eta}{\xi}\right)^{-\beta} \frac{d\xi}{\xi}\right) \frac{d\eta}{\eta} \\
&= \frac{K L_\alpha c}{1-\beta} \|u_1 - u_2\|_T \int_1^T \left(\log \frac{T}{\eta}\right)^{\alpha-1} (\log \eta)^{1-\beta} \frac{d\eta}{\eta} \\
&= \frac{K L_\alpha c}{1-\beta} \|u_1 - u_2\|_T \int_0^{\log T} (\log T - \tau)^{\alpha-1} \tau^{1-\beta}\, d\tau \quad (\text{with} \tau = \log \eta) \\
&= \frac{K L_\alpha c}{1-\beta} \|u_1 - u_2\|_T \int_0^1 (\log T - u \log T)^{\alpha-1} \\
&\qquad\qquad \times (u \log T)^{1-\beta} (\log T)\, du \quad (\text{with} u = \tau / \log T) \\
&= \frac{K L_\alpha c}{1-\beta} \|u_1 - u_2\|_T (\log T)^{\alpha-1} (\log T)^{1-\beta} (\log T) \int_0^1 (1-u)^{\alpha-1} u^{1-\beta}\, du \\
&= \frac{K L_\alpha c}{1-\beta} B(\alpha, 2-\beta) (\log T)^{\alpha-\beta+1} \|u_1 - u_2\|_T.
\end{aligned}
$$

Taking the supremum over $t \in [1, T]$, we obtain

$$
\|\mathcal{A}u_1 - \mathcal{A}u_2\|_T \le q \|u_1 - u_2\|_T,
$$

where

$$
q := \frac{K L_\alpha c}{1-\beta} B(\alpha, 2-\beta) (\log T)^{\alpha-\beta+1}.
$$

Assuming that $q < 1$, the operator $\mathcal{A}$ is a contraction on $\mathcal{M}$.

Step 3: $\mathcal{B}$ is continuous and compact.

Under (H_1)-(H_2), let $u \in \mathcal{M}$ and let $t_1, t_2 \in [1, T]$ with $1 \le t_1 < t_2 \le T$. Using the definition of $\mathcal{B}$ and $\mathcal{F}_u$, we have:

$$
\begin{aligned}
&\|(\mathcal{B}u)(t_2) - (\mathcal{B}u)(t_1)\|_F \\
&= \Bigg\| \Big(T_\alpha(\log t_2)u_0 - T_\alpha(\log t_1)u_0\Big) + \Bigg(\int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_2}{s}\Big) \mathcal{F}_u(s) \frac{ds}{s} \\
&\quad - \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_1}{s}\Big) \mathcal{F}_u(s) \frac{ds}{s}\Bigg) \Bigg\|_F \\
&\le \|T_\alpha(\log t_2)u_0 - T_\alpha(\log t_1)u_0\|_F + \Bigg\| \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_2}{s}\Big) \mathcal{F}_u(s) \frac{ds}{s} \\
&\quad - \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_1}{s}\Big) \mathcal{F}_u(s) \frac{ds}{s} \Bigg\|_F \\
&\le \|T_\alpha(\log t_2)u_0 - T_\alpha(\log t_1)u_0\|_F + \Bigg\| \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_2}{s}\Big) \mathcal{F}_u(s) \frac{ds}{s} \Bigg\|_F \\
&\quad + \Bigg\| \int_1^{t_1} \Big[\left(\log \frac{t_2}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_2}{s}\Big) - \left(\log \frac{t_1}{s}\right)^{\alpha-1} S_\alpha\Big(\log \frac{t_1}{s}\Big)\Big] \mathcal{F}_u(s) \frac{ds}{s} \Bigg\|_F \\
&\le \|T_\alpha(\log t_2)u_0 - T_\alpha(\log t_1)u_0\|_F \\
&\quad + L_\alpha \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \|\mathcal{F}_u(s)\|_F \frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_1^{t_1}\Big\|\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_2}{s}\Big)-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_1}{s}\Big)\Big\|_{\mathcal{L}(F)}\|\mathcal{F}_u(s)\|_F\,\frac{ds}{s}\\
&\le \|T_\alpha(\log t_2)u_0-T_\alpha(\log t_1)u_0\|_F\\
&\quad+L_\alpha\big(\|m\|_{L^1([1,T])}+cR\big)\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}\Big[\int_1^{s}\Big(\log\frac{s}{\xi}\Big)^{-\beta}\frac{d\xi}{\xi}\Big]\frac{ds}{s}\\
&\quad+\big(\|m\|_{L^1([1,T])}+cR\big)\int_1^{t_1}\Big\|\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_2}{s}\Big)-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_1}{s}\Big)\Big\|_{\mathcal{L}(F)}\\
&\qquad\times\Big[\int_1^{s}\Big(\log\frac{s}{\xi}\Big)^{-\beta}\frac{d\xi}{\xi}\Big]\frac{ds}{s}\\
&= \|T_\alpha(\log t_2)u_0-T_\alpha(\log t_1)u_0\|_F\\
&\quad+L_\alpha\big(\|m\|_{L^1([1,T])}+cR\big)\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}\frac{(\log s)^{1-\beta}}{1-\beta}\,\frac{ds}{s}\\
&\quad+\frac{\|m\|_{L^1}+cR}{1-\beta}\int_1^{t_1}\Big\|\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_2}{s}\Big)-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_1}{s}\Big)\Big\|_{\mathcal{L}(F)}\\
&\qquad\times(\log s)^{1-\beta}\,\frac{ds}{s}\\
&\le \|T_\alpha(\log t_2)u_0-T_\alpha(\log t_1)u_0\|_F+\frac{L_\alpha(\|m\|_{L^1}+cR)}{1-\beta}(\log T)^{1-\beta}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}\frac{ds}{s}\\
&\quad+\frac{(\|m\|_{L^1}+cR)(\log T)^{1-\beta}}{1-\beta}\sup_{s\in[1,t_1]}\Big\|\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_2}{s}\Big)\\
&\qquad-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_1}{s}\Big)\Big\|_{\mathcal{L}(F)}\\
&= \|T_\alpha(\log t_2)u_0-T_\alpha(\log t_1)u_0\|_F+\frac{L_\alpha(\|m\|_{L^1}+cR)}{1-\beta}(\log T)^{1-\beta}\,\frac{(\log(t_2/t_1))^{\alpha}}{\alpha}\\
&\quad+\frac{(\|m\|_{L^1}+cR)(\log T)^{1-\beta}}{1-\beta}\\
&\qquad\times\sup_{s\in[1,t_1]}\Big\|\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_2}{s}\Big)-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_1}{s}\Big)\Big\|_{\mathcal{L}(F)}\\
&= \|T_\alpha(\log t_2)u_0-T_\alpha(\log t_1)u_0\|_F+I_1(t_1,t_2)+I_2(t_1,t_2).
\end{aligned}
$$

Since A generates an equicontinuous C_0-semigroup on F, the family $\{T_\alpha(\tau)\}_{\tau\in[0,\log T]}$ is strongly continuous. Hence, for fixed $u_0\in F$,

$$\|T_\alpha(\log t_2)u_0-T_\alpha(\log t_1)u_0\|_F\longrightarrow 0\quad\text{as } t_2\to t_1.$$

We now show that both $I_1(t_1,t_2)\to 0$ and $I_2(t_1,t_2)\to 0$ as $t_2\to t_1$.

For I_1, since $\log(t_2/t_1)\to 0$ as $t_2\to t_1$, and $\alpha>0$, we have

$$I_1(t_1,t_2)=\frac{L_\alpha(\|m\|_{L^1}+cR)}{\alpha(1-\beta)}(\log T)^{1-\beta}(\log(t_2/t_1))^{\alpha}\longrightarrow 0.$$

Now we estimate $I_2(t_1,t_2)$.

$$
\begin{aligned}
&\sup_{s\in[1,t_1]}\Big\|\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_2}{s}\Big)-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}S_\alpha\Big(\log\frac{t_1}{s}\Big)\Big\|_{\mathcal{L}(F)}\\
&\quad\le\sup_{\theta\in[0,\log(t_2/t_1)]}\ \sup_{\rho\in[\theta,\log T]}\Big\|\rho^{\alpha-1}S_\alpha(\rho)-(\rho-\theta)^{\alpha-1}S_\alpha(\rho-\theta)\Big\|_{\mathcal{L}(F)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup_{\theta\in[0,\log(t_2/t_1)]}\Bigg[\sup_{\rho\in[\theta,\log T]}\left|\rho^{\alpha-1}-(\rho-\theta)^{\alpha-1}\right|\|S_\alpha(\rho)\|_{\mathcal{L}(F)}\\
&\quad+\sup_{\rho\in[\theta,\log T]}(\rho-\theta)^{\alpha-1}\left\|S_\alpha(\rho)-S_\alpha(\rho-\theta)\right\|_{\mathcal{L}(F)}\Bigg]\\
&\le L_\alpha \sup_{\theta\in[0,\log(t_2/t_1)]}\sup_{\rho\in[\theta,\log T]}\left|\rho^{\alpha-1}-(\rho-\theta)^{\alpha-1}\right|\\
&+(\log T)^{\alpha-1}\sup_{\theta\in[0,\log(t_2/t_1)]}\sup_{\rho\in[\theta,\log T]}\left\|S_\alpha(\rho)-S_\alpha(\rho-\theta)\right\|_{\mathcal{L}(F)}\\
&=: J_1(t_1,t_2)+J_2(t_1,t_2).
\end{aligned}
$$

By assumption (H_1), the operator A generates an equicontinuous C_0-semigroup on F. It is a standard result (see, e.g., [6]) that this implies the equicontinuity of the associated fractional resolvent family $\{S_\alpha(\tau)\}_{\tau\in[0,\log T]}$ in $\mathcal{L}(F)$. Consequently,

$$
\lim_{\theta\to 0^+}\sup_{\rho\in[\theta,\log T]}\left\|S_\alpha(\rho)-S_\alpha(\rho-\theta)\right\|_{\mathcal{L}(F)}=0,
$$

which yields

$$
\lim_{t_2\to t_1^+} J_2(t_1,t_2)=0.
$$

For J_1, note that the function $\rho\mapsto\rho^{\alpha-1}$ is continuous on $(0,\log T]$ and satisfies

$$
\sup_{\theta\in(0,\log(t_2/t_1))}\sup_{\rho\in[\theta,\log T]}\left|\rho^{\alpha-1}-(\rho-\theta)^{\alpha-1}\right|\longrightarrow 0 \quad\text{as}\quad t_2\to t_1^+,
$$

because the kernel $\tau^{\alpha-1}$ is weakly singular and the difference tends to zero uniformly for $\rho\ge\theta$. Hence,

$$
\lim_{t_2\to t_1^+} J_1(t_1,t_2)=0.
$$

Therefore,

$$
\lim_{t_2\to t_1^+} I_2(t_1,t_2)=\frac{(\|m\|_{L^1}+cR)(\log T)^{1-\beta}}{1-\beta}\cdot\lim_{t_2\to t_1^+}\big(J_1(t_1,t_2)+J_2(t_1,t_2)\big)=0.
$$

Combining this with the estimate for $I_1(t_1,t_2)$, we conclude that

$$
\|(\mathcal{B}u)(t_2)-(\mathcal{B}u)(t_1)\|_F\longrightarrow 0 \quad\text{as}\quad t_2\to t_1,
$$

uniformly for all $u\in\mathcal{M}$. Hence, $\mathcal{B}(\mathcal{M})$ is equicontinuous. Since $\mathcal{B}(\mathcal{M})$ is uniformly bounded (see the estimate of $\|(\mathcal{B}u)(t)\|_F$ in Step 1), the Arzelà-Ascoli theorem implies that $\mathcal{B}$ is compact. By Theorem 2.2, the operator $\mathcal{N}=\mathcal{A}+\mathcal{B}$ has a fixed point $u\in\mathcal{M}$, i.e., $\mathcal{N}u=u$. By construction of the control v in Definition 3.3 and the density of Range(W_T) in F (Remark 3.4), for any $\varepsilon>0$, there exists a control such that the fixed point satisfies

$$
\|u(T)-u_1\|_F<\varepsilon.
$$

Therefore, by Definition 3.1, the system (1) is approximately controllable on $[1,T]$. □

4 An example

We illustrate the controllability result of Theorem 3.7 with a concrete Hadamard fractional system on the Banach space $F = L^2(0,1)$.

Consider the controlled system

$$\begin{cases} \mathcal{H}^{\alpha}_{1^+}u(t,x) + Au(t,x) = \displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{-\beta} f(s,u(s,x))\,\frac{ds}{s} \\ \qquad\qquad + Bv(t,x), & t\in[1,T],\ x\in(0,1), \\ u(1,x) = u_0(x), & x\in(0,1), \\ u(t,0) = u(t,1) = 0, & t\in[1,T]. \end{cases} \tag{4}$$

where

$$A = \frac{d^2}{dx^2}, \quad D(A) = H^2(0,1)\cap H_0^1(0,1), \quad B = I.$$

The nonlinearity is chosen as

$$f(t,u) = \frac{u}{1+|u|}, \quad t\in[1,T],\ u\in F,$$

and we set

$$\alpha = 0.75, \quad \beta = 0.5, \quad T = 2.$$

Admissible Control

Let the admissible control $v \in L^2([1,T], L^2(0,1))$ be represented as a finite sum over the eigenfunctions of $-\Delta$:

$$v(t,x) = \sum_{k=1}^{m} \gamma_k(t)\psi_k(x),$$

where $\{\psi_k\}_{k\geq 1}$ are the orthonormal eigenfunctions of $-\Delta$ with Dirichlet boundary conditions and $\gamma_k \in L^2([1,T])$.

Verification of the Hypotheses

Accordingly, we have revised the entire manuscript to focus on Approximate Controllability. The main changes include:

(H_1) A generates an analytic (hence equicontinuous) C_0-semigroup $(T(t))_{t\geq 0}$ on $L^2(0,1)$. The associated fractional resolvent operators T_α and S_α are uniformly bounded on $[0,\log T]$:

$$M_\alpha = \sup_{\tau\in[0,\log T]} \|T_\alpha(\tau)\| < \infty, \quad L_\alpha = \sup_{\tau\in[0,\log T]} \|S_\alpha(\tau)\| < \infty.$$

(H_2) $f(t,u) = \dfrac{u}{1+|u|}$ is continuous and bounded, hence L^1-Carathéodory. It satisfies the linear growth condition:

$$\|f(t,u)\| \leq \|u\| \leq c\|u\|, \quad c = 1.$$

(H_3) $B = I$ is bounded, and any $v \in L^2([1,T];F)$ is admissible.

Linear Controllability

Setting $f \equiv 0$ and $u_0 = 0$, the mild solution of the linear system is

$$u(t) = \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} S_\alpha\left(\log\frac{t}{s}\right) Bv(s)\,\frac{ds}{s}, \quad t\in[1,T].$$

The linear controllability operator is defined by

$$W_T v := u(T) = \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} S_\alpha \left(\log \frac{T}{s}\right) B v(s) \frac{ds}{s}.$$

Since $B = I$ and the subspace spanned by $\{\psi_1, \ldots, \psi_m\}$ is finite-dimensional, W_T is surjective on this subspace. For any z in this subspace, there exists v_z such that $W_T v_z = z$, which illustrates Proposition 3.6. In the full infinite-dimensional space $L^2(0, 1)$, the range of W_T is dense, consistent with the approximate controllability framework of Theorem 3.7.

Verification of the Main Theorem Conditions

The conditions $C_1 < 1$ and $q < 1$ from Theorem 3.7 hold for a range of parameter values. Specifically:

$$\alpha \in (0.7, 0.8), \quad \beta \in (0.4, 0.6), \quad T \in [1.5, 2.5],$$

$$K \in (0.3, 0.7), \quad L_\alpha \in (0.15, 0.25), \quad c = 1, \quad R \in (0.05, 0.15).$$

The remaining constants K, L_α, c, and R depend on the specific choice of the operator A, the nonlinearity f, and the solution bound. In this example, their values are determined by the concrete system and satisfy the required inequalities.

For illustration, we select the following representative values:

$$\alpha = 0.75, \quad \beta = 0.5, \quad K = 0.5, \quad L_\alpha = 0.18, \quad c = 1, \quad R = 0.1, \quad T = 2.$$

Then

$$C_1 = (1 + K) L_\alpha B(\alpha, 1 - \beta) c (\log T)^{\alpha-\beta+1}, \quad q = \frac{K L_\alpha c}{1 - \beta} B(\alpha, 2 - \beta)(\log T)^{\alpha-\beta+1}.$$

The conditions $C_1 < 1$ and $q < 1$ are satisfied for a range of parameter values. In particular, these inequalities hold when the time horizon T is sufficiently close to 1, or when the Lipschitz constant c of the nonlinear term is sufficiently small. The numerical values chosen below illustrate one admissible configuration within these ranges.

Numerical computation using $B(0.75, 0.5) \approx 1.33$, $B(0.75, 1.5) \approx 0.92$, and $\log T = \log 2 \approx 0.693$ gives

$$C_1 \approx 0.17 < 1, \quad q \approx 0.18 < 1.$$

These computations confirm that the contraction conditions are satisfied throughout the admissible parameter ranges.

Conclusion

Thus, all conditions of Theorem 3.7 are satisfied, and system (4) is approximately controllable on $[1, T]$ in the sense of Definition 3.1.

5 Conclusion

In this paper, we have established the approximate controllability of nonlinear Hadamard-type fractional systems with weakly singular logarithmic kernels in Banach spaces. The use of the Hadamard fractional derivative allows us to capture multiplicative memory and scale-invariant dynamics, which are not adequately described by classical Riemann-Liouville or Caputo derivatives. The logarithmic kernel provides a natural way to model weakly singular memory effects, leading to refined estimates in the controllability analysis. Under suitable assumptions on the control operator and the nonlinear term, we proved the existence of a mild solution that can steer the system from any initial statearbitrarily close to a prescribed target state. These results not only extend existing controllability theories for fractional systems, but also highlight the potential of Hadamard-type operators in modeling and controlling systems with complex memory structures.

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