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EXAMINATION OF AN ANALOGOUS HYBRID-TYPE FRACTIONAL DIFFERENTIAL PROBLEM UNDER NONSINGULAR KERNEL

MUHAMMAD SHER
Department of Mathematics
University of Malakand
Chakdara Dir(L) 18000
Khyber Pakhtunkhwa, Pakistan
msherpak55@gmail.com

KAMAL SHAH* and MUHAMMAD SARWAR†
Department of Mathematics, University of Malakand
Chakdara Dir(L) 18000
Khyber Pakhtunkhwa, Pakistan

Department of Mathematics and Sciences
Prince Sultan University
Riyadh 11586, Saudi Arabia
**kshah@psu.edu.sa*
†sarwarswati@gmail.com

MANEL HLEILI
Department of Mathematics
Faculty of Science, University of Tabuk
71491 Tabuk, Saudi Arabia
mhleili@ut.edu.sa

‡Corresponding author.

BAHAAELDIN ABDALLA
Department of Mathematics and Sciences
Prince Sultan University, Riyadh 11586, Saudi Arabia
babdallah@psu.edu.sa

THABET ABDELJAWAD[‡]
Department of Fundamental Sciences
Faculty of Engineering and Architecture
Istanbul Gelisim University
Avcilar 34310, Istanbul, Turkey

Department of Mathematics and Sciences
Prince Sultan University, Riyadh 11586, Saudi Arabia

Department of Medical Research
China Medical University, Taichung 40402, Taiwan
tabdeljawad@psu.edu.sa

MANAR A. ALQUDAH
Department of Mathematical Sciences
College of Sciences
Princess Nourah bint Abdulrahman University
P.O. Box 84428, Riyadh 11671, Saudi Arabia
maalqudah@pnu.edu.sa

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Abstract

A class of hybrid fractional differential equations is the subject of the current investigation. Differential operator with Mittag-Leffler-type kernel is used. Proportional delay, a crucial component in electrodynamics and infectious illnesses, is included in the problem. Several factors, including qualitative analyses, Ulam–Hyer-type stability, and a general numerical scheme, have been taken into consideration for the suggested problem. We used functional analysis concepts, in particular the theory of fixed points, to general analysis. Lagrange's interpolation approach, an existence methodology, has been used to study numerical aspect of the considered model. By using the adopted results, an example is studied at the conclusion to support the proposed theory.

Keywords: Theory and Analysis; Exponential Kernel-Type Operators; Numerical Tools.

1. INTRODUCTION

The area devoted to Fractional Calculus (FC) has countless applications in scientific and engineering fields such as physics,[1] Biology,[2] Chemistry[3] and a lots of other fields.[4] The invention of noninteger order derivative came into existence at the same age as integer order derivative being mentioned. Following the introduction of certain newly sophisticated functions and technological advancements, the sector was highly appreciated. As of now, a variety of complex scientific issues have been successfully solved with the aforementioned concept. For example, the model of State-space for lithium-ion batteries,[5] Zika virus transmission dynamics model under Caputo derivative,[6] fractional-order modeling of electric circuits,[7] these model are successfully studied by using FC.

It is important to note that there is no one commonly agreed definition for the term "fractional derivative" is one of its beauties. In the literature, there are several definitions that may be found, including the Riemann–Liouville (RL) concept, Caputo definition,[8] conformable fractional derivative,[9] Atangana–Baleanu definition,[10] We refer to Refs. 6, 7, 10–12 for a recent breakdown of a number of mathematical issues using these concepts as the basis. It has been discovered that RL or Caputo notions are insufficient to explain problems pertaining to material heterogeneities.[13] Authors in Ref. 14 created a fractional derivative with a nonsingular kernel, known as the Caputo–Fabrizio derivative, to get around this problem. Losada and Nieto examined a few of the new notions and properties in Ref. 13. The Caputo–Fabrizio notion applied mathematical analysis to analyze the existence and uniqueness of various problems. See Refs. 15 and 16, respectively, for properties and research of particular boundary value problems employing specified derivative. Locality issues were discovered by Atangana and Baleanu in the Caputo–Fabrizio operator's kernel. They extended the exponential kernel of the Caputo–Fabrizio operator to the Mittag-Leffler kernel[10] in order to get around this problem. The Atangana–Baleanu–Caputo (ABC) derivative is the operator with the new kernel as a result. There are numerous uses for this new term in various scientific domains. For example, Xu *et al.* in Ref. 17 studied HIV-1 model with ABC derivative. Khan and his co-authors in Ref. 18 investigated the popular decay equation using discrete fractional order operator. For more applications and discussions on ABC derivative, see Refs. 18–20 and reference there in. Further, it has been shown that the fractional operators with fractal parameter play a significant role in modeling problems related to fractal and irregular structures. For instance, authors in Ref. 21 studied "the heat transfer of ZnO + Al_2O_3 + TiO_2/DW based ternary hybrid nanofluid under fractal fractional derivative", in Ref. 22 the scholars investigated "The modeling of magnetohydrodynamic Casson hybrid nanofluid with heat transfer enhancement in the context of fractal "fractional operator", in Ref. 22 authors analyzed "The transmission dynamics and sensitivity analysis of pine wilt disease with asymptomatic carriers via fractal-fractional differential operator of Mittag-Leffler kernel" and in Ref. 22, "The dynamics of cooperative reactions based on chemical kinetics with reaction speed: A comparative analysis with singular and nonsingular kernels" has been studied using different fractional concepts.

Delay Differential Equations (DDEs) are a unique class of differential equations where the unknown function depends on both its past and present times. Physical phenomena have a time delay as a result of these issues. DDEs are used in many domains, but particularly in electrodynamics. It is also successfully applied to problems in economics, physics, engineering, and infectious diseases (delay can be inserted for the time a disease takes to display its symptoms). It is especially helpful for modelling issues with memory effects. Finding answers to delay-type problems has taken a great deal of time and effort from many academics (see Refs. 25 and 26). Authors of Refs. 27 and 28 have provided a qualitative analysis of delay differential equations.

The existence, stability, and numerical aspects are very crucial terms of mathematical analysis particularly for differential equations, to see wether the physical problem based on the model really exist and also to check its validity and behavior. The existence theory answer, weather a functional equation has a solution or not if yes then how many solution(s) exist. This mentioned area has been investigated for various important dynamical problems like Refs. 29–31. It also studies that wether a problem of applied fields is correctly model or not. If we insure that the solution exist and unique, so one can say that the considered problem is well posed, stable, and predictable in nature and make the foundation for finding the solution of the dynamical system numerically. Further, Ulam–Hayer's stability shows the relation between the exact solution and any approximate solution and tells that if some one perturbed the original system from initial state, then wether it re-bounce to its initial state or not. Further, the concept insures under what condition the exact solution of the dynamical system must be nearly equal to exact solution. The mentioned stability is studied by different scholars, see, for example, Refs. 32 and 33. Also, the exact solution of fractional problems is quite difficult to find, hence existing are newly adopted and the results are generalized to find the approximate solution of fractional differential problem especially of hybrid type. In the past, different techniques have been utilized to formulate such a theory. However, in our work, the fixed point results will be applied to get the fundamental results of this study. In short, the theory of existence is a backbone of functional equations to guarantee that a mathematical problem of applied sciences and engineering exist, reliable, and valid as well.

A new concept for the modelings of complex problems has been introduced which is term as Hybrid Differential Equations (HDEs) under various initial or boundary value conditions and gained significant interest of many researcher, for instance see Refs. 34 and 35 and references there. The basic idea of HDEs is to combine both discrete and continuous dynamics. The mentioned differential equations can be used to describe and model physical problems having nonhomogeneous behavior. This class of differential equations includes the derivative of unknown function hybrid with the nonlinearity depending on it. The HDEs are important class of differentia equations because they includes several dynamical systems as special cases. Recently, remarkable works have been done in the aforementioned area, see Refs. 36–41. A detailed work on fractals fractional theory was recently published by Atangana and his co-author.[42] These differential equations can be used to study different problems in physics, chemistry, control theory, Biology and other fields of science. This concept can be utilized to study issues that show sudden changes, such as in automated systems that are affected by both continuous and discrete signals. On the application of fractional operators these differential equations ca handel very complicated problem with memory, hereditary and time dependency with discrete and continuous behavior. In this regard, in our study, a class of HDEs of fractional order of linear perturbation type is selected for qualitative analysis and approximate solution. The general problem with order $\omega \in (0, 1]$ is defined as follows:

$$\begin{cases} {}^{\mathrm{ABC}}{}_{0}D_{\theta}^{\omega}[\mathbf{\Phi}(\theta) - \mathbf{F}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta))] \\ \quad = \mathbf{G}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)), \\ \mathbf{\Phi}(0) = \mathbf{\Phi}_0 + \int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}(\xi))d\xi, \end{cases} \tag{1}$$

where $0 < \lambda < 1, \theta \in \mathbf{L} = [0, \mathbf{T}], \mathbf{T} < \infty$, $\mathbf{F}, \mathbf{G} \in C[\mathbf{L} \times \mathbf{R}^2, \mathbf{R}]$ and $\mathbf{H} \in C[\mathbf{R}, \mathbf{R}]$.

By utilizing hybrid theory of fixed point, we set up some important results associated to existence of solution and stability theory to the mentioned problem. Then, by using Lagrange's interpolation method, a general numerical scheme will be developed. At the final stage, an example will be constructed to justify the adopted results. Here, we used the generalized ABC-type fractional operator with non singular kernel. The ABC derivative of real order is considered in the current analysis which has non-local and non-singular characteristics. The non-local and non-singular behavior of the mentioned operator addresses limitations of other fractional operators. The other operators, such as RL, Caputo, Hilfer, have kernels (power-law kernels) which are singular in nature, which some times faces difficulties when analysing memory, hereditary and time dependence effects for real world problems. Unlike other fractional definitions, the ABC concept utilizes the kernel of Mittag-Leffler-type, hence offering continuous memory effects which make this operator useful in the formulation of complex physical issues where other singular operator make problems in its modeling. Further, due to the nonlocal behavior of the kernel of ABC derivative, it can be applied to a variety of problems of different nature.

2. PRELIMINARIES

Here, some basic results are recalled which are important in our study. Let $\Omega = C(\mathbf{L})$ be the Banach space, then for any $\mathbf{\Phi} \in \Omega$, we use the norm as follows:

$$\|\mathbf{\Phi}\| = \sup_{\theta \in \mathbf{L}} |\mathbf{\Phi}(\theta)|.$$

Definition 1 (Ref. 8). For $\mathbf{\Phi} \in C(\mathbf{L})$, the RL integral of arbitrary order $\omega \in (0, 1]$ is given by

$$^{\mathrm{RL}}\mathbf{I}^{\omega}\mathbf{\Phi}(\theta) = \frac{1}{\Gamma(\omega)} \int_0^{\theta} (\theta - \xi)^{\omega - 1}\mathbf{\Phi}(\xi)d\xi, \tag{2}$$

Definition 2 (Ref. 10). ABC-type arbitrary order integral for $\omega(0 < \omega \leq 1)$ is defined as follows:

$$^{\mathrm{ABC}}\mathbf{I}_{0^+}^{\omega}\mathbf{\Phi}(\theta) = \begin{cases} \dfrac{1 - \omega}{M(\omega)}\mathbf{\Phi}(\theta) + \dfrac{\omega}{\Gamma(\omega)M(\omega)} \\ \quad \times \displaystyle\int_0^{\theta} (\theta - \xi)^{\omega - 1}\mathbf{\Phi}(\xi)d\xi. \end{cases} \tag{3}$$

Function $M(\omega)$ in the definition is called normalization such that $M(0) = M(1) = 1$.

Definition 3 (Ref. 10). ABC-type arbitrary order derivative of order $\omega(0 < \omega \leq 1)$ is defined as follows:

$$
{}^{\mathrm{ABC}}\mathbf{D}_0^{\omega}\mathbf{\Phi}(\theta) = \begin{cases} \dfrac{M(\omega)}{(1-\omega)}\displaystyle\int_0^{\theta} \times E_{\omega} \\ \quad \times \left[-\dfrac{\omega}{1-\omega}(\theta-\xi)^{\omega}\right]\mathbf{\Phi}'(\xi)d\xi. \end{cases} \tag{4}
$$

In the above definition, E_ω is Mittag-Leffler function, which is the generalization of the exponential function,

$$
E_{\omega}(\theta) = \sum_{k=0}^{\infty}\frac{\theta^k}{\Gamma(\omega k+1)}, \tag{5}
$$

is the one parameter ($\omega \in R^+$) Mittag-Leffler function and the two-parameter ($\omega, \beta \in R^+$) Mittag-Leffler function is defined as

$$
\mathbb{E}_{\omega,\beta}(\theta) = \sum_{k=0}^{\infty}\frac{\theta^k}{\Gamma(\omega k+\beta)}. \tag{6}
$$

Lemma 4 (Ref. 42). *Corresponding to the problem, we have*

$$
\begin{aligned} {}^{\mathrm{ABC}}D^{\omega}\mathbf{\Phi}(\theta) &= \mathrm{X}(\theta), \quad \textit{with } \omega \in (0,1], \\ \mathbf{\Phi}(0) &= \mathbf{\Phi}_0, \end{aligned}
$$

where $\lim_{\theta\to 0}\mathrm{X}(\theta) = 0$, *we have an equivalent integral form as*

$$
\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \dfrac{1-\omega}{M(\omega)}\mathrm{X}(\theta) + \dfrac{\omega}{M(\omega)\Gamma(\omega)} \\ \quad \times \displaystyle\int_0^{\theta}(\theta-\xi)^{\omega-1}\mathrm{X}(\xi)d\xi, \quad \textit{if } \theta \in (0,\mathbf{T}]. \end{cases}
$$

3. THEORETICAL RESULTS

We will examine our consideration of the existence theory of problem (1) in this section of our paper.

Lemma 5. *In view of Lemma* 4, *the problem*

$$
\begin{cases} {}^{\mathrm{ABC}}{}_0D_{\theta}^{\omega}[\mathbf{\Phi}(\theta) - \mathbf{F}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta))] \\ \quad = \mathbf{G}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)), \\ \mathbf{\Phi}(0) = \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}}\mathbf{H}(\mathbf{\Phi}(\xi))d\xi, \end{cases} \tag{7}
$$

is equivalent to the integral form given by

$$
\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}}\mathbf{H}(\mathbf{\Phi}(\xi))d\xi + \mathbf{F}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) \\ \quad + \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) + \dfrac{\omega}{M(\omega)\Gamma(\omega)} \\ \quad \times \displaystyle\int_0^{\theta}(\theta-\xi)^{\omega-1}\mathbf{G}(\xi,\mathbf{\Phi}(\xi),\mathbf{\Phi}(\lambda\xi))d\xi. \end{cases}
$$

Proof. By using Lemma 4, we get

$$
\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}}\mathbf{H}(\mathbf{\Phi}(\xi))d\xi + \mathbf{F}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) \\ \quad + \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) + \dfrac{\omega}{M(\omega)\Gamma(\omega)} \\ \quad \times \displaystyle\int_0^{\theta}(\theta-\xi)^{\omega-1}\mathbf{G}(\xi,\mathbf{\Phi}(\xi),\mathbf{\Phi}(\lambda\xi))d\xi. \end{cases}
$$

□

Describe an operator as $\mathbf{Z} : \Omega \to \Omega$ by

$$
\mathbf{Z}\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}}\mathbf{H}(\mathbf{\Phi}(\xi))d\xi + \mathbf{F}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) \\ \quad + \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) + \dfrac{\omega}{M(\omega)\Gamma(\omega)} \\ \quad \times \displaystyle\int_0^{\theta}(\theta-\xi)^{\omega-1}\mathbf{G}(\xi,\mathbf{\Phi}(\xi),\mathbf{\Phi}(\lambda\xi))d\xi. \end{cases} \tag{8}
$$

Divide the above operator (8) into two sub-operators as follows:

$$
\mathbf{Z_1}\mathbf{\Phi}(\theta) = \begin{cases} \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) \\ \quad + \mathbf{F}(\theta,\mathbf{\Phi}(\theta),\mathbf{\Phi}(\lambda\theta)) \end{cases} \tag{9}
$$

and

$$
\mathbf{Z_2}\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}}\mathbf{H}(\mathbf{\Phi}(\xi))d\xi + \dfrac{\omega}{M(\omega)\Gamma(\omega)} \\ \quad \times \displaystyle\int_0^{\theta}(\theta-\xi)^{\omega-1}\mathbf{G}(\xi,\mathbf{\Phi}(\xi),\mathbf{\Phi}(\lambda\xi))d\xi. \end{cases} \tag{10}
$$

From Eqs. (9) and (10), $\mathbf{Z}(\mathbf{\Phi})$ implies

$$
\mathbf{Z}(\mathbf{\Phi}) = \mathbf{Z_1}(\mathbf{\Phi}) + \mathbf{Z_2}(\mathbf{\Phi}). \tag{11}
$$

We need the hypothesis deduced as follows:

$\mathbf{A_1}$ Corresponding to $\boldsymbol{\ell} > 0$, we have
$$\|\mathbf{H}\| \leq \boldsymbol{\ell}\|\boldsymbol{\Phi}\|.$$

$\mathbf{A_2}$ Let $\mathbf{v} > 0$ be a real constant, then one has
$$|\mathbf{H}(\boldsymbol{\Phi}) - \mathbf{H}(\boldsymbol{\Phi}')| \leq \mathbf{v}|\boldsymbol{\Phi} - \boldsymbol{\Phi}'|.$$

$\mathbf{A_3}$ For $\mathbf{m}_1 > 0$, $\mathbf{m}_2 > 0$ and $\mathbf{m}_3 > 0$, we get
$$\|\mathbf{G}\| \leq \mathbf{m}_1\|\boldsymbol{\Phi}\| + \mathbf{m}_2\|\boldsymbol{\Phi}'\| + \mathbf{m}_3.$$

$\mathbf{A_4}$ Corresponding to constants $\mathbf{q}_1 > 0$, $\mathbf{q}_2 > 0$, one has
$$\begin{aligned}&|\mathbf{G}(\theta, \boldsymbol{\Phi}, \boldsymbol{\Phi}') - \mathbf{G}(\theta, \bar{\boldsymbol{\Phi}}, \bar{\boldsymbol{\Phi}}')|\\ &\quad \leq \mathbf{q}_1|\boldsymbol{\Phi} - \bar{\boldsymbol{\Phi}}| + \mathbf{q}_2|\boldsymbol{\Phi}' - \bar{\boldsymbol{\Phi}}'|.\end{aligned}$$

$\mathbf{A_5}$ There exist constants $\mathbf{k}_1, \mathbf{k}_2 > 0$, for whom, we have
$$\begin{aligned}&|\mathbf{F}(\theta, \boldsymbol{\Phi}, \boldsymbol{\Phi}') - \mathbf{F}(\theta, \bar{\boldsymbol{\Phi}}, \bar{\boldsymbol{\Phi}}')|\\ &\quad \leq \mathbf{k}_1|\boldsymbol{\Phi} - \bar{\boldsymbol{\Phi}}| + \mathbf{k}_2|\boldsymbol{\Phi}' - \bar{\boldsymbol{\Phi}}'|.\end{aligned}$$

Theorem 6. *The problem* (1) *has at least one solution if the condition* $\Pi = \mathbf{k}_1 + \frac{1-\omega}{M(\omega)}(\mathbf{q}_1 + \mathbf{q}_2) + \mathbf{k}_2 < 1$ *together with hypothesis* $\mathbf{A_1}$, $\mathbf{A_3}$ *and* $\mathbf{A_5}$ *hold.*

Proof. Let $\mathbf{D} = \{\boldsymbol{\Phi} \in \Omega : \|\boldsymbol{\Phi}\| \leq \sigma\}$ be a bounded, closed, and convex subset of Ω. It's obvious that $\mathbf{Z_1}$ is continuous. If we assume that $\boldsymbol{\Phi}, \boldsymbol{\Phi}' \in \mathbf{D}$,
$$\begin{aligned}
&\|\mathbf{Z_1}(\boldsymbol{\Phi}) - \mathbf{Z_1}(\boldsymbol{\Phi}')\|\\
&= \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\Bigg|\mathbf{F}(\theta, \boldsymbol{\Phi}(\theta), \boldsymbol{\Phi}(\lambda\theta))\\
&\quad + \frac{(1-\omega)\mathbf{G}(\theta, \boldsymbol{\Phi}(\theta), \boldsymbol{\Phi}(\lambda\theta))}{M(\omega)}\\
&\quad - \Bigg(\mathbf{F}(\theta, \boldsymbol{\Phi}'(\theta), \boldsymbol{\Phi}'(\lambda\theta))\\
&\quad + \frac{1-\omega}{M(\omega)}\mathbf{G}(\theta, \boldsymbol{\Phi}'(\theta), \boldsymbol{\Phi}'(\lambda\theta))\Bigg)\Bigg|\Bigg\},\\
&\leq \sup_{\theta\in[0,\mathbf{T}]} \{\mathbf{k}_1|\boldsymbol{\Phi}(\theta) - \boldsymbol{\Phi}'(\theta)|\\
&\quad + \mathbf{k}_2|\boldsymbol{\Phi}(\lambda\theta) - \boldsymbol{\Phi}'(\lambda\theta)|\}\\
&\quad + \frac{1-\omega}{M(\omega)} \sup_{\theta\in[0,\mathbf{T}]} \{\mathbf{q}_1|\boldsymbol{\Phi}(\theta) - \boldsymbol{\Phi}'(\theta)|\\
&\quad + \mathbf{q}_2|\boldsymbol{\Phi}(\lambda\theta) - \boldsymbol{\Phi}'(\lambda\theta)|\},\\
&\leq [\mathbf{k}_1 + \mathbf{k}_2]\|\boldsymbol{\Phi} - \boldsymbol{\Phi}'\|\\
&\quad + \frac{1-\omega}{M(\omega)}[\mathbf{q}_1 + \mathbf{q}_2]\|\boldsymbol{\Phi} - \boldsymbol{\Phi}'\|,\\
&\leq \left[\mathbf{k}_1 + \mathbf{k}_2 + \frac{1-\omega}{M(\omega)}(\mathbf{q}_1 + \mathbf{q}_2)\right]\|\boldsymbol{\Phi} - \boldsymbol{\Phi}'\|,\\
&\leq \Pi\|\boldsymbol{\Phi} - \boldsymbol{\Phi}'\|.
\end{aligned}$$
Hence, $\mathbf{Z_1}$ is a contraction.

It is now demonstrated that $\mathbf{Z_2}$ is equi-continuous. By using assumption ($\mathbf{A_1}, \mathbf{A_3}$), and take $\boldsymbol{\Phi} \in \mathbf{D}$, we proceed as
$$\begin{aligned}
&\|\mathbf{Z_2}(\boldsymbol{\Phi})\|\\
&= \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\Bigg|\boldsymbol{\Phi}_0 + \int_0^{\mathbf{T}} \mathbf{H}(\boldsymbol{\Phi}(\xi))d\xi + \frac{\omega}{M(\omega)\Gamma(\omega)}\\
&\quad \times \int_0^{\theta} (\theta - \xi)^{\omega-1}\mathbf{G}(\xi, \boldsymbol{\Phi}(\xi), \boldsymbol{\Phi}(\lambda\xi))d\xi\Bigg|\Bigg\},\\
&\leq \boldsymbol{\Phi}_0 + \mathbf{T}\|\mathbf{H}\| + \frac{\mathbf{T}^{\omega}}{M(\omega)\Gamma(\omega)}\|\mathbf{G}\|,\\
&< \infty.
\end{aligned}$$
Hence, $\mathbf{Z_2}$ is bounded and continuous as well. Additionally, if $\theta_1 < \theta_2 \in [0, \mathbf{T}]$, by taking
$$\begin{aligned}
&\|\mathbf{Z_2}\boldsymbol{\Phi}(\theta_2) - \mathbf{Z_2}\boldsymbol{\Phi}(\theta_1)\|\\
&= \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\Bigg|\boldsymbol{\Phi}_0 + \int_0^{\mathbf{T}} \mathbf{H}(\boldsymbol{\Phi}(\xi))d\xi + \frac{\omega}{M(\omega)\Gamma(\omega)}\\
&\quad \times \int_0^{\theta_2} (\theta_2 - \xi)^{\omega-1}\mathbf{G}(\xi, \boldsymbol{\Phi}(\xi), \boldsymbol{\Phi}(\lambda\xi))d\xi\\
&\quad - \Bigg(\boldsymbol{\Phi}_0 \int_0^{\mathbf{T}} \mathbf{H}(\boldsymbol{\Phi}(\xi))d\xi + \frac{\omega}{M(\omega)\Gamma(\omega)}\\
&\quad \times \int_0^{\theta_1} (\theta_1 - \xi)^{\omega-1}\mathbf{G}(\xi, \boldsymbol{\Phi}(\xi), \boldsymbol{\Phi}(\lambda\xi))d\xi\Bigg)\Bigg|\Bigg\},\\
&\leq \frac{\omega\|\mathbf{G}\|}{M(\omega)\Gamma(\omega)}\\
&\quad \times \left[\left|\int_0^{\theta_2} (\theta_2 - \xi)^{\omega-1} - \int_0^{\theta_1} (\theta_1 - \xi)^{\omega-1}\right|\right],\\
&\leq \frac{\|\mathbf{G}\|}{M(\omega)\Gamma(\omega)} \times [(\theta_2^{\omega} - \theta_1^{\omega})],
\end{aligned}$$
as a result,
$$\begin{aligned}|\mathbf{Z_2}\boldsymbol{\Phi}(\theta_2) - \mathbf{Z_2}\boldsymbol{\Phi}(\theta_1)\| &\leq \frac{\|\mathbf{G}\|}{M(\omega)\Gamma(\omega)}\\ &\quad \times [\theta_2^{\omega} - \theta_1^{\omega}] \to 0,\end{aligned}$$
as $\theta_1 \to \theta_2$.

Hence, $\mathbf{Z_2}$ is uniformly continuous. We concluded in view of Arzelá–Ascoli and Schauder's fixed point theorems that the problem (1) has at least one solution. □

Theorem 7. *The proposed problem has a unique solution if the condition* $\mho = (\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{T}\mathbf{v}) + (\frac{\Gamma(\omega)(1-\omega)+\mathbf{T}^\omega}{M(\omega)\Gamma(\omega)})(\mathbf{q}_1 + \mathbf{q}_2) < 1$, *together with assumptions* $\mathbf{A_2}, \mathbf{A_4}$, *and* $\mathbf{A_5}$ *holds.*

Proof. By taking $\mathbf{\Phi}, \bar{\mathbf{\Phi}} \in \mathbf{D}$, then

$$\begin{aligned}
&\|\mathbf{Z}(\mathbf{\Phi}) - \mathbf{Z}(\bar{\mathbf{\Phi}})\| \\
&\quad \le \|\mathbf{Z_1}(\mathbf{\Phi}) - \mathbf{Z_1}(\bar{\mathbf{\Phi}})\| + \|\mathbf{Z_2}(\mathbf{\Phi}) - \mathbf{Z_2}(\bar{\mathbf{\Phi}})\|, \\
&\quad \le \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\Bigg|\mathbf{F}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)) \\
&\qquad + \frac{(1-\omega)\mathbf{G}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta))}{M(\omega)} \\
&\qquad - \Bigg(\mathbf{F}(\theta, \bar{\mathbf{\Phi}}(\theta), \bar{\mathbf{\Phi}}(\lambda\theta)) \\
&\qquad + \frac{1-\omega}{M(\omega)}\mathbf{G}(\theta, \bar{\mathbf{\Phi}}(\theta), \bar{\mathbf{\Phi}}(\lambda\theta))\Bigg)\Bigg|\Bigg\} \\
&\qquad + \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\Bigg|\mathbf{\Phi}_0 + \int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}(\xi))d\xi \\
&\qquad + \frac{\omega}{M(\omega)\Gamma(\omega)}\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi))d\xi \\
&\qquad - \Bigg(\mathbf{\Phi}_0 + \int_0^{\mathbf{T}} \mathbf{H}(\bar{\mathbf{\Phi}}(\xi))d\xi + \frac{\omega}{M(\omega)\Gamma(\omega)} \\
&\qquad \times \int_0^\theta (\theta-\xi)^{\omega-1}\mathbf{G}(\xi, \bar{\mathbf{\Phi}}(\xi), \bar{\mathbf{\Phi}}(\lambda\xi))d\xi\Bigg)\Bigg|\Bigg\}, \\
&\quad \le \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\mathbf{k}_1|\mathbf{\Phi}(\theta) - \bar{\mathbf{\Phi}}(\theta)| \\
&\qquad + \mathbf{k}_2|\mathbf{\Phi}(\lambda\theta) - \bar{\mathbf{\Phi}}(\lambda\theta)| \\
&\qquad + \frac{1-\omega}{M(\omega)}(\mathbf{q}_1|\mathbf{\Phi}(\theta) - \bar{\mathbf{\Phi}}(\theta)| \\
&\qquad + \mathbf{q}_2|\mathbf{\Phi}(\lambda\theta) - \bar{\mathbf{\Phi}}(\lambda\theta)|)\Bigg\} \\
&\qquad + \sup_{\theta\in[0,\mathbf{T}]} \Bigg\{\mathbf{v}\int_0^{\mathbf{T}} |\mathbf{\Phi}(\theta) - \bar{\mathbf{\Phi}}(\theta)|d\xi \\
&\qquad + \frac{\omega}{M(\omega)\Gamma(\omega)}\int_0^\theta (\theta-\xi)^{\omega-1} \\
&\qquad \times(\mathbf{q}_1|\mathbf{\Phi}(\theta) - \bar{\mathbf{\Phi}}(\theta)|
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\mathbf{Z}(\mathbf{\Phi}) - \mathbf{Z}(\bar{\mathbf{\Phi}})\| \\
&\quad \le \Bigg[(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{v}\mathbf{T}) + (\mathbf{q}_1 + \mathbf{q}_2) \\
&\qquad \times \left(\frac{\Gamma(\omega)(1-\omega) + \mathbf{T}^\omega}{M(\omega)\Gamma(\omega)}\right)\Bigg]\|\mathbf{\Phi} - \bar{\mathbf{\Phi}}\| \\
&\quad = \mho\|\mathbf{\Phi} - \bar{\mathbf{\Phi}}\|.
\end{aligned}$$

Thus proved. □

4. ULAM–HYER'S STABILITY

We will establish the conditions required for Ulam–Hyer type stability. The numerical solution of a problem depends on this kind of stability. The basic question about an approximation is "Under what conditions will the problem's approximate curve be close to the exact curve"? In the 1940s, Ulam offered a thorough analysis of this important topic (see Ref. 37). The stability that was previously discussed was extended and generalized by Hyers and Rassiass to become known as generalized Ulam–Hyer, Ulam–Hyer–Rassias stability (see Refs. 38 and 39). For more details, see Refs. 40 and 41.

Remark 8. Let there exist an independent function $\varphi : \mathbf{L} \to \mathbf{R}$, such that for $\epsilon > 0$, one has

(i) $|\varphi(\theta)| \le \epsilon$;
(ii) The perturbed problem associated with (1) can be presented as follows:

$$\begin{cases} {}^{\mathrm{ABC}}{}_0D_\theta^\omega[\mathbf{\Phi}(\theta) - \mathbf{F}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta))] \\ \quad = \mathbf{G}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)) + \varphi(\theta), \\ \mathbf{\Phi}(0) = \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}(\xi))d\xi. \end{cases} \tag{12}$$

According to Lemma 5, the problem (12) is equivalent to the integral form as

$$\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}(\xi))d\xi \\ \quad + \mathbf{F}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)) + \dfrac{1-\omega}{M(\omega)} \\ \quad \times [\mathbf{G}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)) + \varphi(\theta)] \\ \quad + \dfrac{\omega}{M(\omega)\Gamma(\omega)}\displaystyle\int_0^\theta (\theta-\xi)^{\omega-1} \\ \quad \times [\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi)) + \varphi(\xi)]d\xi. \end{cases} \tag{13}$$

From (13) and (8), we can write

$$\|\mathbf{Z}\mathbf{\Phi}(\theta) - \mathbf{\Phi}(\theta)\| \leq \left[\frac{\Gamma(\omega)(1-\omega) + \mathbf{T}^{\omega}}{M(\omega)\Gamma(\omega)}\right]\epsilon,$$

$$\leq \Theta\epsilon,$$

where $\Theta = [\frac{\Gamma(\omega)(1-\omega)+\mathbf{T}^{\omega}}{M(\omega)\Gamma(\omega)}]$.

Theorem 9. *Ulam–Hyer and generalized Ulam–Hyer stable solutions exist to problem* (1)*if* $\mho < 1$ *holds.*

Proof. For any $\bar{\mathbf{\Phi}} \in \mathbf{D}$, let there $\mathbf{\Phi} \in \mathbf{D}$ a unique solution of problem (1), one has

$$\begin{aligned}\|\mathbf{\Phi} - \bar{\mathbf{\Phi}}\| &= \|\mathbf{\Phi} - \mathbf{Z}\bar{\mathbf{\Phi}}\| \\ &\leq \|\mathbf{\Phi} - \mathbf{Z}\mathbf{\Phi}\| + \|\mathbf{Z}\mathbf{\Phi} - \mathbf{Z}\bar{\mathbf{\Phi}}\| \\ &\leq \Theta\epsilon + \mho\|\mathbf{\Phi} - \bar{\mathbf{\Phi}}\| \\ &\leq \frac{\Theta\epsilon}{1-\mho}.\end{aligned}$$

Therefore, (1) is stable in the Ulam–Hyer and generalized Ulam–Hyer types. Likewise, several forms of Ulam–Hyer stability may be discussed in the same way. □

5. COMPUTATIONAL TECHNIQUE

We will use a numerical approach in this section of the paper to solve our considered problem. The Lagrange interpolation method will be used to build the numerical scheme, in accordance with the numerical method developed in Ref. 42. The integral solution of problem (1) is given by

$$\mathbf{\Phi}(\theta) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}(\xi))d\xi + \mathbf{F}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)) \\ \quad + \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta, \mathbf{\Phi}(\theta), \mathbf{\Phi}(\lambda\theta)) + \dfrac{\omega}{M(\omega)\Gamma(\omega)} \\ \quad \times \displaystyle\int_0^{\theta} (\theta-\xi)^{\omega-1}\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi))d\xi. \end{cases} \tag{14}$$

By using $\theta = \theta_{j+1}$, Eq. (14) can be written as

$$\mathbf{\Phi}(\theta_{j+1}) = \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}_j)d\xi + \mathbf{F}(\theta_j, \mathbf{\Phi}(\theta_j), \mathbf{\Phi}(\lambda\theta_j)) + \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta_j, \mathbf{\Phi}(\theta_j), \mathbf{\Phi}(\lambda\theta_j)) \\ + \dfrac{\omega}{M(\omega)\Gamma(\omega)}\displaystyle\int_0^{\theta_{j+1}} (\theta_{j+1}-\xi)^{\omega-1}\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi))d\xi, \end{cases}$$

$$= \begin{cases} \mathbf{\Phi}_0 + \displaystyle\int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}_j)d\xi + \mathbf{F}(\theta_j, \mathbf{\Phi}(\theta_j), \mathbf{\Phi}(\lambda\theta_j)) + \dfrac{1-\omega}{M(\omega)}\mathbf{G}(\theta_j, \mathbf{\Phi}(\theta_j), \mathbf{\Phi}(\lambda\theta_j)) \\ + \dfrac{\omega}{M(\omega)\Gamma(\omega)}\displaystyle\sum_{r=0}^{j+1}\int_{\theta_r}^{\theta_{r+1}} (\theta_{r+1}-\xi)^{\omega-1}\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi))d\xi. \end{cases}$$

To approximate the function $\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi))$, use Lagrange-type interpolation with equal spacing of inside the integral:

$$\begin{aligned}\mathbf{G}(\xi, \mathbf{\Phi}(\xi), \mathbf{\Phi}(\lambda\xi)) = {}& \frac{\xi-\theta_{r-1}}{h}\mathbf{G}(\theta_r, \mathbf{\Phi}(\theta_r), \mathbf{\Phi}(\lambda\theta_r)) \\ & - \frac{\xi-\theta_r}{h}\mathbf{G}(\theta_{r-1}, \mathbf{\Phi}(\theta_{r-1}), \mathbf{\Phi}(\lambda\theta_{r-1}))\end{aligned}$$

and

$$\int_0^{\mathbf{T}} \mathbf{H}(\mathbf{\Phi}_j)d\xi = \frac{1}{h}\sum_{r=0}^{j}[\mathbf{H}(\mathbf{\Phi}(\theta_{r+1})) - \mathbf{H}(\mathbf{\Phi}(\theta_r))].$$

Hence

$$\Phi(\theta_{j+1}) = \begin{cases} \Phi_0 + \frac{1}{h}\sum_{r=0}^{j}[\mathbf{H}(\Phi(\theta_{r+1})) - \mathbf{H}(\Phi(\theta_r))] + \mathbf{F}(\theta_j, \Phi(\theta_j), \Phi(\lambda\theta_j)) \\ \quad + \frac{1-\omega}{M(\omega)}\mathbf{G}(\theta_j, \Phi(\theta_j), \Phi(\lambda\theta_j)) + \frac{\omega}{M(\omega)\Gamma(\omega)}\sum_{r=1}^{j}\int_{\theta_r}^{\theta_{r+1}}(\theta_{j+1}-\xi)^{\omega-1} \\ \quad \times\left[\frac{\xi-\theta_{r-1}}{h}\mathbf{G}(\theta_r, \Phi(\theta_r), \Phi(\lambda\theta_r)) - \frac{\xi-\theta_r}{h}\mathbf{G}(\theta_{r-1}, \Phi(\theta_{r-1}), \Phi(\lambda\theta_{r-1}))\right]d\xi. \end{cases}$$

$$\Phi(\theta_{j+1}) = \begin{cases} \Phi_0 + \frac{1}{h}\sum_{r=0}^{j}[\mathbf{H}(\Phi(\theta_{r+1})) - \mathbf{H}(\Phi(\theta_r))] + \mathbf{F}(\theta_j, \Phi(\theta_j), \Phi(\lambda\theta_j)) \\ + \frac{1-\omega}{M(\omega)}\mathbf{G}(\theta_j, \Phi(\theta_j), \Phi(\lambda\theta_j)) + \frac{\omega}{M(\omega)\Gamma(\omega)} \\ \times\left[\frac{1}{h}\sum_{r=1}^{j}\mathbf{G}(\theta_r, \Phi(\theta_r), \Phi(\lambda\theta_r))\int_{\theta_r}^{\theta_{r+1}}(\theta_{j+1}-\xi)^{\omega-1}(\xi-\theta_{r-1})d\xi\right. \\ \left. - \frac{1}{h}\sum_{r=1}^{j}\mathbf{G}(\theta_{r-1}, \Phi(\theta_{r-1}), \Phi(\lambda\theta_{r-1}))\int_{\theta_r}^{\theta_{r+1}}(\theta_{j+1}-\xi)^{\omega-1}(\xi-\theta_r)d\xi\right]. \end{cases}$$

$$\int_{\theta_r}^{\theta_{r+1}}(\theta_{j+1}-\xi)^{\omega-1}(\xi-\theta_{r-1})d\xi \approx \frac{h^{\omega+1}}{\omega(\omega+1)}[((j-r+1)^{\omega}(j-r+2+\omega) - (j-r)^{\omega}(j-r+2+2\omega)] \tag{15}$$

and

$$\int_{\theta_r}^{\theta_{r+1}}(\theta_{j+1}-\xi)^{\omega-1}(\xi-\theta_r)d\xi \approx \frac{h^{\omega+1}}{\omega(\omega+1)}[(j-r+1)^{\omega+1} - (j-r)^{\omega}(j-r+1+\omega)]. \tag{16}$$

By using the above equations, we have

$$\Phi(\theta_{j+1}) = \begin{cases} \Phi_0 + \sum_{r=1}^{j}\left[\frac{(\mathbf{H}(\theta_r) - \mathbf{H}(\theta_{r-1}))}{h}\right] + \mathbf{F}(\theta_j, \Phi(\theta_j), \Phi(\lambda\theta_j)) + \frac{1-\omega}{M(\omega)}\mathbf{G}(\theta_j, \Phi(\theta_j), \Phi(\lambda\theta_j)) \\ + \frac{h^{\omega}}{M(\omega)\Gamma(\omega+2)}\sum_{r=1}^{j}[\mathbf{G}(\theta_r, \Phi(\theta_r), \Phi(\lambda\theta_r))(j+1-r)^{\omega}(j-r+2+\omega) \\ - (j-r)^{\omega}(j+2(1+\omega)-r) - \mathbf{G}(\theta_{r-1}, \Phi(\theta_{r-1}), \Phi(\lambda\theta_{r-1}))(j+1-r)^{\omega+1} \\ - (j-r)^{\omega}(j+1+\omega-r)]. \end{cases}$$

6. ILLUSTRATIVE PROBLEM

Example 10. Suppose we have a nonlocal initial condition for Cauchy-type problem described as follows:

$$\begin{cases} {}^{\mathrm{ABC}}{}_{0}D_{\theta}^{\omega}\left(\Phi(\theta) - \left[\frac{\theta}{15} + \frac{\sin(\Phi(\theta))}{10} + \frac{\cos(\Phi(\lambda\theta))}{20}\right]\right) = \frac{1}{100}e^{-(\Phi(\theta)+\Phi(\lambda\theta))}, \quad \theta \in \mathbf{L} = [0,1], \\ \Phi(0) = 1 + \int_0^1 \frac{\sin(\Phi(\xi))}{5}d\xi. \end{cases} \tag{17}$$

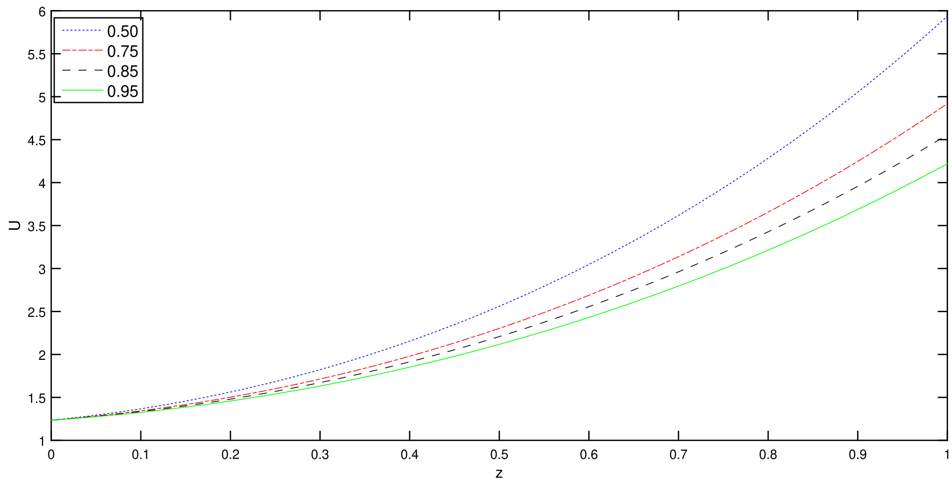

Fig. 1 Graphical interpretation of solution for fractional order $\omega \in [0.5, 0.95]$.

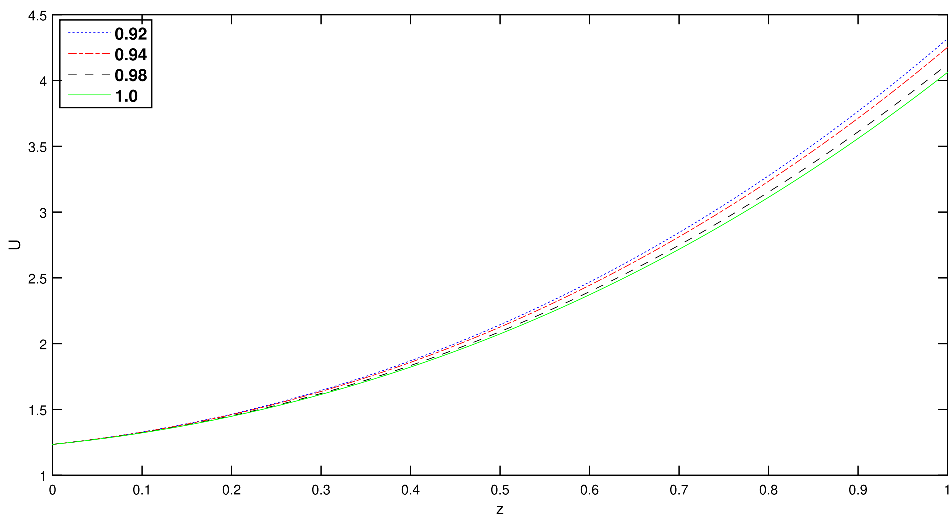

Fig. 2 Graphical interpretation of solution for fractional order $\omega \in [0.92, 1.0]$.

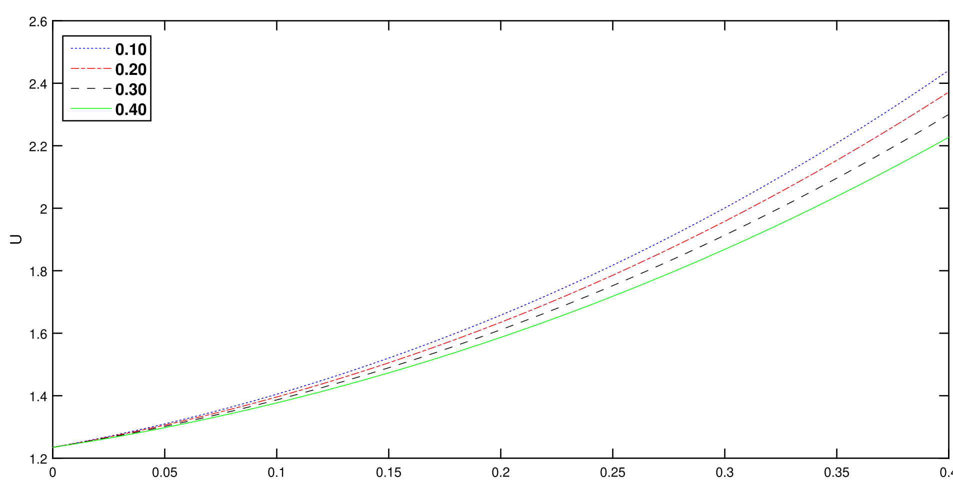

Fig. 3 Graphical interpretation of solution for fractional order $\omega \in [0.10, 0.40]$.

Here,

$$\mathbf{F}(\theta,\boldsymbol{\Phi}(\theta),\boldsymbol{\Phi}(\lambda\theta)) = \frac{\theta}{15} + \frac{1}{10}\sin(\boldsymbol{\Phi}(\theta)) + \frac{1}{20}\cos(\boldsymbol{\Phi}(\lambda\theta),$$

$$\mathbf{G}(\theta,\boldsymbol{\Phi}(\theta),\boldsymbol{\Phi}(\lambda\theta)) = \frac{1}{100}e^{-(\boldsymbol{\Phi}(\theta)+\boldsymbol{\Phi}(\lambda\theta))}$$

and

$$\mathbf{H}(\boldsymbol{\Phi}(\theta)) = \frac{\sin(\boldsymbol{\Phi}(\theta))}{5}.$$

Now

$$\begin{aligned}
&|\mathbf{H}(\boldsymbol{\Phi}(\theta))| = \left|\frac{\sin(\boldsymbol{\Phi}(\theta))}{5}\right| \leq \frac{1}{5}|\boldsymbol{\Phi}(\theta)|,\\
&|\mathbf{H}(\boldsymbol{\Phi}(\theta)) - \mathbf{H}(\bar{\boldsymbol{\Phi}}(\theta))|\\
&\quad = \frac{1}{5}|\sin(\boldsymbol{\Phi}(\theta)) - \sin(\bar{\boldsymbol{\Phi}}(\theta))|\\
&\quad \leq \frac{1}{5}|\boldsymbol{\Phi}(\theta) - \bar{\boldsymbol{\Phi}}(\theta)|,\\
&|\mathbf{G}(\theta,\boldsymbol{\Phi}(\theta),\boldsymbol{\Phi}(\lambda\theta))|\\
&\quad = \left|\frac{1}{100}e^{-(\boldsymbol{\Phi}(\theta)+\boldsymbol{\Phi}(\lambda\theta))}\right|\\
&\quad \leq \frac{1}{100}[|\boldsymbol{\Phi}(\theta)| + |\boldsymbol{\Phi}(\lambda\theta)|],\\
&|\mathbf{G}(\theta,\boldsymbol{\Phi}(\theta),\boldsymbol{\Phi}(\lambda\theta)) - \mathbf{G}(\theta,\bar{\boldsymbol{\Phi}}(\theta),\bar{\boldsymbol{\Phi}}(\lambda\theta))|\\
&\quad = \left|\frac{1}{100}e^{-(\boldsymbol{\Phi}(\theta)+\boldsymbol{\Phi}(\lambda\theta))} - \frac{1}{100}e^{-(\bar{\boldsymbol{\Phi}}(\theta)+\bar{\boldsymbol{\Phi}}(\lambda\theta))}\right|,\\
&\quad \leq \frac{1}{100}[|\boldsymbol{\Phi}(\theta) - \bar{\boldsymbol{\Phi}}(\theta)| + |\boldsymbol{\Phi}(\lambda\theta) - \bar{\boldsymbol{\Phi}}(\lambda\theta)|].\\
&|\mathbf{F}(\theta,\boldsymbol{\Phi}(\theta),\boldsymbol{\Phi}(\lambda\theta)) - \mathbf{F}(\theta,\bar{\boldsymbol{\Phi}}(\theta),\bar{\boldsymbol{\Phi}}(\lambda\theta))|\\
&\quad = \left|\frac{\theta}{15} + \frac{1}{10}\sin(\boldsymbol{\Phi}(\theta)) + \frac{1}{20}\cos(\boldsymbol{\Phi}(\lambda\theta)\right.\\
&\qquad \left. - \left[\frac{\theta}{15} + \frac{1}{10}\sin(\bar{\boldsymbol{\Phi}}(\theta)) + \frac{1}{20}\cos(\bar{\boldsymbol{\Phi}}(\lambda\theta))\right]\right|,\\
&\quad \leq \frac{1}{10}|\boldsymbol{\Phi}(\theta) - \bar{\boldsymbol{\Phi}}(\theta)| + \frac{1}{20}|\boldsymbol{\Phi}(\lambda\theta) - \bar{\boldsymbol{\Phi}}(\lambda\theta)|.
\end{aligned}$$

We deduce the values as $\boldsymbol{\ell} = \frac{1}{5} = \mathbf{v}$, $\mathbf{m}_1 = \mathbf{m}_2 = \frac{1}{100} = \mathbf{q}_1 = \mathbf{q}_2$, $\mathbf{m}_3 = 0$, $\mathbf{k}_1 = \frac{1}{10}$, $\mathbf{k}_2 = \frac{1}{20}$, and also use $M(\omega) = 1 - \omega(1 - \frac{1}{\Gamma(\omega+1)})$, one has

$$\Pi = \frac{1}{5} + \frac{\omega}{50M(\omega)} < 1, \quad \forall\,\omega \in (0,1].$$

Thus by Theorem 7, the given problem has a unique solution. The requirement of Theorems 6 and 9 also fulfils. For step size $h = 0.01$, the solution for various fractional orders is shown in Figs. 1–3.

7. CONCLUSION

In this paper, we examined the qualitative characteristics and Ulam–Hyer-type stability of a large class of HFDEs. The considered problem has been taken under nonsingular fractional-order derivative. We have applied fixed point analysis for the existence theory, while tools of nonlinear analysis were used to deduce the Ulam–Hyer stability results. Additionally, a computation scheme based on Lagrange-interpolation was built to compute the approximate numerical solution. Such crucial analysis plays a significant role in investigating dynamical problems of high interest. In future, such problems can be considered for the different analyses like existence theory, stability theory, numerical aspect, and control theory under different fractional operators introduced recently. We have studied the proposed model under fractional derivative of order $(0,1]$, hence one can extend the analysis to higher i.e. $(n-1,n]$ order in the future. Also, the analysis can be extended in the future to the system of such fractional differential equations. Further, the analysis developed here can be applied to different physical and engineering problems.

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ORCID

Muhammad Sher
https://orcid.org/0009-0007-9632-1843
Kamal Shah
https://orcid.org/0000-0002-8851-4844
Muhammad Sarwar
https://orcid.org/0000-0003-3904-8442
Manel Hleili
https://orcid.org/0009-0002-0547-0821
Bahaaeldin Abdalla
https://orcid.org/0000-0001-8867-0612

Thabet Abdeljawad
https://orcid.org/0000-0002-8889-3768
Manar A. Alqudah
https://orcid.org/0000-0002-9066-5334

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