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On the oscillation of fractional order nonlinear differential equations

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ABSTRACT

In the article, we are concerned with the oscillatory solutions of a class of fractional differential equations. By using generalized Riccati function and Hardy inequalities, we present some oscillation criterias. As a result we give some examples that validity of the established results.

Anahtar Kelimeler: Oscillation, Oscillation Criterias, Fractional Derivative, Generalized Riccati Function.

Kesirli mertebeden doğrusal olmayan diferensiyel denklemlerin salınımlılığı üzerine

ÖZ

Bu makalede, kesirli mertebeden diferensiyel denklemlerin bir sınıfının salınımlı çözümleriyle ilgilenildi. Genelleştirilmiş Riccati fonksiyonu ve Hardy eşitsizlikleri kullanılarak, baz salınımlılık kriterleri sunuldu. Sonuç olarak, kurulan sonuçları sağlayan bazı örnekler verildi.

Keywords: Salınımlılık, Salınımlılık Kriterleri, Kesirli Türev, Genelleştirilmiş Riccati Fonksiyonu.

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1. INTRODUCTION

Fractional differential equations have been proved to be valuable tools in the modelling of many physical and engineering phenomena such as viscous damping, diffusion and wave propagation, electromagnetism, polymer physics, chaos and fractals, electronics, electrical networks, fluid flows, heat transfer, traffic systems, signal processing, system identification, industrial robotics, genetic algorithms.economics, etc, [1-3]. For the many theories and applications of fractional differential equations, we refer to the books [4-7]. Recently, many authors studied the numerical methods for fractional differential equations, the existence, uniqueness, and stability of solutions of fractional differential equations [8-13].

Research on oscillation of various equations like ordinary and partial differential equations, difference equations, dynamic equations on time scales and fractional differential equations has been a hot topic in the literature, and much effort has been made to establish new oscillation criteria for these equations [14-24]. In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations [25-31].

In [32], Jumarie proposed a definition for a fractional derivative which is known as the modified Riemann-Liouville derivative in the literature. In the later years, many researchers have studied several applications of the modified Riemann- Liouville derivative [33-35].

In [27,29], authors have established some new oscillation criteria for the following equations:

$$\left.\begin{aligned} D_t^{\alpha}\left(r(t)\left(D_t^{\alpha}x(t)\right)^{\gamma}\right)+p(t)\left(D_t^{\alpha}x(t)\right)^{\gamma} \\ +q(t)f\left(x(t)\right)=0 \end{aligned}\right\},$$

$$D_t^{\alpha}\left[D_t^{\alpha}\left(r(t)D_t^{\alpha}x(t)\right)\right]+q(t)x(t)=0,$$

$$D_t^{\alpha}\left(a(t)\left(D_t^{\alpha}\left(r(t)D_t^{\alpha}x(t)\right)\right)^{\gamma}\right)+q(t)f\left(x(t)\right)=0,$$

for $t\in[t_0,\infty)$, $0<\alpha<1$ and where $D_t^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative with respect to variable t.

In this study, we are concerned with the oscillation of following fractional differential equations:

$$\left.\begin{aligned} D_t^{\alpha}\left(a(t)\left(D_t^{\alpha}\left(r(t)\left[D_t^{\alpha}x(t)\right]^{\gamma_1}\right)\right)^{\gamma_2}\right) \\ +q(t)f\left(x(t)\right)=0 \end{aligned}\right\} \quad (1.1)$$

where $t\in[t_0,\infty)$, $0<\alpha<1$ and $D_t^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative with respect to the variable t, γ_1 and γ_2 are the quotient of two odd positive number,the function $a\in C^{\alpha}\left([t_0,\infty),\mathrm{R}_+\right)$, $r\in C^{2\alpha}\left([t_0,\infty),\mathrm{R}_+\right)$, $q\in C\left([t_0,\infty),\mathrm{R}_+\right)$, the function of f belong to $C(\mathrm{R},\mathrm{R})$, $f(x)/x\geq k>0$ for all $x\neq 0$, and C^{α} denotes continuous derivative of order α.

Some of the key properties of the Jumarie's modified Riemann-Liouville derivative of order α are listed as follows:

$$D_t^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)d\xi, 0<\alpha<1\\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, 1\leq n\leq\alpha\leq n+1\end{cases}$$

$$D_t^{\alpha}\left(f(t)g(t)\right)=g(t)D_t^{\alpha}f(t)+f(t)D_t^{\alpha}g(t)$$

$$D_t^{\alpha}f\left(g(t)\right)=f'_g\left(g(t)\right)D_t^{\alpha}g(t)=D_t^{\alpha}f\left(g(t)\right)\left(g'(t)\right)^{\alpha}$$

$$D_t^{\alpha}t^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}$$

As usual, a solution $x(t)$ of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillaory.

In the rest of this paper, we denote for the sake of convenience:

$\xi=t^{\alpha}/\Gamma(1+\alpha)$; $\xi_i=t_i^{\alpha}/\Gamma(1+\alpha)$, $i=0,1,2,3,4,5$; $a(t)=\tilde{a}(\xi)$; $r(t)=\tilde{r}(\xi)$; $q(t)=\tilde{q}(\xi)$; $\tilde{\delta}_1(\xi,\xi_i)=\int_{\xi_i}^{\xi}\left(1/\tilde{a}^{1/\gamma_2}(s)\right)ds$; $\delta_1(t,t_i)=\tilde{\delta}_1(\xi,\xi_i)$.

And we use class of averaging functions $H\in C(D,\mathrm{R})$ which satisfy

$$H(t,t)=0, H(t,s)>0 \text{ for } t>s$$

Let H has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H(t,s)}{\partial t} = -h_1(t,s)\sqrt{H(t,s)},$$
$$\frac{\partial H(t,s)}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}$$

where $D = \{(t,s) : t_0 \le s \le t < \infty\}$ and h_1, $h_2 \in L_{loc}(D, \mathrm{R}_+)$.

2. MAIN RESULTS

Lemma 2. 1. Assume $x(t)$ is an eventually positive solution of (1.1), and

$$\int_{\xi_0}^{\infty} \frac{1}{\tilde{a}^{1/\gamma_2}(s)} ds = \infty \tag{2.1}$$

$$\int_{\xi_0}^{\infty} \frac{1}{\tilde{r}^{1/\gamma_1}(s)} ds = \infty \tag{2.2}$$

$$\left.\begin{aligned}&\int_{\xi_0}^{\infty}\left[\frac{1}{\tilde{r}(\zeta)}\int_{\zeta}^{\infty}\left[\frac{1}{\tilde{a}(\tau)}\int_{\tau}^{\infty}\tilde{q}(s)ds\right]^{1/\gamma_2} d\tau\right]^{1/\gamma_1} d\zeta\\ &= \infty\end{aligned}\right\} \tag{2.3}$$

Then, there exist a sufficiently large T such that $D_t^{\alpha}\left(r(t)\left[D_t^{\alpha}x(t)\right]^{\gamma_1}\right) > 0$ on $[T,\infty)$ and either $D_t^{\alpha}x(t) > 0$ on $[T,\infty)$ or $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose $x(t)$ is an eventually solution of (1). Let $a(t) = \tilde{a}(\xi)$, $r(t) = \tilde{r}(\xi)$, $x(t) = \tilde{x}(\xi)$, $q(t) = \tilde{q}(\xi)$ where $\xi = t^{\alpha}/\Gamma(1+\alpha)$. Then, we know that $D_t^{\alpha}\xi(t) = 1$, and furthermore, we have

$$D_t^{\alpha}a(t) = D_t^{\alpha}\tilde{a}(\xi) = \tilde{a}'(\xi)D_t^{\alpha}\xi(t) = \tilde{a}'(\xi)$$

Similarly we have $D_t^{\alpha}r(t) = \tilde{r}'(\xi)$, $D_t^{\alpha}x(t) = \tilde{x}'(\xi)$, $D_t^{\alpha}q(t) = \tilde{q}'(\xi)$. So, (1.1) can be transformed into following form:

$$\left.\begin{aligned}&\left[\tilde{a}(\xi)\left(\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\right)^{\gamma_2}\right]'\\ &\qquad + \tilde{q}(\xi)f\left(\tilde{x}(\xi)\right) = 0, \xi \ge \xi_0 > 0\end{aligned}\right\} \tag{2.4}$$

Then $\tilde{x}(\xi)$ is an eventually positive solution of (2.4), and there exists $\xi_1 > \xi_0$ such that $\tilde{x}(\xi) > 0$ on $[\xi_1,\infty)$. So, $f\left(\tilde{x}(\xi)\right) > 0$ and we have

$$\left.\begin{aligned}&\left[\tilde{a}(\xi)\left(\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\right)^{\gamma_2}\right]'\\ &\qquad = -\tilde{q}(\xi)f\left(\tilde{x}(\xi)\right) < 0, \xi \ge \xi_1\end{aligned}\right\} \tag{2.5}$$

Then, $\tilde{a}(\xi)\left(\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\right)^{\gamma_2}$ is strictly decreasing on $[\xi_1,\infty)$, thus we know that $\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'$ is eventually of one sign. For $\xi_2 > \xi_1$ is sufficiently large, we claim $\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)' > 0$ on $[\xi_2,\infty)$. Otherwise, assume that there exists a sufficiently large $\xi_3 > \xi_2$ such that $\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)' < 0$ on $[\xi_3,\infty)$. Thus, $\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}$ is strictly decreasing on $[\xi_3,\infty)$, and we get that

$$\left.\begin{aligned}&\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1} - \tilde{r}(\xi_3)\left[\tilde{x}'(\xi_3)\right]^{\gamma_1}\\ &= \int_{\xi_3}^{\xi}\frac{\tilde{a}^{1/\gamma_2}(s)\left(\tilde{r}(s)\left[\tilde{x}'(s)\right]^{\gamma_1}\right)'}{\tilde{a}^{1/\gamma_2}(s)}ds\\ &\le \tilde{a}^{1/\gamma_2}(\xi_3)\left(\tilde{r}(\xi_3)\left[\tilde{x}'(\xi_3)\right]^{\gamma_1}\right)'\int_{\xi_3}^{\xi}\frac{1}{\tilde{a}^{1/\gamma_2}(s)}ds\end{aligned}\right\}$$

$$\left.\begin{aligned}&\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1} \le \tilde{r}(\xi_3)\tilde{x}'(\xi_3)\\ &\qquad + \tilde{a}^{1/\gamma_2}(\xi_3)\left(\tilde{r}(\xi_3)\left[\tilde{x}'(\xi_3)\right]^{\gamma_1}\right)'\int_{\xi_3}^{\xi}\frac{1}{\tilde{a}^{1/\gamma_2}(s)}ds\end{aligned}\right\}$$

By (2.1), we have $\lim_{\xi\to\infty}\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1} = -\infty$. So there exists a sufficiently large $\xi_4 > \xi_3$ such that $\tilde{x}'(\xi) < 0$, $\xi \in [\xi_4,\infty)$. Then, we have

$$\begin{aligned}\tilde{x}(\xi) - \tilde{x}(\xi_4) &= \int_{\xi_4}^{\xi}\tilde{x}'(s)ds = \int_{\xi_4}^{\xi}\frac{\tilde{r}^{1/\gamma_1}(s)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{x}'(s)ds\\ &\le \tilde{r}^{1/\gamma_1}(\xi_4)\tilde{x}'(\xi_4)\int_{\xi_4}^{\xi}\frac{1}{\tilde{r}^{1/\gamma_1}(s)}ds\end{aligned}$$

and so,

$$\tilde{x}(\xi) \le \tilde{r}^{1/\gamma_1}(\xi_4)\tilde{x}'(\xi_4)\int_{\xi_4}^{\xi}\frac{1}{\tilde{r}^{1/\gamma_1}(s)}ds$$

By (2.2), we deduce that $\lim_{\xi\to\infty}\tilde{x}(\xi) = -\infty$, which contradicts the fact that $\tilde{x}(\xi)$ is an eventually positive solution of (2.4). Thus,

$\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)' > 0$ on $[\xi_2,\infty)$, and then $D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right) > 0$ on $[t_2,\infty)$. So, $D_t^\alpha x(t) = \tilde{x}'(\xi)$ is eventually of one sign. Now we assume $\tilde{x}'(\xi) < 0$ on $[\xi_5,\infty)$ where $\xi_5 > \xi_4$ is sufficiently large. Since $\tilde{x}(\xi) > 0$, we have $\lim_{\xi\to\infty}\tilde{x}(\xi) = \beta \geq 0$. We claim $\beta = 0$. Otherwise, assume $\beta > 0$. Then $\tilde{x}(\xi) \geq \beta$ on $[\xi_5,\infty)$, $f(x(\xi)) \geq k.x(\xi) > k\beta \geq M$ for $M \in \mathrm{R}_+$ and by (2.5) we have

$$\left[\tilde{a}(\xi)\left(\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\right)^{\gamma_2}\right]' = -\tilde{q}(\xi) f(\tilde{x}(\xi)) \leq -\tilde{q}(\xi)M$$

Substituting ξ with s in above the inequality, and integrating it with respect to s from ξ to ∞ yields

$$-\tilde{a}(\xi)\left(\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\right)^{\gamma_2} < -M\int_\xi^\infty \tilde{q}(s)ds$$

which means

$$\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)' > \left[M\frac{1}{\tilde{a}(\xi)}\int_\xi^\infty \tilde{q}(s)ds\right]^{1/\gamma_2} \quad (2.6)$$

substituting ξ with τ in (2.6), and integrating it with respect to τ from ξ to ∞ yields

$$-\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1} > M^{1/\gamma_2}\int_\xi^\infty\left[\frac{1}{\tilde{a}(\tau)}\int_\tau^\infty \tilde{q}(s)ds\right]^{1/\gamma_2} d\tau$$

That is,

$$\tilde{x}'(\xi) < \left[-M^{1/\gamma_2}\frac{1}{\tilde{r}(\xi)}\int_\xi^\infty\left[\frac{1}{\tilde{a}(\tau)}\int_\tau^\infty \tilde{q}(s)ds\right]^{1/\gamma_2} d\tau\right]^{1/\gamma_1}$$

substituting ξ with ζ in above the inequality, and integrating it with respect to ζ from ξ_5 to ξ yields

$$\left.\begin{aligned}&\tilde{x}(\xi) < \tilde{x}(\xi_5)\\ &-M^{1/\gamma_1\gamma_2}\int_{\xi_5}^{\xi}\left[\frac{1}{\tilde{r}(\zeta)}\int_\zeta^\infty\left[\frac{1}{\tilde{a}(\tau)}\int_\tau^\infty \tilde{q}(s)ds\right]^{1/\gamma_2} d\tau\right]^{1/\gamma_1} d\zeta\end{aligned}\right\}$$

By (2.3), we have $\lim_{t\to\infty}\tilde{x}(\xi) = -\infty$, which causes a contradiction. So, the proof is complete.

Lemma 2. 2. Assume that $x(t)$ is an eventually positive solution of (1) such that $D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right) > 0$, $D_t^\alpha x(t) > 0$ on $[t_1,\infty)$, where $t_1 > t_0$ is sufficiently large. Then, for $t \geq t_1$, we have

$$D_t^\alpha x(t) \geq \frac{a^{1/\gamma_1\gamma_2}(t)\left[D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\right]^{1/\gamma_1}\delta_1^{1/\gamma_1}(t,t_1)}{r^{1/\gamma_1}(t)}$$

Proof. Assume that x is an eventually positive solution of (1). So, by (2.5), we obtain that $\tilde{a}(\xi)\left(\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\right)^{\gamma_2}$ is strictly decreasing on $[\xi_1,\infty)$. Then,

$$\begin{aligned}\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1} &\geq \tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1} - \tilde{r}(\xi_1)\left[\tilde{x}'(\xi_1)\right]^{\gamma_1}\\ &= \int_{\xi_1}^{\xi}\frac{\tilde{a}^{1/\gamma_2}(s)\left(\tilde{r}(s)\left[\tilde{x}'(s)\right]^{\gamma_1}\right)'}{\tilde{a}^{1/\gamma_2}(s)}ds\\ &\geq \tilde{a}^{1/\gamma_2}(\xi)\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\int_{\xi_1}^{\xi}\frac{1}{\tilde{a}^{1/\gamma_2}(s)}ds\\ &= \tilde{a}^{1/\gamma_2}(\xi)\left(\tilde{r}(\xi)\left[\tilde{x}'(\xi)\right]^{\gamma_1}\right)'\tilde{\delta}_1(\xi,\xi_1)\end{aligned}$$

and so,

$$r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1} \geq a^{1/\gamma_2}(t)D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\delta_1(t,t_1)$$

multiplying both sides of above the inequality by $1/r(t)$, we obtain

$$D_t^\alpha x(t) \geq \frac{a^{1/\gamma_1\gamma_2}(t)\left[D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\right]^{1/\gamma_1}\delta_1^{1/\gamma_1}(t,t_1)}{r^{1/\gamma_1}(t)}$$

So, the proof is complete.

Lemma 2. 3. [36]: Assume that A and B are nonnegative real numbers. Then,

$$\lambda AB^{\lambda-1} - A^\lambda \leq (\lambda-1)B^\lambda$$

for all $\lambda > 1$.

Theorem 2. 4. Assume that (2.1)-(2.3) and $\gamma_1\gamma_2 = 1$ hold. If there exists $\varphi \in C^\alpha([t_0,\infty),\mathrm{R}_+)$ such that

$$\left.\begin{aligned}&\int_{\xi_2}^{\xi}\Big\{k\tilde{q}(s)\tilde{\varphi}(s)\\&-\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{4\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\&+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\Big\}ds\\&=\infty\end{aligned}\right\}\quad(2.7)$$

where $k\in\mathrm{R}_+$; $\tilde{\varphi}(\xi)=\varphi(t)$; then, every solution of (1) is oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$.

Proof. Suppose the contrary that $x(t)$ is non-oscillatory solution of (1.1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2. 1, we have $D_t^\alpha\left(r(t)D_t^\alpha x(t)\right)>0$, $t\in[t_2,\infty)$, where $t_2>t_1$ is sufficiently large, and either $D_t^\alpha x(t)>0$ on $[t_2,\infty)$ or $\lim_{t\to\infty}x(t)=0$. Then, define the following generalized Riccati function:

$$\omega(t)=\varphi(t)\left\{\frac{a(t)\left(D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\right)^{\gamma_2}}{x(t)}+\rho(t)\right\}$$

For $t\in[t_2,\infty)$, we have

$$\left.\begin{aligned}D_t^\alpha\omega(t)&=D_t^\alpha\varphi(t)\frac{a(t)\left(D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\right)^{\gamma_2}}{x(t)}\\&+D_t^\alpha\varphi(t)\rho(t)+\varphi(t)D_t^\alpha\rho(t)\\&+\varphi(t)D_t^\alpha\left\{\frac{a(t)\left(D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\right)^{\gamma_2}}{x(t)}\right\}\end{aligned}\right\}$$

So,

$$\begin{aligned}D_t^\alpha\omega(t)&=D_t^\alpha\varphi(t)\frac{\omega(t)}{\varphi(t)}\\&-\varphi(t)\frac{q(t)f(x(t))}{x(t)}+\varphi(t)D_t^\alpha\rho(t)\\&-\varphi(t)\frac{a(t)\left(D_t^\alpha\left(r(t)\left[D_t^\alpha x(t)\right]^{\gamma_1}\right)\right)^{\gamma_2}D_t^\alpha x(t)}{x^2(t)}\end{aligned}$$

Using Lemma 2.2 and definition of f, we obtain

$$\begin{aligned}D_t^\alpha\omega(t)&\le D_t^\alpha\varphi(t)\frac{\omega(t)}{\varphi(t)}\\&-k\varphi(t)q(t)+\varphi(t)D_t^\alpha\rho(t)\\&-\varphi(t)\frac{\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\left\{\frac{\omega(t)}{\varphi(t)}-\rho(t)\right\}^2\end{aligned}$$

and so,

$$\left.\begin{aligned}D_t^\alpha\omega(t)&\le\omega(t)\left\{\frac{D_t^\alpha\varphi(t)}{\varphi(t)}+\frac{2\delta_1^{1/\gamma_1}(t,t_2)\rho(t)}{r^{1/\gamma_1}(t)}\right\}\\&-k\varphi(t)q(t)+\varphi(t)D_t^\alpha\rho(t)\\&-\frac{\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)\varphi(t)}\omega^2(t)\\&-\varphi(t)\frac{\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\rho^2(t)\end{aligned}\right\}\quad(2.8)$$

Setting $\lambda=2$, $A=\left(\frac{\delta_1^{1/\gamma_1}(t,t_2)}{\varphi(t)r^{1/\gamma_1}(t)}\right)^{1/2}\omega(t)$, $B=\frac{2\varphi(t)\delta_1^{1/\gamma_1}(t,t_2)\rho(t)+r^{1/\gamma_1}(t)D_t^\alpha\varphi(t)}{2\left(r^{1/\gamma_1}(t)\varphi(t)\delta_1^{1/\gamma_1}(t,t_2)\right)^{1/2}}$ by a combination of Lemma 2. 3 and (2.8), we get that

$$\begin{aligned}D_t^\alpha\omega(t)&\le-kq(t)\varphi(t)+\varphi(t)D_t^\alpha\rho(t)\\&-\varphi(t)\frac{\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\rho^2(t)\\&+\frac{\left(\begin{aligned}&2\varphi(t)\delta_1^{1/\gamma_1}(t,t_2)\rho(t)\\&+r^{1/\gamma_1}(t)D_t^\alpha\varphi(t)\end{aligned}\right)^2}{4r^{1/\gamma_1}(t)\varphi(t)\delta_1^{1/\gamma_1}(t,t_2)}\end{aligned}\quad(2.9)$$

Now, let $\omega(t)=\tilde{\omega}(\xi)$. Then we have $D_t^\alpha\omega(t)=\tilde{\omega}'(\xi)$ and $D_t^\alpha\varphi(t)=\tilde{\varphi}'(\xi)$. Thus (2.9) is transformed into

$$\tilde{\omega}'(\xi) \le -k\tilde{q}(\xi)\tilde{\varphi}(\xi) + \tilde{\varphi}(\xi)\tilde{\rho}'(\xi) - \tilde{\varphi}(\xi)\frac{\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)}{\tilde{r}^{1/\gamma_1}(\xi)}\tilde{\rho}^2(\xi) + \frac{\left(2\tilde{\varphi}(\xi)\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)\tilde{\rho}(\xi) + \tilde{r}^{1/\gamma_1}(\xi)\tilde{\varphi}'(\xi)\right)^2}{4\tilde{r}^{1/\gamma_1}(\xi)\tilde{\varphi}(\xi)\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)}$$

Substituting ξ with s in above the inequality and integrating two sides of it from ξ_2 to ξ, we have

$$\left.\begin{aligned} &\int_{\xi_2}^{\xi}\Bigg\{k\tilde{q}(s)\tilde{\varphi}(s) - \frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s) + \tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)} \\ &\quad + \tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) - \tilde{\varphi}(s)\tilde{\rho}'(s)\Bigg\}ds \\ &\le \tilde{\omega}(\xi_2) - \tilde{\omega}(\xi) \\ &\le \tilde{\omega}(\xi_2) \\ &< \infty \end{aligned}\right\}$$

which contradicts (2.7). So, the proof is complete.

Theorem 2. 5. Assume that (2.1)-(2.3) and $\gamma_1\gamma_2 = 1$ hold. If there exists $\varphi \in C^{\alpha}\left([t_0,\infty), \mathrm{R}_+\right)$, such that for any sufficiently large $T \ge \xi_0$ there exists a,b,c with $T \le a < c < b$ satisfying

$$\left.\begin{aligned} &\frac{1}{H(c,a)}\int_a^c H(s,a)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds \\ &\frac{1}{H(b,c)}\int_c^b H(b,s)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds \\ &> \frac{1}{H(c,a)}\int_a^c \frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)} \\ &\times\left(h_1(s,a) - \left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)} + \frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(s,a)}\right)^2 ds \\ &+\frac{1}{H(b,c)}\int_c^b \frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)} \\ &\times\left(h_2(b,s) - \left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)} + \frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(b,s)}\right)^2 ds \end{aligned}\right\}$$

where $\tilde{\varphi}$ is defined as in Theorem 2. 4. Then, every solution of (1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose the contrary that $x(t)$ is non-oscillatory solution of (1.1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2. 1, we have $D_t^{\alpha}\left(r(t)\left[D_t^{\alpha}x(t)\right]^{\gamma_1}\right) > 0$, $t \in [t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_t^{\alpha}x(t) > 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} x(t) = 0$. Then (2.8) holds. Let $\omega(t)$, $\tilde{\omega}(\xi)$ be defined as in Theorem 2. 4. Then we have $D_t^{\alpha}\omega(t) = \tilde{\omega}'(\xi)$ and $D_t^{\alpha}\varphi(t) = \tilde{\varphi}'(\xi)$, so

$$\left.\begin{aligned} \tilde{\omega}'(\xi) &\le \tilde{\omega}(\xi)\left\{\frac{\tilde{\varphi}'(\xi)}{\tilde{\varphi}(\xi)} + \frac{2\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)\tilde{\rho}(\xi)}{\tilde{r}^{1/\gamma_1}(\xi)}\right\} \\ &\quad - k\tilde{\varphi}(\xi)\tilde{q}(\xi) + \tilde{\varphi}(\xi)\tilde{\rho}'(\xi) \\ &\quad - \frac{\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)}{\tilde{r}^{1/\gamma_1}(\xi)\tilde{\varphi}(\xi)}\tilde{\omega}^2(\xi) \\ &\quad - \tilde{\varphi}(\xi)\frac{\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)}{\tilde{r}^{1/\gamma_1}(\xi)}\tilde{\rho}^2(\xi) \end{aligned}\right\} \tag{2.10}$$

Choosing a, b, c arbitrary with $a > b > c$ in $[\xi_2,\infty)$. Substituting ξ with s and multiplying two sides of (2.10) by $H(\xi,s)$ and integrating it from c to ξ, we get

$$\begin{aligned} &\int_c^{\xi} H(\xi,s)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds \\ &\le -\int_c^{\xi} H(\xi,s)\tilde{\omega}'(s)ds \\ &\quad + \int_c^{\xi} H(\xi,s)\left[\begin{aligned}&\tilde{\omega}(s)\left\{\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)} + \frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right\} \\ &-\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}\tilde{\omega}^2(s)\end{aligned}\right]ds \end{aligned}$$

Using the method of integration by parts

$$\left.\begin{aligned}&\int_c^{\xi} H(\xi,s)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\&+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds\\&\leq H(\xi,c)\tilde{\omega}(c)\\&-\int_c^{\xi} h_2(\xi,s)\sqrt{H(\xi,s)}\tilde{\omega}(s)\,ds\\&+\int_c^{\xi} H(\xi,s)\left[\begin{aligned}&\tilde{\omega}(s)\left\{\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right\}\\&-\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}\tilde{\omega}^2(s)\end{aligned}\right]ds\end{aligned}\right\}$$

So,

$$\begin{aligned}&\int_c^{\xi} H(\xi,s)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\&+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds\\&\leq H(\xi,c)\tilde{\omega}(c)\\&-\int_c^{\xi}\left\{\begin{aligned}&\left(\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)H(\xi,s)}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}\right)^{1/2}\tilde{\omega}(s)\\&-\frac{1}{2}\left(\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\right)^{1/2}\\&\times\left(h_2(\xi,s)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(\xi,s)}\right)\end{aligned}\right\}^2 ds\\&+\int_c^{\xi}\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\&\times\left(h_2(\xi,s)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(\xi,s)}\right)^2 ds\\&\leq H(\xi,c)\tilde{\omega}(c)\\&+\int_c^{\xi}\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\&\times\left(h_2(\xi,s)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(\xi,s)}\right)^2 ds\end{aligned}$$

Now letting $\xi\to b^-$ and dividing both sides by $H(b,c)$, we obtain,

$$\left.\begin{aligned}&\frac{1}{H(b,c)}\int_c^{b} H(b,s)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\&+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds\\&\leq\tilde{\omega}(c)\\&+\frac{1}{H(b,c)}\int_c^{b}\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\&\times\left(h_2(b,s)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(b,s)}\right)^2 ds\end{aligned}\right\}$$

On the other hand, substituting ξ with s and multiplying two sides of (2.10) by $H(s,\xi)$ and integrating it from ξ to c, with similar calculations, we get

$$\left.\begin{aligned}&\int_{\xi}^{c} H(s,\xi)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\&+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds\\&\leq -H(c,\xi)\tilde{\omega}(c)\\&+\int_{\xi}^{c}\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\&\times\left(h_1(s,\xi)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(s,\xi)}\right)^2 ds\end{aligned}\right\}$$

Now letting $\xi\to a^+$ and dividing both sides by $H(c,a)$, we obtain,

$$\left.\begin{aligned}&\frac{1}{H(c,a)}\int_a^{c} H(s,a)\left\{\begin{aligned}&k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\&+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{aligned}\right\}ds\\&\leq -\tilde{\omega}(c)\\&+\frac{1}{H(c,a)}\int_a^{c}\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\&\times\left(h_1(s,a)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(s,a)}\right)^2 ds\end{aligned}\right\}$$

So, we get the inequality

$$\left.\begin{array}{l}\frac{1}{H(c,a)}\int_a^c H(s,a)\left\{\begin{array}{l}k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{array}\right\}ds\\ \frac{1}{H(b,c)}\int_c^b H(b,s)\left\{\begin{array}{l}k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\+\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\end{array}\right\}ds\\ \le\frac{1}{H(c,a)}\int_a^c\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\ \times\left(h_1(s,a)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(s,a)}\right)^2ds\\ +\frac{1}{H(b,c)}\int_c^b\frac{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)}{4\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\\ \times\left(h_2(b,s)-\left(\frac{\tilde{\varphi}'(s)}{\tilde{\varphi}(s)}+\frac{2\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)}{\tilde{r}^{1/\gamma_1}(s)}\right)\sqrt{H(b,s)}\right)^2ds\end{array}\right)$$

This is a contradiction. Thus, the proof is complete.

Theorem 2. 6. Assume that (2.1)-(2.3), $\gamma_1\gamma_2=1$ hold and there exists a function $G\in C([\xi_0,\infty),\mathrm{R})$ such that $G(\xi,\xi)=0$, for $\xi\ge\xi_0$, $G(\xi,s)\ge 0$ for $\xi>s\ge\xi_0$ and G has non-positive continuous partial derivative $G_s'(\xi,s)$. If $\tilde{\varphi}$ is defined as in Theorem 2. 4 and

$$\left.\begin{array}{l}\limsup_{\xi\to\infty}\frac{1}{G(\xi,\xi_0)}\int_{\xi_0}^{\xi}G(\xi,s)\Big\{k\tilde{q}(s)\tilde{\varphi}(s)\\ -\tilde{\varphi}(s)\tilde{\rho}'(s)-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\\ -\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{4\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\}ds=\infty\end{array}\right\}$$

Then every solution of (1.1) is oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$.

Proof. Suppose the contrary that $x(t)$ is non-oscillatory solution of (1.1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1.1) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 2. 1, we have $D_t^{\alpha}\left(r(t)\left[D_t^{\alpha}x(t)\right]^{\gamma_1}\right)>0$, $t\in[t_2,\infty)$, where $t_2>t_1$ is sufficiently large, and either $D_t^{\alpha}x(t)>0$ on $[t_2,\infty)$ or $\lim_{t\to\infty}x(t)=0$. Then (2.9) holds. Let $\omega(t)=\tilde{\omega}(\xi)$. Then we have $D_t^{\alpha}\omega(t)=\tilde{\omega}'(\xi)$ and $D_t^{\alpha}\varphi(t)=\tilde{\varphi}'(\xi)$, so

$$\begin{aligned}\tilde{\omega}'(\xi)\le&-k\tilde{q}(\xi)\tilde{\varphi}(\xi)+\tilde{\varphi}(\xi)\tilde{\rho}'(\xi)\\&-\tilde{\varphi}(\xi)\frac{\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)}{\tilde{r}^{1/\gamma_1}(\xi)}\tilde{\rho}^2(\xi)\\&+\frac{\left(2\tilde{\varphi}(\xi)\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)\tilde{\rho}(\xi)+\tilde{r}^{1/\gamma_1}(\xi)\tilde{\varphi}'(\xi)\right)^2}{4\tilde{r}^{1/\gamma_1}(\xi)\tilde{\varphi}(\xi)\tilde{\delta}_1^{1/\gamma_1}(\xi,\xi_2)}\end{aligned}$$

Substituting ξ with s in above the inequality and multiplying two sides of it by $G(\xi,s)$ and integrating it from ξ_2 to ξ, we get

$$\begin{aligned}&\int_{\xi_2}^{\xi}G(\xi,s)\Big\{k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\&-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\\&-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\}ds\\&\le-\int_{\xi_2}^{\xi}G(\xi,s)\tilde{\omega}'(s)ds\end{aligned}$$

$$\left.\begin{array}{l}I=\int_{\xi_0}^{\xi}G(\xi,s)\Big\{k\tilde{q}(s)\tilde{\varphi}(s)-\tilde{\varphi}(s)\tilde{\rho}'(s)\\ \quad-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s)\\ \quad-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\}ds\end{array}\right\}$$

$$\begin{aligned}&\le G(\xi,\xi_2)\tilde{\omega}(\xi_2)+\int_{\xi_2}^{\xi}G_s'(\xi,s)\tilde{\omega}(s)\Delta s\\&\le G(\xi,\xi_2)\tilde{\omega}(\xi_2)\\&\le G(\xi,\xi_0)\tilde{\omega}(\xi_2)\end{aligned}$$

Then,

$$\begin{aligned} I = &\int_{\xi_0}^{\xi_2} G(\xi,s)\Big\{ k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) \\ &-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\} ds \\ &+\int_{\xi_2}^{\xi} G(\xi,s)\Big\{ k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) \\ &-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\} ds \end{aligned}$$

$$\begin{aligned} I \le &G(\xi,\xi_0)\tilde{\omega}(\xi_2) \\ &+G(\xi,\xi_0)\int_{\xi_2}^{\xi} \Big| k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) \\ &-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big| \end{aligned}$$

Thus, we get

$$\begin{aligned} &\limsup_{\xi\to\infty} \frac{1}{G(\xi,\xi_0)} I \\ &\le \tilde{\omega}(\xi_2) \\ &+\int_{\xi_2}^{\xi} \Big| k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) \\ &-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big| \\ &\le \tilde{\omega}(\xi_2) < \infty \end{aligned}$$

This is a contradiction. So, the proof is complete.

From the Theorems, one can derive a lot of oscillation criteria. For instance, consider $G(\xi,s)=(\xi-s)^{\lambda}$, or $G(\xi,s)=\ln\left(\frac{\xi}{s}\right)$ in the Theorem 2. 6. Then, we have the following results.

Corollary 2. 7. Under the conditions of Theorem 2. 6, if

$$\begin{aligned} &\limsup_{\xi\to\infty} \frac{1}{(\xi-\xi_0)^{\lambda}} \int_{\xi_0}^{\xi} (\xi-s)^{\lambda} \Big\{ k\tilde{q}(s)\tilde{\varphi}(s) - \tilde{\varphi}(s)\tilde{\rho}'(s) \\ &-\tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) \\ &-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\} ds = \infty \end{aligned}$$

Then, every solution of (1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$.

Corollary 2. 8. Under the conditions of Theorem 2. 6, if

$$\begin{aligned} &\limsup_{\xi\to\infty} \frac{1}{\ln(\xi)-\ln(\xi_0)} \int_{\xi_0}^{\xi} (\ln(\xi)-\ln(s)) \Big\{ k\tilde{q}(s)\tilde{\varphi}(s) \\ &-\tilde{\varphi}(s)\tilde{\rho}'(s) - \tilde{\varphi}(s)\frac{\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}{\tilde{r}^{1/\gamma_1}(s)}\tilde{\rho}^2(s) \\ &-\frac{1}{4}\frac{\left(2\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)\tilde{\rho}(s)+\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}'(s)\right)^2}{\tilde{r}^{1/\gamma_1}(s)\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\Big\} ds = \infty \end{aligned}$$

Then, every solution of (1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$.

3. APPLICATIONS

Example 3. 1. Consider the fractional differential equation,

$$\begin{aligned} &D_t^{1/3}\left[t^{1/5}\left(D_t^{1/3}\left[D_t^{1/3}x(t)\right]^{5/3}\right)^{3/5}\right] \\ &\qquad + t^{-2/3}x(t)\left(1+\sin^2(x(t))\right)=0 \end{aligned} \quad (3.1)$$

for $t \ge 3$. This corresponds to (1.1) with $t_0=3$, $\alpha=\frac{1}{3}$, $\gamma_1=\frac{5}{3}$, $\gamma_2=\frac{3}{5}$, $a(t)=t^{1/5}$, $r(t)=1$, $q(t)=t^{-2/3}$ and $f(x)=x\left(1+\sin^2 x\right)$. So, $f(x)/x=1+\sin^2 x \ge 1 = k$, $\xi_0 = 3^{1/3}/\Gamma(4/3)$, $\tilde{a}(\xi)=\left(\xi\Gamma(4/3)\right)^{3/5}$, $\tilde{q}(\xi)=\left(\xi\Gamma(4/3)\right)^{-2}$. So,

$$\tilde{\delta}_1(\xi,\xi_2)=\int_{\xi_2}^{\xi}1/\tilde{a}^{1/\gamma_2}(s)ds$$
$$=\left[\Gamma(4/3)\right]^{-1}\int_{\xi_2}^{\xi}\frac{1}{s}ds$$
$$=\left[\Gamma(4/3)\right]^{-1}\left(\ln(\xi)-\ln(\xi_2)\right)$$

which implies $\lim_{\xi\to\infty}\tilde{\delta}_1(\xi,\xi_2)=\infty$, and so, (2.1) holds. Then, there exists a sufficiently large $T>\xi_2$ such that $\tilde{\delta}_1(\xi,\xi_2)>1$ on $[T,\infty)$. In (2.2),

$$\int_{\xi_0}^{\infty}\frac{1}{r^{1/\gamma_1}(s)}ds=\int_{\xi_0}^{\infty}ds=\infty$$

In (2.3),

$$\int_{\xi_0}^{\infty}\left[\frac{1}{\tilde{r}(\zeta)}\int_{\zeta}^{\infty}\left[\frac{1}{\tilde{a}(\tau)}\int_{\tau}^{\infty}\tilde{q}(s)ds\right]^{1/\gamma_2}d\tau\right]^{1/\gamma_1}d\zeta$$
$$=\left[\Gamma(4/3)\right]^{-13/5}\int_{\xi_0}^{\xi}\left[\int_{\zeta}^{\infty}\left[\frac{1}{\tau^{3/5}}\int_{\tau}^{\infty}s^{-2}ds\right]^{5/3}d\tau\right]^{3/5}d\zeta$$
$$=-\left[\Gamma(4/3)\right]^{-7/5}\int_{\xi_0}^{\xi}\left[\int_{\zeta}^{\infty}\tau^{-8/3}d\tau\right]^{3/5}d\zeta$$
$$=(3/5)^{3/5}\left[\Gamma(4/3)\right]^{-7/5}\int_{\xi_0}^{\xi}\zeta^{-1}d\zeta$$
$$=\infty$$

Letting $\tilde{\varphi}(\xi)=\xi$ and $\tilde{\rho}(\xi)=0$ in Theorem 2.4,

$$\int_{\xi_0}^{\infty}\left\{k\tilde{q}(s)\tilde{\varphi}(s)-\frac{1}{4}\frac{\tilde{r}^{1/\gamma_1}(s)\left[\tilde{\varphi}'(s)\right]^2}{\tilde{\varphi}(s)\tilde{\delta}_1^{1/\gamma_1}(s,\xi_2)}\right\}ds$$
$$=\int_{\xi_0}^{T}\left[\left(\Gamma(4/3)\right)^{-2}-\frac{1}{4\tilde{\delta}_1^{3/5}(s,\xi_2)}\right]\frac{1}{s}ds$$
$$+\int_{T}^{\infty}\left[\left(\Gamma(4/3)\right)^{-2}-\frac{1}{4\tilde{\delta}_1^{3/5}(s,\xi_2)}\right]\frac{1}{s}ds$$
$$\geq\int_{\xi_0}^{T}\left[\left(\Gamma(4/3)\right)^{-2}-\frac{1}{4\tilde{\delta}_1^{3/5}(s,\xi_2)}\right]\frac{1}{s}ds$$
$$+\int_{T}^{\infty}\left[\left(\Gamma(4/3)\right)^{-2}-\frac{1}{4}\right]\frac{1}{s}ds$$
$$=\infty$$

So, (3.1) is oscillatory by Theorem 2. 4.

Example 3. 2. Consider the fractional differential equation,

$$\left.\begin{aligned}&D_t^{1/7}\left[t^{3/7}\left(D_t^{1/7}\left[t^{1/21}\left(D_t^{1/7}x(t)\right)^{1/3}\right]\right)^3\right]\\&\quad+t^{-3/7}\left(\Gamma(8/7)\right)^3\exp\left(x^2(t)\right)x(t)=0\end{aligned}\right\}\quad(3.2)$$

for $t\geq 2$. This corresponds to (1.1) with $t_0=2$, $\alpha=\frac{1}{7}$, $\gamma_1=\frac{1}{3}$, $\gamma_2=3$, $a(t)=t^{3/7}$, $r(t)=t^{-1/21}$, $q(t)=t^{-3/7}\left(\Gamma(8/7)\right)^3$ and $f(x)=\exp\left(x^2\right)x$. So, $f(x)/x\geq 1=k$, $\xi_0=2^{1/7}/\Gamma(8/7)$, $\tilde{a}(\xi)=\left(\xi\Gamma(8/7)\right)^3$, $\tilde{r}(\xi)=\left(\xi\Gamma(8/7)\right)^{-1/3}$, $\tilde{q}(\xi)=\xi^{-3}$. So,

$$\tilde{\delta}_1(\xi,\xi_2)=\int_{\xi_2}^{\xi}1/\tilde{a}^{1/\gamma_2}(s)ds$$
$$=\left[\Gamma(8/7)\right]^{-1}\int_{\xi_2}^{\xi}s^{-1}ds$$
$$=\left[\Gamma(8/7)\right]^{-1}\left(\ln(\xi)-\ln(\xi_2)\right)$$

which implies $\lim_{\xi\to\infty}\tilde{\delta}_1(\xi,\xi_2)=\infty$ and so (2.1) holds. Then, there exists a sufficiently large $T>\xi_2$ such that $\tilde{\delta}_1(\xi,\xi_2)>1$ on $[T,\infty)$. In (2.2),

$$\int_{\xi_0}^{\infty}\frac{1}{\tilde{r}^{1/\gamma_1}(s)}ds=\left[\Gamma(8/7)\right]\int_{\xi_0}^{\infty}sds=\infty$$

In (2.3),

$$\int_{\xi_0}^{\infty}\left[\frac{1}{\tilde{r}(\zeta)}\int_{\zeta}^{\infty}\left[\frac{1}{\tilde{a}(\tau)}\int_{\tau}^{\infty}\tilde{q}(s)ds\right]^{1/\gamma_2}d\tau\right]^{1/\gamma_1}d\zeta$$
$$=\left[\Gamma(8/7)\right]^{-8/3}\int_{\xi_0}^{\infty}\left[\frac{1}{\zeta^{-1/3}}\int_{\zeta}^{\infty}\left[\frac{1}{\tau^3}\int_{\tau}^{\infty}s^{-3}ds\right]^{1/3}d\tau\right]^3d\zeta$$
$$=-\frac{\left[\Gamma(8/7)\right]^{-10/3}}{2}\int_{\xi_0}^{\infty}\left[\frac{1}{\zeta^{1/3}}\int_{\zeta}^{\infty}\tau^{-5/3}d\tau\right]^3d\zeta$$
$$=3\frac{\left[\Gamma(8/7)\right]^{-10/3}}{4}\int_{\xi_0}^{\infty}\zeta^{-1}d\zeta$$
$$=\infty$$

Letting $\tilde{\varphi}(\xi)=s^2$, $\lambda=1$ and $\tilde{\rho}(\xi)=0$ in Corollary 2. 7, we have

$$A=\limsup_{\xi\to\infty}\frac{1}{(\xi-\xi_0)}\int_{\xi_0}^{\xi}(\xi-s)\left[s^{-1}-\frac{\left(s\Gamma(8/7)\right)^{-1}}{\tilde{\delta}_1^3(s,\xi_2)}\right]ds$$
$$=\limsup_{\xi\to\infty}\frac{1}{(\xi-\xi_0)}\int_{\xi_0}^{\xi}(\xi-s)\left[1-\frac{1}{\Gamma(8/7)}\frac{1}{\tilde{\delta}_1^3(s,\xi_2)}\right]\frac{1}{s}ds$$

$$A=\limsup_{\xi\to\infty}\frac{1}{(\xi-\xi_0)}\times\left[\begin{array}{l}\int_{\xi_0}^{T}(\xi-s)\left[1-\frac{1}{\Gamma(8/7)}\frac{1}{\tilde{\delta}_1^3(s,\xi_2)}\right]\frac{1}{s}ds\\+\int_{T}^{\xi}(\xi-s)\left[1-\frac{1}{\Gamma(8/7)}\frac{1}{\tilde{\delta}_1^3(s,\xi_2)}\right]\frac{1}{s}ds\end{array}\right]$$
$$\geq\limsup_{\xi\to\infty}\frac{1}{(\xi-\xi_0)}\times\left[\begin{array}{l}\int_{\xi_0}^{T}(\xi-s)\left[1-\frac{1}{\Gamma(8/7)}\frac{1}{\tilde{\delta}_1^3(s,\xi_2)}\right]\frac{1}{s}ds\\+\int_{T}^{\xi}(\xi-s)\left[1-\frac{1}{\Gamma(8/7)}\right]\frac{1}{s}ds\end{array}\right]$$
$$=\infty$$

So, we deduce that (3.2) is oscillatory by Corollary 2. 7.

4. CONCLUSION

In this paper, we are concerned with the oscillation for a kind of fractional differential equations. The fractional differential equation is defined in the sense of the modified Riemann-Liouville fractional derivative. By use of the properties of the fractional derivative, we consider a variable transformation that the fractional differential equations are converted into another differential equation of integer order. Then, some oscillation criteria for the equation (1.1) are established. Finally, we give some examples to illustrate the main results.

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