

PARAMETER ESTIMATION IN A BLACK SCHOLES

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In this paper we discuss parameter estimation in Black Scholes model. A nonparametric estimation method and well known maximum likelihood estimator are considered. Our aim is to estimate the unknown parameters for stochastic differential equation with discrete time observation data. In simulation study we compare the nonparametric method with maximum likelihood method using stochastic numerical scheme named with Euler Maruyama.

Key words: nonparametric estimation method, maximum likelihood estimation method, stochastic differential equations, Black Scholes model.

1. Introduction

When the stochastic differential equation is given with certain coefficients, it is easy to solve this equation with numerical methods, Euler Maruyama method, Milstein method, Runge Kutta method, etc. [1],[7]. However if we have only observed discrete data over any time interval it has difficulty. To reach diffusion and drift coefficients are required in this case, so we need estimation methods. [3],[11],[1],[8] mention about estimation methods for stochastic differential equations. A number of researchers have used nonparametric techniques [11],[12]. Nonparametric method is simple for to implement and estimate the coefficients, but it is reasonably if we have frequent data. Maximum likelihood method [10] is other estimation method which is more efficient than nonparametric method.

In many practical cases where a diffusion process has been observed at discrete time points an explicit likelihood function is rarely available. These types of data have recently received great interest. A simple method to obtain an estimator for discrete time is construct from the stale data, an approximation to the estimator found in the theory for continuous observations which is includes discrete time approximations to stochastic integrals. [9],[4] etc. study such estimators analytically, in general it is difficult but it can be done easily by numerical simulation.

In this paper our aim is discuss the differences between nonparametric estimation method and maximum likelihood estimation method with numerical application in finance.

This article is organized as follows. In Section 2 and Section 3 we discuss the nonparemetric and maximum likelihood estimation methods respectively. Coefficients of Black Scholes model [2], are obtained in Section 4 using monthly YHOO stock, from 01.01.2005 to 01.01.2015 which model is very handy in finance. Then, we use the MATLAB package program to solve the stochastic differential equation numerically with its obtained parameters and compare original data with our approximate solutions. Our results are supported via graphs at the end of the paper.

2. Nonparametric Estimation Method

We consider $X(t)$ diffusion process, which satisfies the stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) \quad (1)$$

Under proper restrictions on μ , σ and arbitrary function ϕ , from [5] conditional expectation $E_t[\phi(X_{t+\Delta t}, t)]$ in the Taylor series form can be written;

$$E_t[\phi(X_{t+\Delta t}, t)] = \phi(X_t, t) + \mathcal{L}\phi(X_t, t)\Delta t + \frac{1}{2}\mathcal{L}^2\phi(X_t, t)(\Delta t)^2 + \dots \quad (2)$$

$$\frac{1}{n!}\mathcal{L}^n\phi(X_t, t)(\Delta t)^n + O((\Delta t)^{n+1})$$

where

$$\begin{aligned} \mathcal{L}\phi(x, t) &= \lim_{\zeta \downarrow t} \frac{E(\phi(X_\zeta, \zeta) | X_t = x) - \phi(x, t)}{\zeta - t} \\ &= \frac{\partial\phi(x, t)}{\partial t} + \frac{\partial\phi(x, t)}{\partial x}\mu(x) + \frac{1}{2}\frac{\partial^2\phi(x, t)}{\partial x^2}\sigma^2(x) \end{aligned} \quad (3)$$

from Eq.(2);

$$\mathcal{L}\phi(X_t, t) = \frac{1}{\Delta t} E_t[\phi(X_{t+\Delta t}, t + \Delta t) - \phi(X_t, t)] - \frac{1}{2}\mathcal{L}^2\phi(X_t, t)\Delta t - \dots \quad (4)$$

is obtained. Taking the first term on the right side of Eq.(4) and ignoring other terms as an error we obtain first order approximate for $\mathcal{L}\phi$. So we can write;

$$\mathcal{L}\phi(X_t, t) = \frac{1}{\Delta t} E_t[\phi(X_{t+\Delta t}, t + \Delta t) - \phi(X_t, t)] + O(\Delta t) \quad (5)$$

If we want approximate a specific $\rho(x, t)$ function, we need only to specify ϕ function which provide $\mathcal{L}\phi(X_t, t) = \rho(x, t)$. To find $\mu(X_t)$ coefficient we take $\phi_1(x, t) \equiv x$ and to find $\sigma(X_t)$ coefficient, $\phi_2(x, t) \equiv (x - X_t)^2$ is determined in [11]. Then from Eq.(3);

$$\mathcal{L}\phi_1(X_t, t) = \mu(X_t) \quad (6)$$

$$\mathcal{L}\phi_2(X_t, t) = \sigma^2(X_t) \quad (7)$$

are obtained. Accordingly

$$\mu(X_t) = \frac{1}{\Delta t} E_t[X_{t+\Delta t} - X_t] + O(\Delta t) \quad (8)$$

$$\sigma^2(X_t) = \frac{1}{\Delta t} E_t[(X_{t+\Delta t} - X_t)^2] + O(\Delta t) \quad (9)$$

are found.

If we want to find θ parameter vector via nonparametric estimation method in

$$dX(t) = \mu(X(t); \theta)dt + \sigma(X(t); \theta)dW(t) \quad (10)$$

and we have x_0, x_1, \dots, x_N observed data of X_t at the respectively uniformly distributed times $t_i = i\Delta t$ for $i = 0, 1, \dots, N$ where $\Delta t = \frac{T}{N}$, from Eq.(8) and Eq.(9) we can find θ using following equations;

$$\sum_{i=0}^{N-1} \mu(t_i, x_i, \theta) = \frac{1}{\Delta t} \sum_{i=0}^{N-1} (x_{i+1} - x_i) \quad (11)$$

$$\sum_{i=0}^{N-1} \sigma^2(t_i, x_i, \theta) = \frac{1}{\Delta t} \sum_{i=0}^{N-1} (x_{i+1} - x_i)^2. \quad (12)$$

3. Maximum Likelihood Estimation Method

We consider Eq.(10). Assume that for $j = 0, 1, 2, \dots, N$, x_j are known, density of the initial case is $g_0(x_0 | \theta)$ and transition probability density of (t_j, x_j) beginning from (t_{j-1}, x_{j-1}) is $g(t_j, x_j | t_{j-1}, x_{j-1}; \theta)$. Then from [6], the maximum likelihood estimation (MLE) of θ is value of θ which maximizes following joint density equation;

$$\mathcal{D}(\theta) = g_0(x_0 | \theta) \prod_{j=1}^N g(t_j, x_j | t_{j-1}, x_{j-1}; \theta) \quad (13)$$

with $L(\theta) = -\ln(\mathcal{D}(\theta))$ transformation we can rewrite Eq. (13) as following;

$$L(\theta) = -\ln(g_0(x_0 | \theta)) - \sum_{j=1}^N \ln(g(t_j, x_j | t_{j-1}, x_{j-1}; \theta)) \quad (14)$$

Our aim is find the minimum value of θ which make $L(\theta)$ function minimum. This proper θ value demonstrate with $\bar{\theta}$. From Euler Maruyama approximation schema we can write;

$$x_{t_j} \approx x_{t_{j-1}} + \mu(t_{j-1}, x_{t_{j-1}}; \theta)\Delta t + g(t_{j-1}, x_{t_{j-1}}; \theta)\sqrt{\Delta t}\xi_j \quad (15)$$

where $\xi_i \square N(0,1)$ and transition probability density is following;

$$g(t_j, x_j | t_{j-1}, x_{j-1}; \theta) \approx \frac{1}{\sqrt{2\pi\delta_j^2}} \exp\left[-\frac{(x_j - \gamma_j)^2}{2\delta_j^2}\right] \quad (16)$$

where

$$\gamma_j = x_{t_{j-1}} + \mu(t_{j-1}, x_{t_{j-1}}; \theta)\Delta t$$

$$\delta_j = g(t_{j-1}, x_{t_{j-1}}; \theta)\sqrt{\Delta t}.$$

We can approximate density function as in [6], for arbitrary value of θ , we can find solution of (10) numerically. This process is repeated M times and we show this estimated values with \bar{x}_i . Then the transition density $g(t_j, x_j | t_{j-1}, x_{j-1}; \theta)$ can be estimated by

$$g^{(M)}(t_j, x_j | t_{j-1}, x_{j-1}; \theta) = \frac{1}{Mh} \sum_{i=1}^M K\left[\frac{x_j - \bar{x}_i}{h}\right] \quad (17)$$

formula where K is a non-negative kernel function which demonstrate following;

$$K(\vartheta) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\vartheta^2}{2}\right]$$

and h is bandwidth equal to

$$h = 0.9\varepsilon M^{-1/5}.$$

where

$$\varepsilon^2 = \frac{1}{M-1} \left(\sum_{i=1}^M x_i^2 - \frac{1}{M} \left(\sum_{i=1}^M x_i \right)^2 \right).$$

4. Analysis of Experimental Data

In this section we tackle the YHOO stock. The stock data set is graphed for every month in Fig.1 over the years 2005 to 2015.

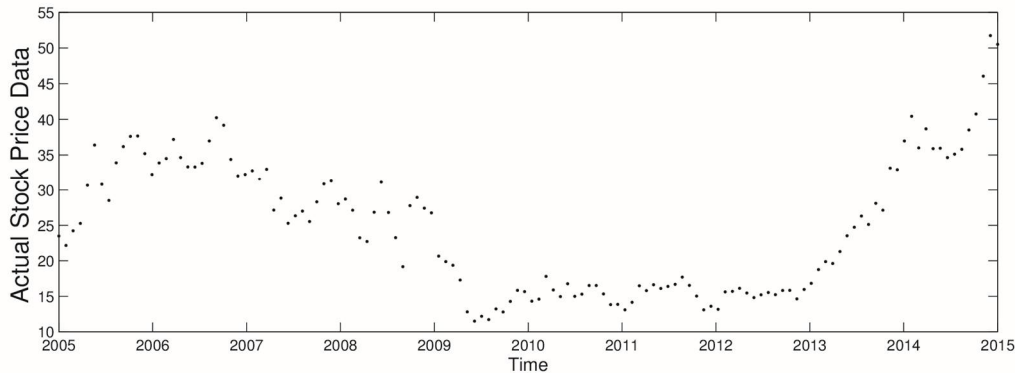


Figure 1: Monthly the YHOO stock, January 2005 to January 2015

We suit this data to stochastic differential equation named Black Scholes model

$$\begin{aligned} X(t) &= \theta_1 X(t)dt + \theta_2 X(t)dW(t) \\ X(0) &= 23.49 \end{aligned} \quad (18)$$

where $X(t)$ is the stock price at time t and $\theta = [\theta_1, \theta_2]^T$ is to be determined. We need to estimate the parameter θ_1 and θ_2 using MLE and nonparametric parameter estimation methods. Firstly we estimate the θ_1 and θ_2 with MLE method. If we use MLE procedure we obtain the optimal values $\bar{\theta}_1 = 0.0107$ and $\bar{\theta}_2 = 0.1000$ approximately. Therefore, a reasonable stochastic differential equation based on the data for the YHOO stock for 132 months is

$$\begin{aligned} dX(t) &= 0.0107X(t)dt + 0.1000X(t)dW(t) \\ X(0) &= 23.49 \end{aligned} \quad (19)$$

Now we estimate the θ_1 and θ_2 with nonparametric estimation method. If we use non-parametric estimation method procedure for $N = 131$ and $\Delta t = 1$, we obtain the optimal values $\hat{\theta}_1 = 0.0084$ and $\hat{\theta}_2 = 0.0952$ approximately. Therefore, a reasonable stochastic differential equation based on the data for the YHOO stock for 132 months is

$$\begin{aligned} dX(t) &= 0.0084X(t)dt + 0.0952X(t)dW(t) \\ X(0) &= 23.49 \end{aligned} \quad (20)$$

The mean of stock price of actual data is $\mu = 24.8445$ and $(23.2291, 26.4600)$ gives its 95% confidence interval. Solving Eq.(19) and Eq.(20) with Euler Maruyama approximation method we obtain the mean of stock price forecasting with MLE is $\bar{\mu} = 24.4767$ and $(22.7030, 26.2504)$ gives its 95% confidence interval and the mean of stock price forecasting with non-parametric parameter estimation is $\hat{\mu} = 22.1242$ and $(20.5504, 23.6980)$ gives its 95% confidence interval respectively.

In Fig. 2, the simulated data, evaluated with MLE using the SDE model, and the monthly actual data are plotted for every month over the year 2005-2015. "Actual" holds real data for the YHOO stock, which is plotted as red straight lines. "Forecast" keeps MLE estimation using Euler Maruyama approximations, which is plotted as blue straight lines.

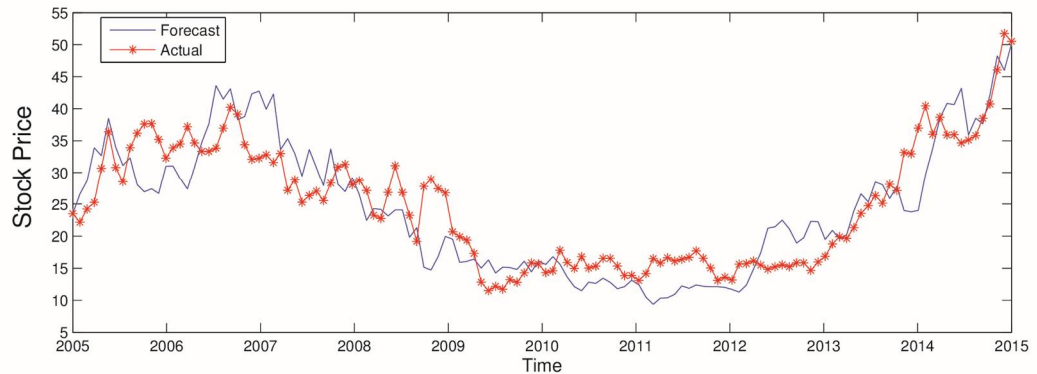


Figure 2: Estimated data using MLE and actual data of the YHOO stock, January 2005 to January 2015

In Fig. 3, the simulated data, evaluated with nonparametric parameter estimation using the SDE model, and the monthly actual data are plotted for every month over the year 2005-2015. "Actual" holds real data for the YHOO stock, which is plotted as red straight lines. "Forecast" keeps nonparametric estimation using Euler Maruyama approximations, which is plotted as blue straight lines.

All of the graphs indicate that the Stochastic differential equation model supplies sensible fit to the data.

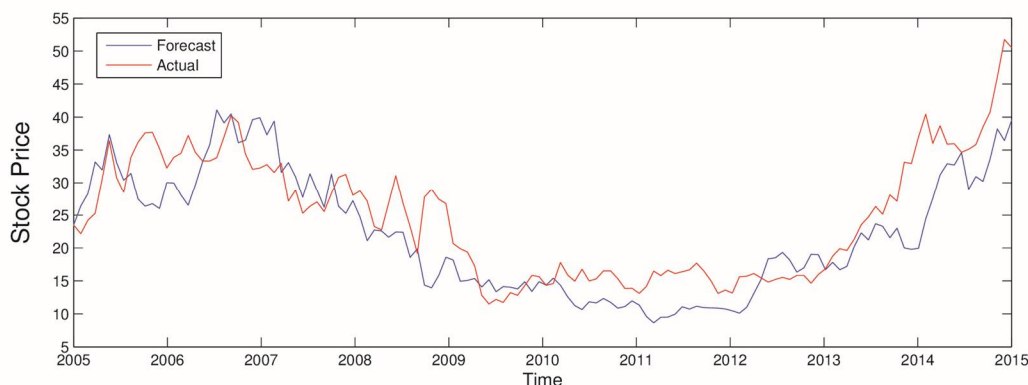


Figure 3: Estimated data using nonparametric estimation and actual data of the YHOO stock January 2005 to January 2015

5. Conclusion

In this paper we have been concerned with the estimation of the θ_1 and θ_2 in the drift coefficient and in the diffusion coefficient respectively of a Black Scholes model, when the observation data known. Using the YHOO stock data monthly between 01.01.2005 and 01.01.2015 maximum likelihood estimation parameters $\bar{\theta}_1$, $\bar{\theta}_2$ and nonparametric estimation parameters $\hat{\mu}_1$, $\hat{\mu}_2$ are obtained. This obtained parameters written in the Black Scholes model. Then applying Euler Maruyama method this stochastic differential equation with its initial value, the simulated solution is obtained for each estimation method at each time. After that, we compared actual data with numerical solutions for each estimation method. According to our results we can say that maximum likelihood estimation method is have a good approximation to observation data via nonparametric estimation method

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