Some properties of the Mittag-Leffler functions and their relation with the Wright functions

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Abstract

This paper is a short description of our recent results on an important class of the so-called Mittag-Leffler functions, which became important as solutions of fractional order differential and integral equations, control systems and refined mathematical models of various physical, chemical, economical, management and bioengineering phenomena. We have studied the Mittag-Leffler functions as their typical representatives, including many interesting special cases that have already proven their usefulness in fractional calculus and its applications. We obtained a number of useful relationships between the Mittag-Leffler functions and the Wright functions. The Wright function plays an important role in the solution of a linear partial differential equation. The Wright function, which we denote by $W(z; \alpha, \beta)$, is so named in honor of Wright who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions.
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1 Introduction
1.1 The Mittag-Leffler function

The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. In fact, the exponential function itself is a very special form, one of an infinite set of these seemingly ubiquitous functions. The standard definition of Mittag-Leffler [1] is given as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in C, R(\alpha) > 0, z \in C. \tag{1}$$

A two-parameter function of the M-L (Mittag-Leffler) type is defined by the series expansion [2]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0, z \in C). \tag{2}$$

The M-L function provides a simple generalization of the exponential function because of the substitution of $n! = \Gamma(n + 1)n!$ with $(n\alpha)! = \Gamma(n\alpha + 1)$. Particular cases of (2) recover elementary functions are recovered

$$E_{1,2} = \frac{e^z - 1}{z}, \qquad E_{2,2}(z^2) = \frac{\sinh(z)}{z},$$

and

$$E_{1/2,1} = e^{z^2} \operatorname{erfc}(-z),$$

where erf (erfc) denotes the (complementary) error function defined as [3]

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt.$$

By means of the series representation, a generalization of (1) and (2) is introduced by Prabhakar [4] as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha k + \beta)} \cdot \frac{z^k}{k!} \quad (\alpha, \beta, \gamma \in C, R(\alpha) > 0, R(\beta), R(\gamma) > 0, z \in C), \tag{3}$$

where $(\gamma)_n$ is the Pochhammer symbol [5] given by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0; \lambda \neq 0), \\ (\gamma)_n = \gamma(\gamma + 1)(\gamma + 2) \cdots (\gamma + n - 1) & (n \in N; \lambda \in C). \end{cases}$$

Note that

$$E_{1,1}^1(z) = e^z, \qquad E_{\alpha,1}^1(z) = E_\alpha(z), \qquad E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z).$$

Some new properties of the Mittag-Leffler function, including a definite integral and recurrence relation, were investigated in [6, 7].

1.2 The Wright function

The Wright function plays an important role in the solution of a linear partial differential equation. The Wright function, which we denote by $W(z; \alpha, \beta)$, is so named in honor of Wright, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions. This function was introduced that related Mittag-Leffler [8–11]. We obtained a number of useful relationships between the Mittag-Leffler functions and the Wright functions.

1.2.1 Definition

The Wright function is defined by the series representation, convergent in the whole z-complex plane [12]

$$W(z; \alpha, \beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}.$$

1.2.2 The integral representation of the Wright function

$$W(z; \alpha, \beta) = \frac{1}{2\pi i} \int_{Ha} \exp\{u + zu^{-\alpha}\} u^{-\beta} \, du,$$

where Ha denotes the Hankel path. To prove the Hankel path, let us write the integrated function in the form of a power series in z and perform term-by-term integration using the integral representation formula for the reciprocal gamma function

$$\frac{1}{\Gamma(z)} = \int_{Ha} e^{\xi} \zeta^z \, d\zeta.$$

In fact,

$$W(z; \alpha, \beta) = \frac{1}{2\pi i} \int_{Ha} \exp\{u + zu^{-\alpha}\} u^{-\beta} \, du = \frac{1}{2\pi i} \int_{Ha} e^u \left[\sum_{n=0}^{\infty} \frac{z^n}{n!} u^{-\alpha n} \right] u^{-\beta} \, du$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[\frac{1}{2\pi i} \int_{Ha} e^u u^{-\alpha n - \beta} \, du \right] = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\alpha n + \beta)}.$$

1.2.3 The Laplace transform of the Wright function

We recall that the Mittag-Leffler function plays fundamental roles in applications of fractional calculus like fractional relaxation and fractional oscillation [13–16]. Kiryakova introduced and studied the multi-index Mittag-Leffler functions as their typical representatives, including many interesting special cases that have already proven their usefulness in FC and its applications [17]. Srivatava and Tomovski introduced and investigated the fractional calculus with an integral operator which contains the following family of generalized Mittag-Leffler functions [18]. Haubold, Mathaian and Saxena studied the Mittag-Leffler functions and their applications [19]. There is an interesting link between the Wright function and the Mittag-Leffler function. We now point out that the Wright function is related to the Mittag-Leffler function through the following Laplace transform pair:

$$L\{W(t; \alpha, \beta); s\} = L\left\{ \sum_{k=0}^{\infty} \frac{t^k}{k! \Gamma(\alpha k + \beta)}; s \right\}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\alpha k + \beta)} \cdot \frac{1}{s^{k+1}}$$

$$= s^{-1} E_{\alpha, \beta}\left(s^{-1}\right).$$

2 Some properties of the Mittag-Leffler functions

Theorem 1 (Derivative of the Mittag-Leffler function) *If* $\alpha, \beta, \gamma \in C$, $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma) > 0$, $z \in C$ *and* $r, n \in N$, *then*

$$\frac{d^n}{dx^n}\left[z^{\beta-1} E^{\gamma}_{\alpha, \beta+r\alpha}(z)\right] = z^{\beta-n-1} E^{\gamma}_{\alpha, \beta+r\alpha-n}(z).$$

Proof Using definition (3), we have that

$$\frac{d^n}{dx^n}\left[z^{\beta-1} E^{\gamma}_{\alpha, \beta+r\alpha}(z)\right] = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha(k+r) + \beta - n)} \frac{z^{k+\beta-1-n}}{k!} = z^{\beta-n-1} E^{\gamma}_{\alpha, \beta+r\alpha-n}(z).$$

The Wright function is expressed with help of the Mittag-Leffler function:

$$\frac{d^n}{dx^n}\big[z^{\beta-1}E_{\alpha,\beta+r\alpha}^{\gamma}(z)\big] = \sum_{k=0}^{\infty} z^{\beta-1-n}(\gamma)_k W(z;\alpha,\beta+r\alpha-n).$$

□

Theorem 2 (Integration of the Mittag-Leffler function) *If* $\alpha,\beta,\gamma \in C$, $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma) > 0$, $z \in C$ *and* $r \in N$, *then*

$$\int_0^z z^{\beta+r\alpha-1}E_{\alpha,\beta+r\alpha}^{\gamma}(z)\,dz = z^{\beta+r\alpha}E_{\alpha,\beta+r\alpha+1}^{\gamma}(z).$$

Proof According to the Mittag-Leffler function, we have

$$z^{\beta+r\alpha-1}E_{\alpha,\beta+r\alpha}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha(k+r)+\beta)}\cdot\frac{z^{k+r\alpha+\beta-1}}{k!}.$$

Integrating both sides gives

$$\int_0^z z^{\beta+r\alpha-1}E_{\alpha,\beta+r\alpha}^{\gamma}(z)\,dz = \int_0^z \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha(k+r)+\beta)}\cdot\frac{z^{k+r\alpha+\beta-1}}{k!}\,dz$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha(k+r)+\beta+1)}\cdot\frac{z^{k+r\alpha+\beta}}{k!}$$

$$= z^{\beta+r\alpha}E_{\alpha,\beta+r\alpha+1}^{\gamma}(z).$$

Relation with the Wright functions is as follows:

$$\int_0^z z^{\beta+r\alpha-1}E_{\alpha,\beta+r\alpha}^{\gamma}(z)\,dz = \int_0^z \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha(k+r)+\beta)}\cdot\frac{z^{k+r\alpha+\beta-1}}{k!}\,dz$$

$$= \sum_{k=0}^{\infty} z^{\beta+r\alpha}(\gamma)_k W(z;\alpha,\beta+r\alpha+1).$$

□

Theorem 3 *Let* $\alpha,\beta,\gamma \in C$, $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma) > 0$, $z \in C$, $r \in N$, *then*

$$E_{\alpha,\beta}^{\gamma}(z) = \beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma}(z).$$

Proof By definition (3), we have that

$$\beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma}(z) = \beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z\frac{d}{dz}\sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta+1)}\cdot\frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\beta(\gamma)_n}{\Gamma(\alpha n+\beta+1)}\cdot\frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{\alpha n(\gamma)_n}{\Gamma(\alpha n+\beta+1)}\cdot\frac{z^n}{n!}$$

$$= E_{\alpha,\beta}^{\gamma}(z).$$

Relation with the Wright functions is as follows:

$$\beta E^{\gamma}_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E^{\gamma}_{\alpha,\beta+1}(z) = \beta E^{\gamma}_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta + 1)} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\beta(\gamma)_n}{\Gamma(\alpha n + \beta + 1)} \cdot \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{\alpha n (\gamma)_n}{\Gamma(\alpha n + \beta + 1)} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} (\gamma)_n W(z; \alpha, \beta). \qquad \square$$

Theorem 4 *For $\alpha, \beta, \gamma \in C$, $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma) > 0$, $z \in C$, $r \in N$. 'Note that'*

$$E^{\gamma}_{\alpha,\beta+r\alpha+1}(z) - E^{\gamma}_{\alpha,\beta+r\alpha+2}(z)$$

$$= (\beta + r\alpha)(\beta + r\alpha + 2) E^{\gamma}_{\alpha,\beta+r\alpha+3}(z) + \alpha^2 z^2 \frac{d^2}{dz^2} E^{\gamma}_{\alpha,\beta+r\alpha+3}(z)$$

$$+ \alpha \{\alpha + 2 + 2(\beta + r\alpha)\} z \frac{d}{dz} E^{\gamma}_{\alpha,\beta+r\alpha+3}(z). \qquad (4)$$

Proof We have

$$E^{\gamma}_{\alpha,\beta+r\alpha+1}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\alpha(k+r) + \beta)\Gamma(\alpha(k+r) + \beta)} \cdot \frac{z^k}{k!}, \qquad (5)$$

$$E^{\gamma}_{\alpha,\beta+r\alpha+2}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\alpha(k+r) + \beta)(\alpha(k+r) + \beta + 1)\Gamma(\alpha(k+r) + \beta)} \cdot \frac{z^k}{k!}. \qquad (6)$$

Equation (6) can be written as follows:

$$E^{\gamma}_{\alpha,\beta+r\alpha+2}(z) = \sum_{k=0}^{\infty} \left[\frac{1}{\alpha(k+r) + \beta} - \frac{1}{\alpha(k+r) + \beta + 1} \right] \frac{z^k (\gamma)_k}{\Gamma(\alpha(k+r) + \beta) k!}$$

$$= E^{\gamma}_{\alpha,\beta+r\alpha+1}(z) - \sum_{k=0}^{\infty} \frac{z^k (\gamma)_k}{(\alpha(k+r) + \beta + 1)\Gamma(\alpha(k+r) + \beta) k!}. \qquad (7)$$

We find from equation (7) that

$$E^{\gamma}_{\alpha,\beta+r\alpha+1}(z) - E^{\gamma}_{\alpha,\beta+r\alpha+2}(z) = \sum_{k=0}^{\infty} \frac{z^k (\gamma)_k}{(\alpha(k+r) + \beta + 1)\Gamma(\alpha(k+r) + \beta) k!}$$

$$= \frac{z^k (\gamma)_k}{k!} \left(\frac{1}{(\alpha(k+r) + \beta + 2)(\alpha(k+r) + \beta + 1)\Gamma(\alpha(k+r) + \beta)} \right.$$

$$\left. + \frac{1}{(\alpha(k+r) + \beta + 2)\Gamma(\alpha(k+r) + \beta)} \right) \qquad (8)$$

or

$$\sum_{k=0}^{\infty} \frac{z^k (\gamma)_k}{(\alpha(k+r) + \beta + 1)\Gamma(\alpha(k+r) + \beta) k!}$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha(k+r) + \beta) z^k (\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3) k!} + \sum_{k=0}^{\infty} \frac{(\alpha(k+r) + \beta)(\alpha(k+r) + \beta + 1) z^k (\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3) k!}$$

$$= \sum_{k=0}^{\infty} \left\{ \frac{z^k (\gamma)_k}{k!} \left(\frac{\alpha k}{\Gamma(\alpha(k+r) + \beta + 3)} + \frac{(\beta + r\alpha)}{\Gamma(\alpha(k+r) + \beta + 3)} \right) \right\}$$

$$+ \sum_{k=0}^{\infty} \left\{ \frac{z^k (\gamma)_k}{k!} \left(\frac{(k\alpha)^2}{\Gamma(\alpha(k+r) + \beta + 3)} \right. \right.$$

$$\left. \left. + \frac{k\alpha(2\beta + 2r\alpha + 1)}{\Gamma(\alpha(k+r) + \beta + 3)} + \frac{(\beta + r)^2 + (\beta + r)}{\Gamma(\alpha(k+r) + \beta + 3)} \right) \right\}. \tag{9}$$

We now say each summation on the right-hand side of equation (9) is as follows:

$$\frac{d^2}{dz^2} \left\{ z^2 E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z) \right\} = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)(\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3)} \cdot \frac{z^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{k^2 (\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3)} \cdot \frac{z^k}{k!}$$

$$+ 3 \sum_{k=0}^{\infty} \frac{k(\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3)} \cdot \frac{z^k}{k!} + 2 E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z) \tag{10}$$

or

$$\sum_{k=0}^{\infty} \frac{k^2 (\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3)} \cdot \frac{z^k}{k!}$$

$$= 4z \frac{d}{dz} E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z)$$

$$+ z^2 \frac{d^2}{dz^2} E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z) - 3 \sum_{k=0}^{\infty} \frac{k(\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3)} \cdot \frac{z^k}{k!}. \tag{11}$$

We find from equation (11) that

$$\sum_{k=0}^{\infty} \frac{k^2 (\gamma)_k}{\Gamma(\alpha(k+r) + \beta + 3)} \cdot \frac{z^k}{k!} = z \frac{d}{dz} E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z) + z^2 \frac{d^2}{dz^2} E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z). \tag{12}$$

Using equations (9), (11), and (12), we get

$$E_{\alpha, \beta + r\alpha + 1}^{\gamma}(z) - E_{\alpha, \beta + r\alpha + 2}^{\gamma}(z)$$

$$= (\beta + r\alpha)(\beta + r\alpha + 2) E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z) + \alpha^2 z^2 \frac{d^2}{dz^2} E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z)$$

$$+ \alpha \left\{ \alpha + 2 + 2(\beta + r\alpha) \right\} z \frac{d}{dz} E_{\alpha, \beta + r\alpha + 3}^{\gamma}(z).$$

Relation with the Wright functions is as follows:

$$E_{\alpha, \beta + r\alpha + 1}^{\gamma}(z) - E_{\alpha, \beta + r\alpha + 2}^{\gamma}(z)$$

$$= \sum_{k=0}^{\infty} (\beta + r\alpha)(\beta + r\alpha + 2)(\gamma)_k W(z; \alpha, \beta + r\alpha + 3)$$

$$+ \sum_{k=0}^{\infty} \alpha^2 z^2 (\gamma)_k \frac{d^2}{dz^2} W(z; \alpha, \beta + r\alpha + 3)$$

$$+ \sum_{k=0}^{\infty} \alpha \{\alpha + 2 + 2(\beta + r\alpha)\} (\gamma)_k z \frac{d}{dz} W(z; \alpha, \beta + r\alpha + 3). \qquad \square$$

Theorem 5 *If $\alpha, \beta, \gamma \in C$, $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma) > 0$, $z \in C$, $r \in N$, then*

$$z^r E_{\alpha, \beta + r\alpha}^{\gamma}(z) = E_{\alpha, \beta}^{\gamma}(z) - \sum_{k=0}^{r-1} \frac{(\gamma)_k}{\Gamma(k\alpha + \beta)} \frac{z^k}{k!}. \qquad (13)$$

Proof We have from (13)

$$\sum_{k=r}^{\infty} \frac{(\gamma)_k}{\Gamma(k\alpha + \beta)} \frac{z^k}{k!} = E_{\alpha, \beta}^{\gamma}(z) - \sum_{k=o}^{r-1} \frac{(\gamma)_k}{\Gamma(k\alpha + \beta)} \frac{z^k}{k!}.$$

For $r = 2, 3, 4, \ldots$, we obtain

$$z^2 E_{\alpha, \beta + 2\alpha}^{\gamma}(z) = E_{\alpha, \beta}^{\gamma}(z) - \frac{1}{\Gamma(\beta)} - \frac{(\gamma)_1 z}{\Gamma(\alpha + \beta)},$$

$$z^3 E_{\alpha, \beta + 3\alpha}^{\gamma}(z) = E_{\alpha, \beta}^{\gamma}(z) - \frac{1}{\Gamma(\beta)} - \frac{(\gamma)_1 z}{\Gamma(\alpha + \beta)} - \frac{(\gamma)_2 z^2}{\Gamma(2\alpha + \beta)},$$

$$z^4 E_{\alpha, \beta + 4\alpha}^{\gamma}(z) = E_{\alpha, \beta}^{\gamma}(z) - \frac{1}{\Gamma(\beta)} - \frac{(\gamma)_1 z}{\Gamma(\alpha + \beta)} - \frac{(\gamma)_2 z^2}{\Gamma(2\alpha + \beta)} - \frac{(\gamma)_3 z^3}{\Gamma(3\alpha + \beta)},$$

$$\vdots$$

$$\sum_{k=r}^{\infty} \frac{(\gamma)_k}{\Gamma(k\alpha + \beta)} \frac{z^k}{k!} = E_{\alpha, \beta}^{\gamma}(z) - \sum_{k=o}^{r-1} \frac{(\gamma)_k}{\Gamma(k\alpha + \beta)} \frac{z^k}{k!}.$$

Relation with the Wright functions is as follows:

$$\sum_{k=r}^{\infty} \frac{(\gamma)_k}{\Gamma(k\alpha + \beta)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} (\gamma)_k W(z; \alpha, \beta) - \sum_{k=0}^{r-1} (\gamma)_k W(z; \alpha, \beta). \qquad \square$$

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