Composite Structures 189 (2018) 127–136

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

Free vibration analysis of two directional functionally graded beams using a third order shear deformation theory

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ARTICLE INFO

Keywords:
Two directional functionally beam
Free vibration analysis
Third order shear deformation theory
Lagrange equations

ABSTRACT

This paper presents the free vibration behavior of two directional functionally graded beams subjected to various sets of boundary conditions which are simply supported (SS), clamped-simply supported (CS), clamped-clamped (CC) and clamped-free (CF) by employing a third order shear deformation theory. The material properties of the beam vary exponentially in both directions. In order to investigate the free vibration response, the equations of motion are derived by means of Lagrange equations. The axial, transverse deflections and rotation of the cross sections are expressed in polynomial forms including auxiliary functions which are used to satisfy the boundary conditions. The verification and convergence studies are performed by using computed results from a previous study which is based on the Timoshenko beam formulation. The results for extensive studies are provided to understand the influences of the different gradient indexes, various aspect ratios and boundary conditions on the free vibration responses of the two directional functionally graded beams.

1. Introduction

Functionally Graded Materials (FGMs) are a class of composites that have received great attention in many modern engineering applications such as military, aerospace, nuclear energy, biomedical, automotive, civil engineering and marine. Due to its lower transverse shear stresses, high resistance to temperature shocks and no interface problems through the layer interfaces, the researchers have extensively examined the static, vibration and buckling responses of these structures during the last decade [1–24]. However, the conventional FGMs (or 1D-FGM) with material properties which vary in one direction are not efficient to satisfy the technical requirements such as the temperature and stress distributions in different directions for aerospace craft and shuttles [25].

The mentioned deficiency of the conventional FGM can be eliminated by using a new type FGM with material properties varying in desired directions. The mechanical and thermal behaviors of two-directional FG structures have been investigated so far. The Element Free Galerkin Method is employed to optimize the natural frequencies of 2D two-directional functionally graded beams (FGBs) in [26]. The static and thermal deformations of bi-directional FGBS are investigated by employing the state-space based differential quadrature method obtain the semi-analytical elasticity solutions [27]. A symplectic elasticity solution for static and free vibration analyses of 2D-FGBs with the material properties varying exponentially in [28]. The fully coupled thermo-mechanical behavior of 2D-FGBs is studied using an isogeometric finite element model in [29]. Free and forced vibration of Timoshenko 2D-FGBS under the action of a moving load is investigated in [30]. The buckling of Timoshenko beams composed of 2D-FGM is studied in [31]. The static behavior of the 2D-FGBs by using various beam theories is presented in [32]. An analytical solution for the static deformations of the bi-directional functionally graded thick circular beams is developed based on a new shear deformation theory with a logarithmic function in the postulated expression for the circumferential displacement in [33]. The flexure behavior of the two directional FG sandwich beams by using a quasi-3D theory and the SSPH (Symmetric Smoothed Particle Hydrodynamics) method is studied in [34].

As it is seen from above discussions, most of the studies are related to the static, dynamic and buckling analysis of conventional functionally graded (1D-FG) beams. The studies related to two directional FGBs are still limited. As far as author aware, there is no reported work on the free vibration analysis of the two directional FBGs based on a third order shear deformation theory. Main differences of this paper from the related paper [30] are: the present theory does not require a shear correction factor which depends on the material and geometrical properties as well as boundary conditions [35] of the 2D-FGBs and satisfies the zero traction boundary condition of the top and bottom surfaces of the beam, the second and third natural frequencies of the 2D-FGBs for various end conditions, aspect ratios and gradient indexes are presented within this paper and it is clear that the accuracy of the

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https://doi.org/10.1016/j.compstruct.2018.01.060
Received 24 August 2017; Received in revised form 1 December 2017; Accepted 19 January 2018
Available online 20 February 2018

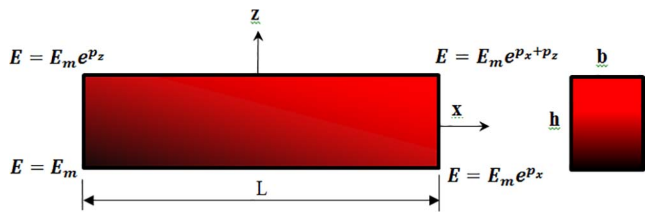

Fig. 1. Geometry and coordinate of a two-directional FGB.

Table 1
Kinematic boundary conditions used for the numerical computations.

BC	$x = -L/2$	$x = L/2$
S-S	$u = 0, w = 0$	$w = 0$
C-S	$u = 0, w = 0, \phi = 0,\ w' = 0$	$w = 0$
C-C	$u = 0, w = 0, \phi = 0,\ w' = 0$	$u = 0, w = 0, \phi = 0,\ w' = 0$
C-F	$u = 0, w = 0, \phi = 0,\ w' = 0$	

Table 2
Boundary exponents for various boundary conditions.

BC	Left end			Right end		
	p_u	p_w	p_ϕ	q_u	q_w	q_ϕ
SS	1	1	0	0	1	0
CS	1	2	1	0	1	0
CC	1	2	1	1	2	1
CF	1	2	1	0	0	0

Timoshenko beam theory decreases as the mode number increases [7]. As a result, a third order shear deformation theory is necessary to have a better prediction of vibration responses of the two directional FGBs. The main novelty of this paper is that the free vibration behavior of the two directional FGBs is analyzed based on a third order shear deformation theory by using the Lagrange equations with four different end conditions for the first time.

2. Theory and formulation

2.1. Homogenization of material properties

A two-directional functionally graded beam of length L, width b and thickness h is shown in Fig. 1. The material properties of the beam vary exponentially not only in the z-direction (thickness direction) but also in the x-direction (along the length of the beam). The Young's modulus E, shear modulus G, Poissons's ratio ν and mass density ρ vary according to the following expressions [30]

$$E(x,z) = E_m e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)+p_z\left(\frac{z}{h}+\frac{1}{2}\right)}$$
$$G(x,z) = \frac{E(x,z)}{2(1+\nu_m)}$$
$$\rho(x,z) = \rho_m e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)+p_z\left(\frac{z}{h}+\frac{1}{2}\right)} \tag{1}$$

where E_m,ν_m and ρ_m are the material properties of the reference material value at the point $(-L/2,\ -h/2)$, p_x and p_z are the gradient indexes which determine the material properties through the thickness (h) and length of the beam (L), respectively. When the p_x and p_z are set to zero then the beam becomes homogeneous.

2.2. Kinetic, strain and stress relations

The following displacement field is given for the third order shear deformation theory (Reddy Beam Theory (RBT))

$$U(x,z,t) = u(x,t) + z\phi(x,t) - \alpha z^3\left(\phi(x,t) + \frac{\partial w(x,t)}{\partial x}\right)$$
$$W(x,z,t) = w(x,t) \tag{2}$$

Here u and w are the axial and transverse displacements of any point on the neutral axis ,ϕ is the rotation of the cross sections, $\alpha = 4/(3h^2)$. By using the Eq. (2), the strain-displacement relations of the RBT are given by

$$\varepsilon_{xx} = \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} - \alpha z^3\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)$$
$$\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \phi + \frac{\partial w}{\partial x} - \beta z^2\left(\phi + \frac{\partial w}{\partial x}\right) \tag{3}$$

Table 3
Verification and convergence studies, dimensionless fundamental frequencies (λ_1) of SS two directional FGBs with respect to gradient index and aspect ratio change.

Beam theory		p_x	L/h = 5 p_z						L/h = 20 p_z					
			0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
Timoshenko [30]		0	2.6767	2.6748	2.6669	2.6533	2.6337	2.6103	2.8369	2.8349	2.8251	2.8115	2.7919	2.7685
RBT	2 terms		2.9433	2.9402	2.9310	2.9157	2.8947	2.8682	3.1468	3.1436	3.1342	3.1187	3.0972	3.0700
	4 terms		2.6780	2.6753	2.6672	2.6539	2.6354	2.6121	2.8380	2.8351	2.8267	2.8127	2.7933	2.7689
	6 terms		2.6773	2.6746	2.6665	2.6532	2.6347	2.6114	2.8371	2.8343	2.8258	2.8118	2.7925	2.7681
	8 terms		2.6773	2.6746	2.6665	2.6532	2.6347	2.6114	2.8371	2.8343	2.8258	2.8118	2.7925	2.7681
	10 terms		2.6773	2.6746	2.6665	2.6532	2.6347	2.6114	2.8371	2.8343	2.8258	2.8118	2.7925	2.7681
	12 terms		2.6773	2.6746	2.6665	2.6532	2.6347	2.6114	2.8371	2.8343	2.8258	2.8118	2.7925	2.7681
Timoshenko [30]		0.4	2.6728	2.6689	2.6611	2.6474	2.6279	2.6044	2.8330	2.8291	2.8212	2.8076	2.7880	2.7626
RBT	2 terms		2.9448	2.9417	2.9325	2.9172	2.8961	2.8695	3.1525	3.1493	3.1399	3.1243	3.1027	3.0755
	4 terms		2.6740	2.6740	2.6713	2.6632	2.6497	2.6312	2.8350	2.8322	2.8237	2.8097	2.7904	2.7660
	6 terms		2.6722	2.6694	2.6613	2.6479	2.6293	2.6059	2.8326	2.8298	2.8213	2.8073	2.7880	2.7636
	8 terms		2.6722	2.6694	2.6613	2.6479	2.6293	2.6059	2.8326	2.8298	2.8213	2.8073	2.7880	2.7636
	10 terms		2.6722	2.6694	2.6613	2.6479	2.6293	2.6059	2.8326	2.8298	2.8213	2.8073	2.7880	2.7636
	12 terms		2.6722	2.6694	2.6613	2.6479	2.6293	2.6059	2.8326	2.8298	2.8213	2.8073	2.7880	2.7636
Timoshenko [30]		1	2.6455	2.6416	2.6337	2.6201	2.6005	2.5771	2.8095	2.8056	2.7978	2.7841	2.7646	2.7412
RBT	2 terms		2.9522	2.9491	2.9398	2.9245	2.9033	2.8766	3.1820	3.1788	3.1693	3.1536	3.1318	3.1044
	4 terms		2.6527	2.6500	2.6418	2.6283	2.6096	2.5860	2.8193	2.8165	2.8080	2.7941	2.7749	2.7505
	6 terms		2.6452	2.6425	2.6343	2.6208	2.6022	2.5788	2.8089	2.8061	2.7977	2.7839	2.7647	2.7405
	8 terms		2.6452	2.6425	2.6343	2.6208	2.6022	2.5788	2.8089	2.8061	2.7977	2.7839	2.7647	2.7405
	10 terms		2.6452	2.6425	2.6343	2.6208	2.6022	2.5788	2.8089	2.8061	2.7977	2.7839	2.7647	2.7405
	12 terms		2.6452	2.6425	2.6343	2.6208	2.6022	2.5788	2.8089	2.8061	2.7977	2.7839	2.7647	2.7405

Table 4
Verification and convergence studies, dimensionless fundamental frequencies (λ_1) of CF two directional FGBs with respect to gradient index and aspect ratio change.

Beam theory		p_x	L/h = 5 p_z						L/h = 20 p_z					
			0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
Timoshenko [30]		0	0.9844	0.9832	0.9796	0.9735	0.9661	0.9576	1.0126	1.0126	1.0087	1.0029	0.9970	0.9873
RBT	2 terms		0.9925	0.9916	0.9887	0.9840	0.9775	0.9693	1.0180	1.0170	1.0140	1.0090	1.0021	0.9933
	4 terms		0.9858	0.9849	0.9821	0.9774	0.9710	0.9628	1.0131	1.0121	1.0090	1.0041	0.9972	0.9885
	6 terms		0.9852	0.9842	0.9814	0.9767	0.9703	0.9622	1.0130	1.0120	1.0090	1.0040	0.9971	0.9884
	8 terms		0.9849	0.9840	0.9811	0.9765	0.9701	0.9619	1.0130	1.0120	1.0090	1.0040	0.9971	0.9884
	10 terms		0.9848	0.9839	0.9810	0.9764	0.9700	0.9618	1.0130	1.0120	1.0090	1.0040	0.9971	0.9884
	12 terms		0.9848	0.9839	0.9810	0.9764	0.9700	0.9618	1.0130	1.0120	1.0090	1.0040	0.9971	0.9884
Timoshenko [30]		0.4	0.8709	0.8697	0.8673	0.8624	0.8564	0.8486	0.8955	0.8935	0.8916	0.8876	0.8798	0.8721
RBT	2 terms		0.8844	0.8835	0.8810	0.8768	0.8710	0.8636	0.9049	0.9040	0.9013	0.8969	0.8907	0.8830
	4 terms		0.8730	0.8722	0.8697	0.8655	0.8598	0.8525	0.8951	0.8942	0.8915	0.8871	0.8810	0.8733
	6 terms		0.8723	0.8715	0.8690	0.8649	0.8592	0.8519	0.8950	0.8941	0.8915	0.8871	0.8810	0.8733
	8 terms		0.8721	0.8713	0.8688	0.8649	0.8589	0.8517	0.8950	0.8941	0.8914	0.8870	0.8810	0.8733
	10 terms		0.8720	0.8712	0.8687	0.8645	0.8588	0.8516	0.8950	0.8941	0.8914	0.8870	0.8810	0.8733
	12 terms		0.8720	0.8712	0.8687	0.8645	0.8588	0.8516	0.8950	0.8941	0.8914	0.8870	0.8810	0.8733
Timoshenko [30]		1	0.7216	0.7216	0.7177	0.7138	0.7099	0.7021	0.7392	0.7392	0.7373	0.7333	0.7275	0.7216
RBT	2 terms		0.7443	0.7436	0.7415	0.7379	0.7330	0.7268	0.7593	0.7585	0.7563	0.7525	0.7474	0.7409
	4 terms		0.7234	0.7227	0.7206	0.7172	0.7124	0.7064	0.7394	0.7387	0.7365	0.7329	0.7278	0.7215
	6 terms		0.7228	0.7221	0.7200	0.7166	0.7118	0.7058	0.7394	0.7387	0.7365	0.7328	0.7278	0.7214
	8 terms		0.7226	0.7219	0.7198	0.7164	0.7116	0.7056	0.7394	0.7386	0.7364	0.7328	0.7278	0.7214
	10 terms		0.7225	0.7218	0.7197	0.7163	0.7115	0.7055	0.7394	0.7386	0.7364	0.7328	0.7278	0.7214
	12 terms		0.7225	0.7218	0.7197	0.7163	0.7115	0.7055	0.7394	0.7386	0.7364	0.7328	0.7278	0.7214

Table 5
Verification and convergence studies, dimensionless fundamental frequencies (λ_1) of CC two directional FGBs with respect to gradient index and aspect ratio change.

Beam theory		p_x	L/h = 5 p_z						L/h = 20 p_z					
			0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
Timoshenko [30]		0	5.1943	5.1904	5.1806	5.1630	5.1396	5.1083	6.3486	6.3427	6.3251	6.2939	6.2529	6.2001
RBT	2 terms		5.3585	5.3543	5.3417	5.3210	5.2922	5.2556	6.3865	6.3803	6.3616	6.3307	6.2881	6.2341
	4 terms		5.2767	5.2727	5.2605	5.2403	5.2124	5.1770	6.3579	6.3517	6.3331	6.3024	6.2600	6.2064
	6 terms		5.2486	5.2446	5.2326	5.2127	5.1851	5.1501	6.3544	6.3482	6.3296	6.2990	6.2566	6.2030
	8 terms		5.2375	5.2335	5.2215	5.2017	5.1743	5.1395	6.3527	6.3465	6.3280	6.2974	6.2550	6.2015
	10 terms		5.2330	5.2290	5.2171	5.1974	5.1701	5.1354	6.3518	6.3456	6.3271	6.2695	6.2542	6.2007
	12 terms		5.2314	5.2274	5.2155	5.1958	5.1685	5.1339	6.3513	6.3451	6.3266	6.2690	6.2537	6.2002
Timoshenko [30]		0.4	5.1982	5.1943	5.1845	5.1669	5.1435	5.1123	6.3564	6.3486	6.3310	6.2998	6.2587	6.2060
RBT	2 terms		5.3741	5.3699	5.3573	5.3365	5.3077	5.2711	6.4133	6.4071	6.3883	6.3573	6.3145	6.2603
	4 terms		5.2814	5.2773	5.2652	5.2450	5.2170	5.1816	6.3642	6.3580	6.3394	6.3087	6.2662	6.2125
	6 terms		5.2530	5.2489	5.2369	5.2170	5.1894	5.1544	6.3605	6.3543	6.3357	6.3051	6.2627	6.2090
	8 terms		5.2418	5.2377	5.2256	5.2059	5.1782	5.1431	6.3589	6.3526	6.3341	6.3034	6.2611	6.2075
	10 terms		5.2373	5.2333	5.2213	5.2016	5.1743	5.1396	6.3580	6.3518	6.3332	6.3026	6.2602	6.2067
	12 terms		5.2356	5.2316	5.2197	5.2000	5.1727	5.1381	6.3575	6.3513	6.3327	6.3021	6.2597	6.2062
Timoshenko [30]		1	5.2197	5.2177	5.2060	5.1884	5.1650	5.1337	6.3876	6.3818	6.3623	6.3330	6.2900	6.2373
RBT	2 terms		5.4549	5.4507	5.4380	5.4170	5.3879	5.3510	6.5533	6.5469	6.5277	6.4961	6.4523	6.3970
	4 terms		5.3062	5.3021	5.2898	5.2696	5.2415	5.2059	6.3973	6.3910	6.3723	6.3415	6.2988	6.2449
	6 terms		5.2761	5.2720	5.2599	5.2400	5.2123	5.1771	6.3930	6.3867	6.3681	6.3372	6.2946	6.2408
	8 terms		5.2644	5.2604	5.2484	5.2285	5.2010	5.1661	6.3913	6.3850	6.3664	6.3356	6.2930	6.2392
	10 terms		5.2597	5.2557	5.2437	5.2239	5.1965	5.1617	6.3903	6.3841	6.3655	6.3347	6.2921	6.2383
	12 terms		5.2580	5.2540	5.2421	5.2223	5.1949	5.1601	6.3896	6.3832	6.3646	6.3338	6.2912	6.2374

where $\beta = 3\alpha = 4/(h^2)$.

The stress-strain relationship of a two directional functionally graded beam in the material coordinate axes is given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E(x,z) & 0 \\ 0 & G(x,z) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where $(\sigma_{xx}, \sigma_{xz})$ are the stresses and $(\varepsilon_{xx}, \gamma_{xz})$ are the strains with respect to the axes.

2.3. Formulation of free vibration

The strain energy of the beam including the energy associated with the shearing strain can be written as

$$U = \frac{1}{2} \int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz}) dV \tag{5}$$

where V is the volume of the beam. By substituting Eqs. (3) and (4) into Eq. (5), the strain energy can be obtained as the form of

Table 6
The first three dimensionless frequencies of SS two directional FGBs with respect to gradient index and aspect ratio change.

λ	p_x	L/h = 5						L/h = 20					
		p_z						p_z					
		0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
λ_1	0	2.6773	2.6746	2.6665	2.6532	2.6347	2.6114	2.8371	2.8343	2.8258	2.8118	2.7925	2.7681
	0.2	2.6760	2.6733	2.6652	2.6518	2.6333	2.6099	2.8360	2.8332	2.8247	2.8107	2.7914	2.7669
	0.4	2.6722	2.6694	2.6613	2.6479	2.6293	2.6059	2.8326	2.8298	2.8213	2.8073	2.7880	2.7636
	0.6	2.6657	2.6630	2.6548	2.6414	2.6228	2.5994	2.8270	2.8241	2.8157	2.8017	2.7824	2.7581
	0.8	2.6567	2.6540	2.6458	2.6324	2.6138	2.5903	2.8191	2.8162	2.8078	2.7939	2.7747	2.7503
	1	2.6452	2.6424	2.6343	2.6208	2.6022	2.5788	2.8089	2.8061	2.7977	2.7838	2.7647	2.7404
λ_2	0	7.8540	7.8478	7.8294	7.7988	7.7561	7.7015	11.2095	11.1982	11.1646	11.1091	11.0324	10.9354
	0.2	7.5387	7.5333	7.5172	7.4904	7.4528	7.4046	11.2105	11.1992	11.1655	11.1098	11.0329	10.9357
	0.4	7.2297	7.2248	7.2104	7.1863	7.1525	7.1091	11.2134	11.2021	11.1683	11.1125	11.0354	10.9379
	0.6	6.9270	6.9226	6.9094	6.8874	6.8566	6.8170	11.2184	11.2070	11.1731	11.1171	11.0397	10.9419
	0.8	6.6308	6.6268	6.6146	6.5943	6.5660	6.5295	11.2254	11.2140	11.1799	11.1236	11.0459	10.9477
	1	6.3414	6.3376	6.3263	6.3075	6.2812	6.2473	11.2344	11.2229	11.1887	11.1321	11.0540	10.9553
λ_3	0	9.2909	9.2876	9.2774	9.2606	9.2374	9.2082	24.7346	24.7081	24.6289	24.4984	24.3188	24.0930
	0.2	9.2914	9.2874	9.2752	9.2551	9.2274	9.1927	24.7365	24.7091	24.6276	24.4935	24.3094	24.0784
	0.4	9.2927	9.2882	9.2747	9.2524	9.2216	9.1829	24.7420	24.7133	24.6281	24.4885	24.2976	24.0592
	0.6	9.2950	9.2902	9.2756	9.2517	9.2186	9.1770	24.7511	24.7202	24.6285	24.4794	24.2774	24.0279
	0.8	9.2982	9.2931	9.2778	9.2527	9.2179	9.1741	24.7638	24.7276	24.6217	24.4537	24.2322	23.9650
	1	9.3023	9.2970	9.2812	9.2551	9.2190	9.1735	24.7801	24.7198	24.5660	24.3539	24.1002	23.8118

Table 7
The first three dimensionless frequencies of CS two directional FGBs with respect to gradient index and aspect ratio change.

λ	p_x	L/h = 5						L/h = 20					
		p_z						p_z					
		0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
λ_1	0	3.8916	3.8883	3.8786	3.8626	3.8404	3.8122	4.4075	4.4031	4.3902	4.3687	4.3391	4.3017
	0.2	3.8371	3.8339	3.8244	3.8086	3.7867	3.7589	4.3472	4.3429	4.3301	4.3090	4.2798	4.2428
	0.4	3.7825	3.7793	3.7699	3.7544	3.7328	3.7055	4.2874	4.2831	4.2705	4.2497	4.2209	4.1844
	0.6	3.7276	3.7245	3.7152	3.6999	3.6787	3.6518	4.2278	4.2237	4.2112	4.1907	4.1622	4.1263
	0.8	3.6722	3.6692	3.6601	3.6450	3.6242	3.5977	4.1685	4.1644	4.1521	4.1318	4.1038	4.0684
	1	3.6165	3.6135	3.6045	3.5897	3.5692	3.5432	4.1092	4.1051	4.0931	4.0731	4.0455	4.0106
λ_2	0	7.8540	7.8540	7.8540	7.8540	7.8540	7.8540	14.0499	14.0363	13.9959	13.9292	13.8369	13.7201
	0.2	7.5387	7.5387	7.5387	7.5387	7.5387	7.5387	13.9955	13.9820	13.9418	13.8753	13.7833	13.6670
	0.4	7.2297	7.2297	7.2297	7.2297	7.2297	7.2297	13.9441	13.9307	13.8906	13.8244	13.7328	13.6169
	0.6	6.9270	6.9270	6.9270	6.9270	6.9270	6.9270	13.8958	13.8824	13.8425	13.7765	13.6852	13.5697
	0.8	6.6308	6.6308	6.6308	6.6308	6.6308	6.6308	13.8505	13.8372	13.7973	13.7315	13.6405	13.5255
	1	6.3414	6.3414	6.3414	6.3414	6.3414	6.3414	13.8082	13.7949	13.7552	13.6896	13.5989	13.4842
λ_3	0	10.7346	10.7272	10.7051	10.6686	10.6178	10.5535	28.6433	28.6167	28.5369	28.4052	28.2229	27.9922
	0.2	10.6954	10.6881	10.6661	10.6296	10.5791	10.5149	28.5909	28.5642	28.4847	28.3531	28.1712	27.9409
	0.4	10.6580	10.6507	10.6287	10.5924	10.5421	10.4781	28.5420	28.5154	28.4360	28.3047	28.1230	27.8932
	0.6	10.6223	10.6150	10.5931	10.5569	10.5067	10.4430	27.7079	27.7079	27.7079	27.7079	27.7079	27.7079
	0.8	10.5884	10.5811	10.5593	10.5232	10.4731	10.4096	26.5233	26.5233	26.5233	26.5233	26.5233	26.5233
	1	10.5562	10.5489	10.5272	10.4912	10.4413	10.3780	25.3656	25.3656	25.3656	25.3656	25.3656	25.3656

$$U = \frac{1}{2}\int_V \left[E(x,z)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + (z^2-2\alpha z^4+\alpha^2 z^6)\left(\frac{\partial \phi}{\partial x}\right)^2 + \alpha^2 z^6\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2(z-\alpha z^3)\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} - 2\alpha z^3\frac{\partial u}{\partial x}\frac{d^2 w}{dx^2} + 2(\alpha^2 z^6-\alpha z^4)\frac{\partial \phi}{\partial x}\frac{d^2 w}{dx^2}\right\} + G(x,z)\left\{(1-2\beta z^2+\beta^2 z^4)\phi^2 + (1-2\beta z^2+\beta^2 z^4)\left(\frac{\partial w}{\partial x}\right)^2 + 2(1-2\beta z^2+\beta^2 z^4)\phi\frac{\partial w}{\partial x}\right\}\right] dV \tag{6}$$

The stiffness coefficients can be introduced as follows

$$(A,B,D,C,F,H) = b\int_{-h/2}^{+h/2} E_m e^{p_z\left(\frac{z}{h}+\frac{1}{2}\right)}(1,z,z^2,z^3,z^4,z^6)dz \tag{7}$$

$$(A_s,D_s,F_s) = b\int_{-h/2}^{+h/2} G_m e^{p_z\left(\frac{z}{h}+\frac{1}{2}\right)}(1,z^2,z^4)dz \tag{8}$$

By using Eqs. (6)–(8), the strain energy can be rewritten as

$$U = \frac{1}{2}\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\left[A\left(\frac{\partial u}{\partial x}\right)^2 + (D+\alpha^2 H-2\alpha F)\left(\frac{\partial \phi}{\partial x}\right)^2 + \alpha^2 H\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2(B-\alpha C)\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} - 2\alpha C\frac{\partial u}{\partial x}\frac{d^2 w}{dx^2} + 2(\alpha^2 H-\alpha F)\frac{\partial \phi}{\partial x}\frac{d^2 w}{dx^2} + (A_s+\beta^2 F_s-2\beta D_s)\left(\phi^2 + \left(\frac{\partial w}{\partial x}\right)^2 + 2\phi\frac{\partial w}{\partial x}\right)\right]dx \tag{9}$$

With the similar procedure, the kinetic energy of the beam can be obtained as

Table 8
The first three dimensionless frequencies of CC two directional FGBs with respect to gradient index and aspect ratio change.

λ	p_x	L/h = 5 p_z						L/h = 20 p_z					
		0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
λ_1	0	5.2330	5.2290	5.2171	5.1974	5.1701	5.1354	6.3518	6.3456	6.3271	6.2965	6.2542	6.2007
	0.2	5.2341	5.2301	5.2182	5.1984	5.1711	5.1364	6.3534	6.3472	6.3286	6.2980	6.2557	6.2022
	0.4	5.2373	5.2333	5.2213	5.2016	5.1743	5.1396	6.3580	6.3518	6.3332	6.3026	6.2602	6.2067
	0.6	5.2426	5.2386	5.2266	5.2069	5.1795	5.1448	6.3656	6.3594	6.3409	6.3102	6.2678	6.2142
	0.8	5.2500	5.2460	5.2341	5.2143	5.1869	5.1522	6.3764	6.3702	6.3516	6.3209	6.2784	6.2247
	1	5.2597	5.2557	5.2438	5.2239	5.1965	5.1617	6.3903	6.3841	6.3655	6.3347	6.2921	6.2383
λ_2	0	12.0849	12.0775	12.0552	12.0182	11.9669	11.9018	17.1470	17.1308	17.0822	17.0019	16.8909	16.7505
	0.2	12.0862	12.0788	12.0565	12.0195	11.9682	11.9031	17.1490	17.1328	17.0842	17.0039	16.8929	16.7524
	0.4	12.0902	12.0828	12.0604	12.0235	11.9722	11.9070	17.1551	17.1388	17.0902	17.0099	16.8989	16.7584
	0.6	12.0969	12.0894	12.0671	12.0301	11.9787	11.9135	17.1652	17.1489	17.1003	17.0199	16.9088	16.7682
	0.8	12.1061	12.0987	12.0763	12.0393	11.9879	11.9226	17.1793	17.1630	17.1144	17.0340	16.9227	16.7820
	1	12.1181	12.1106	12.0882	12.0512	11.9997	11.9344	17.1975	17.1812	17.1325	17.0520	16.9406	16.7998
λ_3	0	15.7080	15.7080	15.7080	15.7080	15.7080	15.7080	32.7454	32.7154	32.6260	32.4781	32.2734	32.0144
	0.2	15.7159	15.7159	15.7159	15.7159	15.7159	15.7159	32.7475	32.7176	32.6281	32.4802	32.2755	32.0164
	0.4	15.7398	15.7398	15.7398	15.7398	15.7398	15.7398	32.7538	32.7239	32.6344	32.4865	32.2817	32.0227
	0.6	15.7794	15.7794	15.7794	15.7794	15.7794	15.7794	32.7644	32.7345	32.6450	32.4970	32.2922	32.0330
	0.8	15.8348	15.8348	15.8348	15.8348	15.8348	15.8348	32.7792	32.7493	32.6597	32.5117	32.3068	32.0475
	1	15.9057	15.9057	15.9057	15.9057	15.9057	15.9057	32.7983	32.7683	32.6787	32.5306	32.3256	32.0662

Table 9
The first three dimensionless frequencies of CF two directional FGBs with respect to gradient index and aspect ratio change.

λ	p_x	L/h = 5 p_z						L/h = 20 p_z					
		0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
λ_1	0	0.9848	0.9839	0.9810	0.9764	0.9700	0.9618	1.0130	1.0120	1.0090	1.0040	0.9971	0.9884
	0.2	0.9270	0.9261	0.9235	0.9191	0.9131	0.9054	0.9525	0.9515	0.9487	0.9440	0.9376	0.9294
	0.4	0.8720	0.8712	0.8687	0.8645	0.8588	0.8516	0.8950	0.8941	0.8914	0.8870	0.8809	0.8733
	0.6	0.8196	0.8188	0.8165	0.8126	0.8072	0.8004	0.8404	0.8395	0.8370	0.8329	0.8272	0.8200
	0.8	0.7698	0.7691	0.7668	0.7632	0.7581	0.7517	0.7885	0.7877	0.7854	0.7815	0.7761	0.7694
	1	0.7225	0.7218	0.7197	0.7163	0.7115	0.7055	0.7394	0.7386	0.7364	0.7328	0.7278	0.7214
λ_2	0	5.3263	5.3222	5.3098	5.2894	5.2611	5.2252	6.2758	6.2697	6.2513	6.2210	6.1790	6.1260
	0.2	5.2236	5.2195	5.2075	5.1875	5.1597	5.1246	6.1584	6.1524	6.1344	6.1046	6.0635	6.0114
	0.4	5.1217	5.1178	5.1060	5.0864	5.0593	5.0248	6.0428	6.0369	6.0192	5.9900	5.9496	5.8986
	0.6	5.0207	5.0168	5.0052	4.9861	4.9596	4.9259	5.9289	5.9231	5.9057	5.8771	5.8375	5.7874
	0.8	4.9204	4.9166	4.9053	4.8866	4.8606	4.8277	5.8165	5.8108	5.7938	5.7657	5.7269	5.6777
	1	4.8207	4.8171	4.8060	4.7877	4.7624	4.7302	5.7057	5.7001	5.6834	5.6558	5.6177	5.5696
λ_3	0	7.8540	7.8540	7.8540	7.8540	7.8540	7.8540	17.2627	17.2462	17.1970	17.1157	17.0032	16.8610
	0.2	7.5387	7.5387	7.5387	7.5387	7.5387	7.5387	17.1531	17.1368	17.0879	17.0071	16.8953	16.7539
	0.4	7.2297	7.2297	7.2297	7.2297	7.2297	7.2297	17.0472	17.0310	16.9824	16.9021	16.7910	16.6505
	0.6	6.9270	6.9270	6.9270	6.9270	6.9270	6.9270	16.9450	16.9288	16.8805	16.8007	16.6903	16.5507
	0.8	6.6308	6.6308	6.6308	6.6308	6.6308	6.6308	16.8464	16.8304	16.7824	16.7030	16.5932	16.4544
	1	6.3414	6.3414	6.3414	6.3414	6.3414	6.3414	16.7516	16.7356	16.6879	16.6090	16.4999	16.3618

$$K = \frac{1}{2}\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}[I_0\left(\frac{\partial u}{\partial t}\right)^2 + (I_2 + \alpha^2K_1 - 2\alpha J_2)\left(\frac{\partial \phi}{\partial t}\right)^2 + I_0\left(\frac{\partial w}{\partial t}\right)^2 + 2(I_1-\alpha J_1)\frac{\partial u}{\partial t}\frac{\partial \phi}{\partial t} + 2(\alpha^2K_1-\alpha J_2)\frac{\partial \phi}{\partial t}\frac{\partial^2 w}{\partial x\partial t} - 2\alpha J_1\frac{\partial u}{\partial t}\frac{\partial^2 w}{\partial x\partial t} + \alpha^2K_1\left(\frac{\partial^2 w}{\partial x\partial t}\right)^2]dx \quad (10)$$

Here t is the time, and the inertial coefficients can be presented by

$$(I_0,I_1,I_2,J_1,J_2,K_1) = b\int_{-h/2}^{+h/2}\rho_m e^{p_z\left(\frac{z}{h}+\frac{1}{2}\right)}(1,z,z^2,z^3,z^4,z^6)dz \quad (11)$$

It is known that Hamilton's principle can be expressed as Lagrange equations when the functions of infinite dimensions can be expressed in terms of generalized coordinates. Therefore, the displacement functions $u(x,t)$, $w(x,t)$ and the rotation function $\phi(x,t)$ are presented by the following polynomial series which are satisfy the kinematic boundary conditions given in Table 1.

$$u(x,t) = \sum_{j=1}^{m} A_j\theta_j(x)e^{i\omega t},\quad \theta_j(x) = \left(x+\frac{L}{2}\right)^{p_u}\left(x-\frac{L}{2}\right)^{q_u}x^{m-1} \quad (12a)$$

$$w(x,t) = \sum_{j=1}^{m} B_j\varphi_j(x)e^{i\omega t},\quad \varphi_j(x) = \left(x+\frac{L}{2}\right)^{p_w}\left(x-\frac{L}{2}\right)^{q_w}x^{m-1} \quad (12b)$$

$$\phi(x,t) = \sum_{j=1}^{m} C_j\psi_j(x)e^{i\omega t},\quad \psi_j(x) = \left(x+\frac{L}{2}\right)^{p_\phi}\left(x-\frac{L}{2}\right)^{q_\phi}x^{m-1} \quad (12c)$$

where A_j, B_j and C_j are unknown coefficients to be determined, ω is the natural frequency of the beam, $i = \sqrt{-1}$ is the complex number, $\theta_j(x)$, $\varphi_j(x)$ and $\psi_j(x)$ are the shape functions which are proposed for the boundary conditions (BC) to be studied within this paper, p_ξ and q_ξ ($\xi = u,w,\phi$) are the boundary exponents given in Table 2, and assigned regarding to the studied boundary condtion. It has to be mentioned that the shape functions which do not satisfy the boundary conditions may cause slow convergence rates and numerical instabilities.

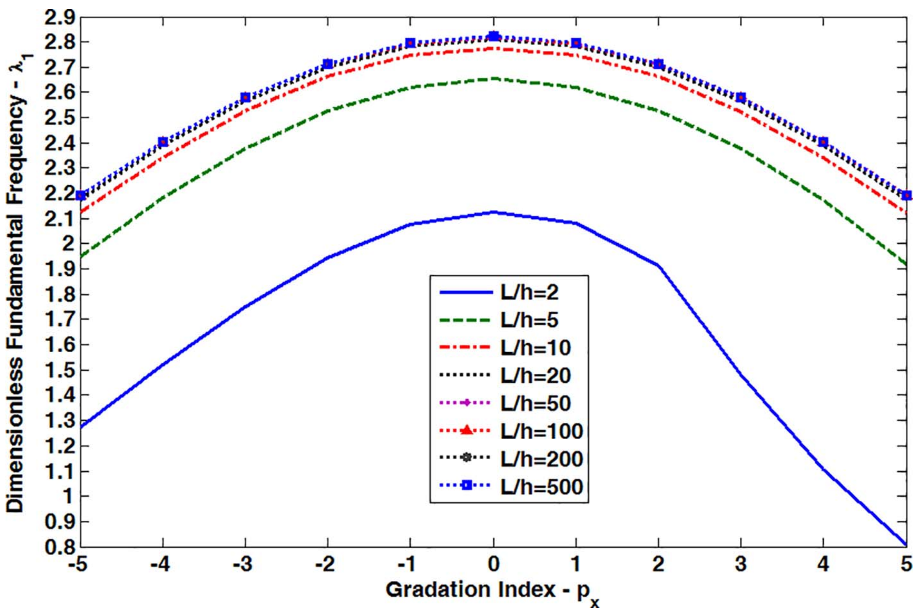

Fig. 2. Variation of the fundamental frequencies with respect to gradient index (p_x) and aspect ratio for two directional FG SS beams. (p_z = 1).

The governing equations of motion can be obtained by substituting Eq. (12) into Eqs. (9) and (10) and then using Lagrange equations

$$\frac{\partial U}{\partial q_j}+\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right)=0 \tag{13}$$

with q_j representing the values of (A_j, B_j and C_j), that leads to

$$\left(\begin{bmatrix}[K_{11}] & [K_{12}] & [K_{13}]\\ [K_{12}]^T & [K_{22}] & [K_{23}]\\ [K_{13}]^T & [K_{23}]^T & [K_{33}]\end{bmatrix}-\omega^2\begin{bmatrix}[M_{11}] & [M_{12}] & [M_{13}]\\ [M_{12}]^T & [M_{22}] & [M_{23}]\\ [M_{13}]^T & [M_{23}]^T & [M_{33}]\end{bmatrix}\right)\begin{Bmatrix}\{A\}\\ \{B\}\\ \{C\}\end{Bmatrix}=\begin{Bmatrix}\{0\}\\ \{0\}\\ \{0\}\end{Bmatrix} \tag{14}$$

where $[K_{kl}]$ are the stiffness matrices and $[M_{kl}]$ are the mass matrices. It should be noted that the stiffness and mass matrices are symmetric and in size mxm. The components of the stiffness and mass matrices are given by:

$$\begin{aligned}
K_{11}(i,j) &= A\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\theta_{i,x}\theta_{j,x}dx\\
K_{12}(i,j) &= -\alpha C\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\theta_{i,x}\varphi_{j,xx}dx\\
K_{13}(i,j) &= (B-\alpha C)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\theta_{i,x}\psi_{j,x}dx\\
K_{22}(i,j) &= \alpha^2 H\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i,xx}\varphi_{j,xx}dx+(A_s+\beta^2F_s\\
&\quad -2\beta D_s)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i,x}\varphi_{j,x}dx\\
K_{23}(i,j) &= (\alpha^2H-\alpha F)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i,xx}\psi_{j,x}dx+(A_s+\beta^2F_s\\
&\quad -2\beta D_s)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i,x}\psi_{j}dx\\
K_{33}(i,j) &= (D+\alpha^2H-2\alpha F)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\psi_{i,x}\psi_{j,x}dx\\
&\quad +(A_s+\beta^2F_s-2\beta D_s)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\psi_{i}\psi_{j}dx\\
M_{11}(i,j) &= I_0\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\theta_{i}\theta_{j}dx, M_{12}(i,j)=-\alpha J_1\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\theta_{i}\varphi_{j,x}dx,\\
M_{13}(i,j) &= (I_1-\alpha J_1)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\theta_{i}\psi_{j}dx\\
M_{22}(i,j) &= I_0\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i}\varphi_{j}dx+\alpha^2K_1\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i,x}\varphi_{j,x}dx\\
M_{23}(i,j) &= (\alpha^2K_1-\alpha J_2)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\varphi_{i,x}\psi_{j}dx\\
M_{33}(i,j) &= (I_2+\alpha^2K_1-2\alpha J_2)\int_{-L/2}^{L/2} e^{p_x\left(\frac{x}{L}+\frac{1}{2}\right)}\psi_{i}\psi_{j}dx \quad i,j=1,2,3,....,m
\end{aligned} \tag{15}$$

3. Numerical results

This section is dedicated to discuss the effects of gradient indexes (or material composition), aspect ratios and boundary conditions on the free vibration behavior of the two directional FGBs. The material and geometrical properties of the beam are defined as

$$E_m = 210GPa,\quad \nu_m = 0.3,\quad \rho_m = 7850\frac{kg}{m^3},\quad h = 1m,\quad b = 0.5m$$

The length of the beam is varied to examine the effect of the shear deformation. Four different boundary conditions, namely simply supported (SS), clamped-simply supported (CS), clamped–clamped (CC) and clamped-free (CF) are considered. The following dimensionless frequency (λ) parameter is used for the representation of the results;

$$\lambda=\frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}} \tag{16}$$

3.1. Verification and convergence studies

The convergence and verification studies are performed by employing different number of terms in the polynomial series expansions. The computed results are presented in terms of dimensionless frequencies considering various gradient indexes in both directions, aspect ratios and boundary conditions, namely SS and CF. The results from the previous study [30] in terms of dimensionless fundamental frequencies (λ_1) are used for comparison purposes. The present results agree well with those from the previous study [30]. It is clear that for the free vibration analysis of SS beams, the responses converge quickly, when the number of terms in polynomial expansion is set to 6 as it is seen from Table 3. However, the agreed results are obtained for CF and CC boundary conditions by employing 8 terms in polynomial expansion as given in Tables 4 and 5. For the sake of accuracy, 10 terms in the polynomial expansion is employed for the extensive free vibration analysis of two directional FGBs.

3.2. Free vibration responses of two directional FGBs

In Tables 6–9, the first three dimensionless frequencies of the two directional FGBs with SS, CS, CC and CF boundary conditions are presented for two different aspect ratios (L/h = 5 and L/h = 20) and various gradient indexes in both directions (p_z and p_x). It is clear from

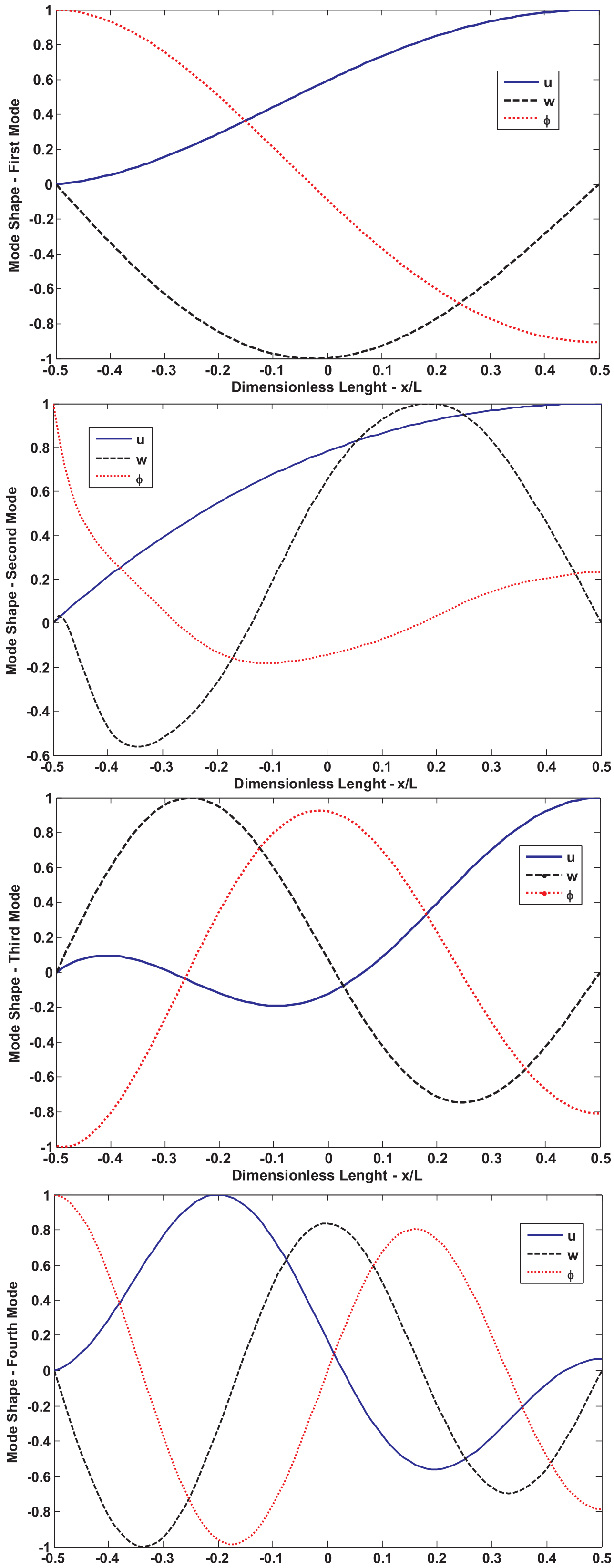

Fig. 3. First four mode shapes of SS two directional FGBs (L/h = 5, p_z = 1and p_x = 1).

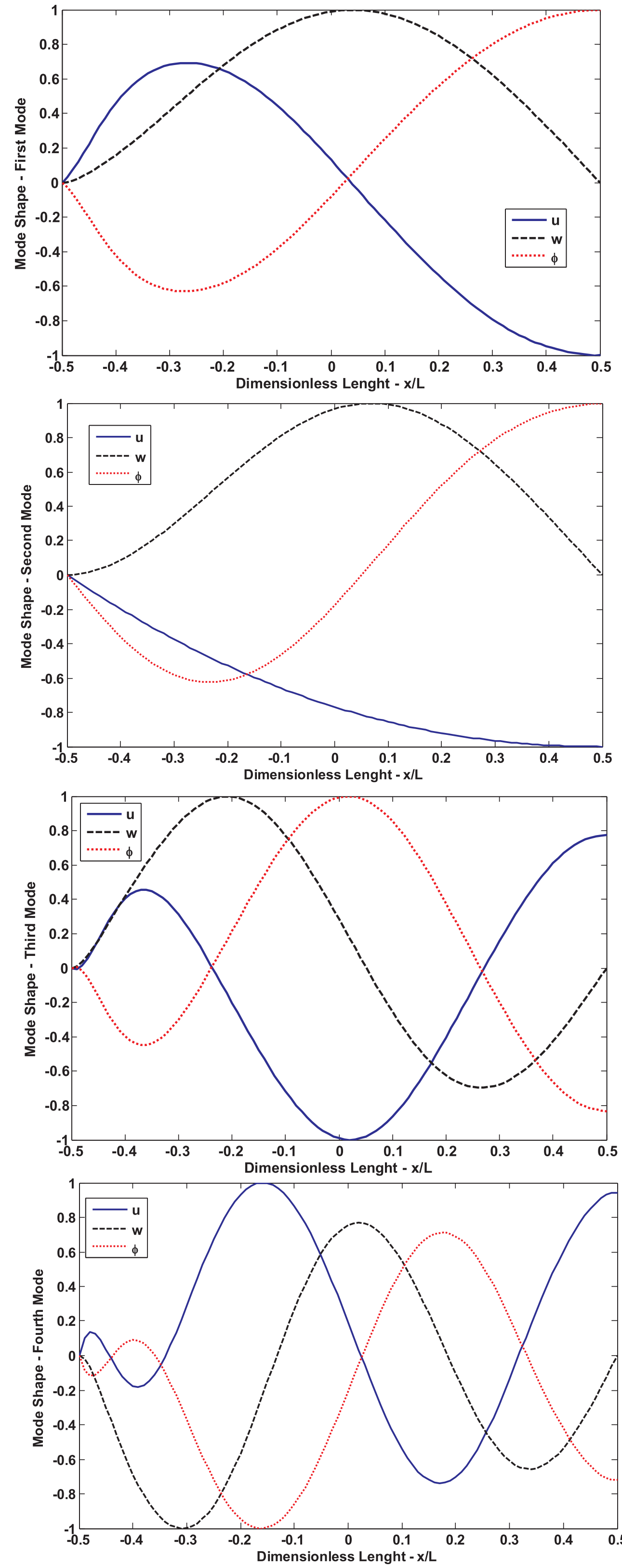

Fig. 4. First four mode shapes of CS two directional FGBs (L/h = 5, p_z = 1and p_x = 1).

the results that the first three dimensionless natural frequencies decrease for SS, CS and CF end conditions while the gradient indexes increase. On the other hand, the free vibration behavior of the CC 2D-FGBs is affected in different ways according to variation of the gradient indexes. The natural frequencies of the CC 2D-FGBs increase as the gradient index in the x-direction increase. However, they are decreasing with an increment of the gradient indexes in the z-direction.

One may expect that the frequencies have to increase since the Young's modulus, ultimately the rigidity of the beam increases with the

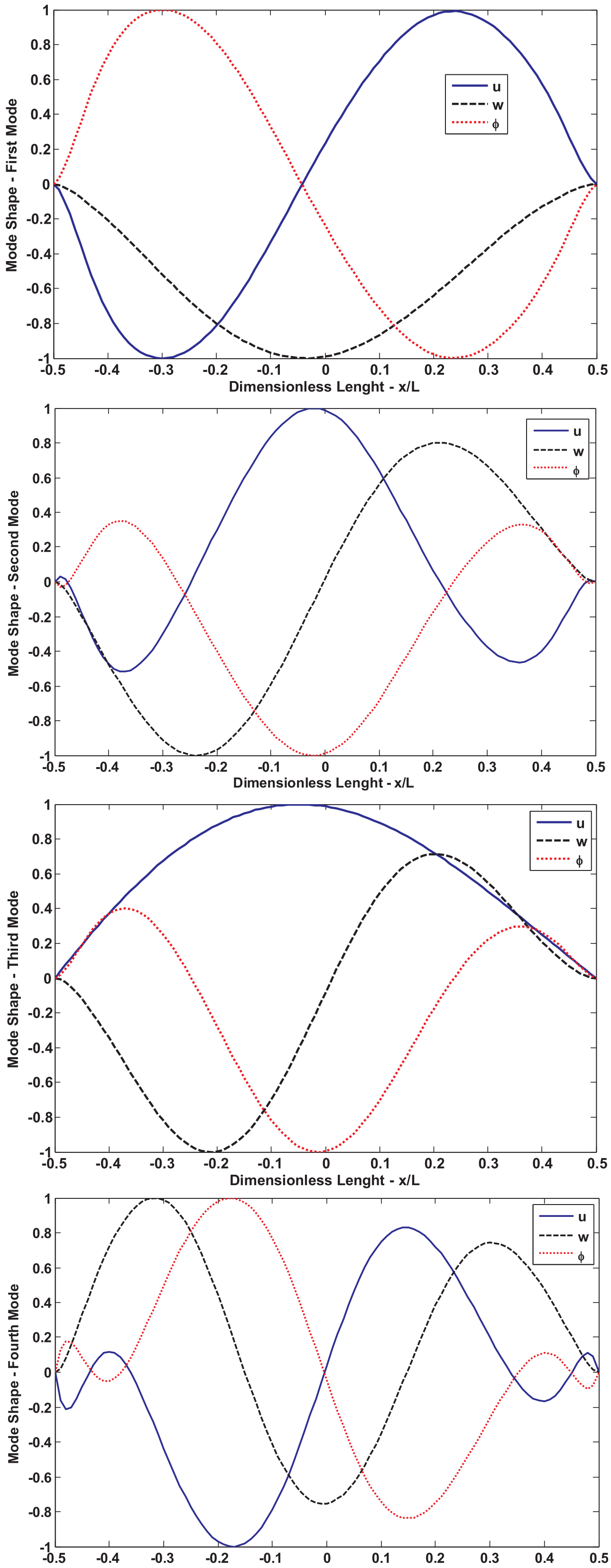

Fig. 5. First four mode shapes of CC two directional FGBs (L/h = 5, p_z = 1and p_x = 1).

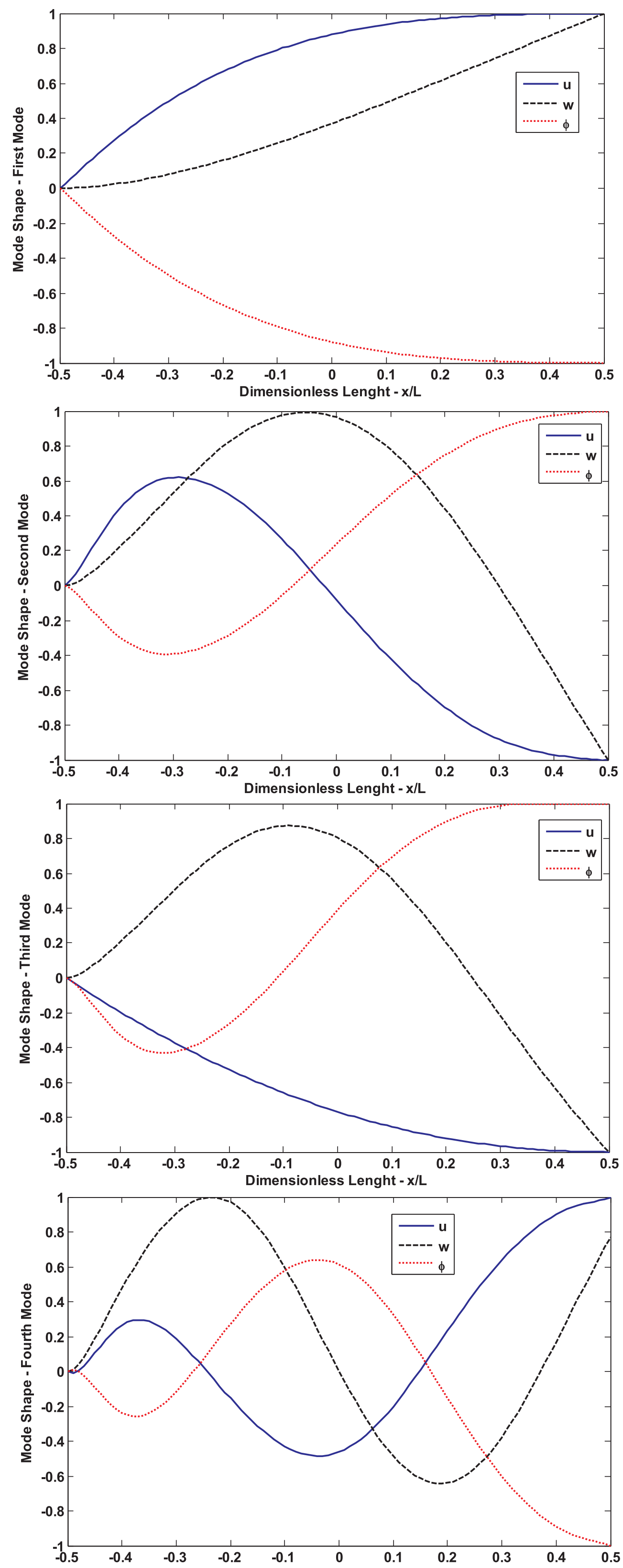

Fig. 6. First four mode shapes of CF two directional FGBs (L/h = 5, p_z = 1and p_x = 1).

increase of the gradient indexes. However, the mass is not constant and increased by the material gradient indexes. It is very well known in vibration theory and should be noted that the frequency is inversely proportional with the mass of the beam. It is found that the effect of the mass on the fundamental frequencies of the beams is a bit more dominant than the effect of the Young's modulus.

It is observed that the gradient indexes have different effects on the vibration responses of the two directional FGBs depending on the

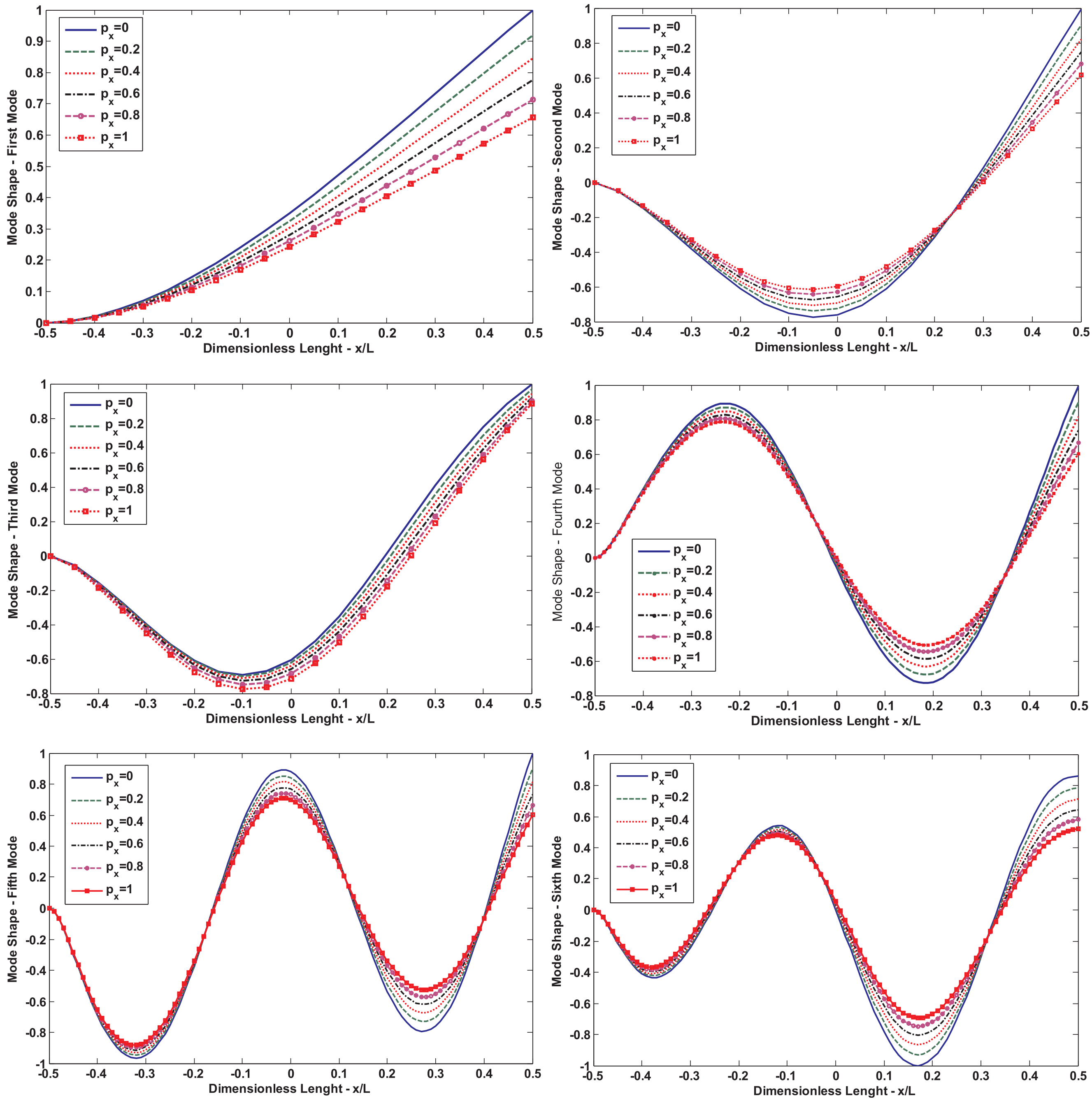

Fig. 7. First six mode shapes (w) of SS two directional FGBs with respect to variation of the gradient index p_x (L/h = 5 and p_z = 1).

boundary conditions. It is observed that for the fundamental frequencies of the SS and CF 2D-FGBs the gradient index p_x is more effective than the p_z. However, for second and third natural frequencies the gradient index p_z is a bit more dominant than p_x. The gradient index in the z-direction is more effective on the vibration responses of CC beams than the gradient index in the x-direction for all aspect ratios and mode number. However, the effects of the gradient indexes on the natural frequencies on the CS 2D-FGBS depend on the aspect ratio and mode number. When the aspect ratio (L/h) is set to 5, the p_x has a bit more effective than p_z for first and second natural frequencies. On the other hand, for the third mode they have almost the same effect. Based on the results obtained for L/h = 20, the p_x is more effective on the first and third natural frequencies of CS 2D-FGBs than the p_z. And finally, the most important and interesting output of the study is that the second natural frequencies of CS 2D-FGBs with L/h = 5, the third natural frequencies of CS 2D-FGBs with L/h = 20, the third natural frequencies of CC 2D-FGBs with L/h = 5 and the third natural frequencies of CF 2D-FGBs with L/h = 5 are not affected by the variation of the gradient index in the z-direction for all values of p_x. They remain constant while the p_z increases.

To illustrate the effect of gradient index – p_x and the aspect ratio (L/h) on the free vibration response of two directional FGBs, the variation of the fundamental frequencies (λ_1) is plotted in Fig. 2. It is observed that the maximum dimensionless fundamental frequency is obtained when the p_x is set to zero. Except the aspect ratios L/h = 2 and L/h = 5, the curves obtained for the fundamental frequencies based on the various aspect ratios are almost symmetrical according to the axis located at p_x = 0. It can be concluded for positive values of p_x that the effect of the mass on the fundamental frequencies of the thick beams is a bit more dominant than the effect of the Young's modulus. On the other hand, for negative values of p_x it is clear that the Young's modulus is more effective than the mass of the beam on the free vibration of the SS two directional FGBs. It is seen that for moderately thick and thin beams, they have the same effect on the vibration response. The first four mode shapes of SS, CS, CC and CF two directional FGBs with the gradient indexes, $p_z = 1$ and $p_x = 1$, are illustrated in Figs. 3–6. It is clear that the resulting mode shape is referred as triply coupled mode, which are substantial involving axial, shear and flexure deformation for all types of end conditions.

The effect of the gradient index p_x on the flexural vibration mode shapes of the CF two directional FGBs are presented in Fig. 7. It is clear that the flexural displacement increases while the gradient index p_x decreases. With an increment on the p_x, the effects of the propagating components of bending waves gradually become smaller and smaller.

4. Conclusion

The free vibration behavior of the two directional functionally graded beams having different boundary conditions is presented. By employing various gradient indexes in both axial and thickness directions, the material properties of the beam are changed. The governing equations of motion are obtained via Lagrange equations in conjunction with polynomials added auxiliary functions which are necessary to satisfy the boundary conditions. Various gradient indexes, aspect ratios and boundary conditions are considered. The computed results in terms of dimensionless fundamental frequencies are compared with the results from a previous study. It is found that computed results show excellent agreement with previous one. Extensive analysis is performed to understand the influence of the material gradation, aspect ratio and the boundary conditions on the dynamic response of the two directional FGBs.

It is found that the dimensionless fundamental frequency decreases as the material gradient index p_z or p_x increases for all type of boundary conditions except CC boundary condition. The natural frequencies of the CC 2D-FGBs decrease as the gradient index in the z-direction increase. However, they are increasing with an increment of the gradient indexes in the x-direction. The beam theory employed within this paper for the solution of the free vibration responses of the two directional FGBs satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam, thus a shear correction factor is not required. It allows having a better prediction of free vibrations response for the two directional FGBs. For thick beams, the shear effect is very important and higher order shear deformation beam theories are necessary. Free vibration behavior of the two directional FGBs can be controlled and optimized to meet the desired goals by choosing suitable gradient index.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.compstruct.2018.01.060.

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