Sigma J Eng & Nat Sci 36 (2), 2018, 465-472

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

SOME PROPERTIES OF PESSIMISTIC AND OPTIMISTIC VALUES FOR UNCERTAIN RANDOM VARIABLES

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Received: 26.02.2018 Accepted: 04.04.2018

ABSTRACT

The optimistic and pessimistic values of uncertain random variable have been presented for handling both uncertainty and randomness. In this paper, some extensions of optimistic and pessimistic values of uncertain random variable are investigated.As a sample, the optimistic and pessimistic values of an uncertain random variable are expressed.
Keywords: $\alpha-$ pessimistic value; $\alpha-$ optimistic value; uncertain random variables.

1. INTRODUCTION

The presented formula which models a system, reaching a result is normally a simple rule which includes evaluating the formula using the estimates for all variable held within the formula. Derived from simple formulas with random variables having insignificant uncertainty will be influenced minimally by not performing uncertainty analysis. However, for complex systems or models that have random variables with non-trivial uncertainties, neglecting uncertainty may cause deceptive conclusions. Estimating the uncertainty of a function of random variables can be performed by many distinct techniques. However, a general application of probability theory is that the estimated probability is near enough to the real frequency. Since the data is deficient generally, we must ask some experts to estimate their belief degree that each case will occur. In this condition, the usage of probability theory is no more practicable. To deal with this difficulty, an uncertainty theory was constructed by Liu Liu [1] and reconstructed by Liu [2]. Thus he introduced to model the information and knowledge In many events, human uncertainty and randomness concurrently appear in a system. In order to describe this event, uncertain random variable is presented by Liu [3].

Some scholars have studied some problems including both human uncertainty and chance theory. Liu and Ralescu [5] introduced uncertain random risk index with applications. Liu [6] presented uncertain random graphs and uncertain random networks. Zhou et al. [7]presented multi-objective optimization in uncertain random environments. Gao and Yao [8] gave some theories and outcomes of uncertain random processes. Ke et al. [10] presented uncertain random multilevel programming with application to product control problem. Yao and Gao [9] offered

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uncertain random alternating renewal processes with applications to interval availability. Ahmadzade et al. [11] derived some properties of uncertain random partial quadratic entropy. Dalman [12] studied uncertain random fixed multi-item solid transportation problem. Sheng and Gao [13] presented a simulation algorithm to solve uncertain random shortest path problem.

By using concepts and theorems of chance theory, this paper investigates the concepts of $\alpha-$ pessimistic value and $\alpha-$ optimistic value for uncertain random variables.

The rest of this paper is organized as follows. Section 2 presents some basic knowledge on uncertainty theory and chance theory. Section 3 proves some formulas for uncertain random variables. a conclusion is given in Section 4.

2. PRELIMINARIES

2.1. Uncertainty Theory

Let Γ be a nonempty set, L be a σ-algebra over Γ and M be an uncertain measure. Then (Γ, L, M) is a measurable space. A set function $M: L \to [0,1]$ is called an uncertain measure if it satisfies the following four axioms:

Axiom 1 (Normality Axiom) (Liu [1])**:** $\mathrm{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (Duality Axiom) (Liu [1])**:** $\mathrm{M}\{\Lambda\} + \mathrm{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom) (Liu [1]): For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n,$ we obtain

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Axiom 3 (Product Axiom) (Liu [14])**:** Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure M is an uncertain measure satisfying $M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$ where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \cdots$, respectively.

Definition 1 (Liu [2]) An uncertain variable is a function ξ from an uncertainty space (Γ, L, M) to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.

Remark 1 Note that the event $\{\xi \in B\}$ is a subset of the universal set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$.

Definition 2 (Liu [2]) An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \lim_{x \to +\infty} \Phi(x) = 1.$$

Definitio n 3 (Liu [2]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x).$ Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of $\xi.$

Theorem 1 (Liu [2]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n,$ then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{1}$$

has an inverse uncertainty distribution.

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \ldots, \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{2}$$

2.2. Chance Theory

Definition 4 (Liu [3]) Let (Γ, L, M) be an uncertainty space and let $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then the product $(\Gamma, L, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space

$$(\Gamma, L, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr) = (\Gamma \times \Omega, L \times \mathcal{A}, \mathcal{M} \times \Pr).$$

Definition 5 (Liu [3]) An uncertain random variable is a function ξ from a chance space $(\Gamma, L, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in an event in $L \times \mathcal{A}$ for any Borel set B of real numbers.

Theorem 2 (Liu [2]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain random variables on the chance space $(\Gamma, L, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ and let f be a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain random variable determined by

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega))$$

for all $(\gamma, \omega) \in \Gamma \times \Omega.$

Definition 6 (Liu [3]) Let ξ be an uncertain random variable on the chance space $(\Gamma, L, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr),$ and let B be a Borel set of real numbers. Then $\{\xi \in B\}$ is an uncertain random event with chance measure

$$\mathrm{Ch}\{\xi \in B\} = \int_0^1 \Pr\left\{\omega \in \Omega \middle| \mathcal{M}\{\gamma \in \Gamma | \xi(\gamma, \omega) \in B\} \geq x\right\}$$

Theorem 3 (Liu [4]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$\$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables.

Assume f is a measurable function. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) d\,\Psi_1(y_1), d\Psi_2(y_2), \cdots, d\Psi_m(y_m) \tag{3}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of the variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n).$

Definition 7 (Liu [3])Let ξ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \mathrm{Ch}\{\xi \le x\} \tag{4}$$

for any $x \in \Re$.

Theorem 4 Let ξ be an uncertain random variable. Then its expected value is

$$E[\xi] = \int_0^{+\infty} Ch\{\xi \ge r\}dr - \int_{-\infty}^{0} Ch\{\xi \le r\}dr \tag{5}$$

provided that at least one of the two integrals is finite.

Theorem 5 Let ξ be an uncertain random variable with regular chance distribution Φ. If the expected value exists, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{6}$$

3. THE CONCEPTS OF $\alpha -$ PESSIMISTIC VALUE AND $\alpha -$ OPTIMISTIC VALUE FOR AN UNCERTAIN RANDOM VARIABLE

In fact, notation (3) is theoretical, which is not easy usage in most conditions because of the complexity of chance distribution function. To cope with the complexity, a simulation uncertain random is introduced to obtain the chance distribution.

First, we introduce the concepts of $\alpha -$ pessimistic value and $\alpha -$ optimistic value for an uncertain random variable. Then we approximate the chance distribution, $\alpha -$ pessimistic value and $\alpha -$ optimistic value by using a numerical integration method.

Definition 8 Let ξ be an uncertain random variable on chance space. Then, $(\Gamma, L, \mathrm{M}) \times (\Omega, \mathrm{A}, \mathrm{Pr})$ and $\alpha \in (0,1]$.

$$\xi_{\inf}(\alpha) = \inf\{r \mid \mathrm{Ch}\{\xi \le r\} \ge \alpha\} \tag{7}$$

and

$$\xi_{\sup}(\alpha) = \sup\{r \mid \mathrm{Ch}\{\xi \ge r\} \ge \alpha\} \tag{8}$$

are called the $\alpha -$ pessimistic value and the $\alpha -$ optimistic value of ξ, respectively.

Note that Random variables and uncertain variables are special uncertain random variables. The $\alpha -$ pessimistic value and the $\alpha -$ optimistic value of linear uncertain variable $L(a,b)$ are $\xi_{\inf}(\alpha) = (1-\alpha)a + \alpha b$ and $\xi_{\sup}(\alpha) = \alpha a + (1-\alpha)b$.

Theorem 6 Let ξ be an ordinary uncertain random variable and $\alpha \in (0,1]$. Then, we have

$$\mathrm{Ch}\{\xi \le \xi_{\inf}(\alpha)\} = \alpha. \tag{9}$$

Proof: Since the chance distribution is continuous, it follows from the definition of the $\alpha -$ pessimistic value for each $\alpha \in (0,1]$, we have $\mathrm{Ch}\{\xi \le \xi_{\inf}(\alpha)\} = \lim_{n\to\infty} \mathrm{Ch}\{\xi \le \xi_{\inf}(\alpha) - 1/n\} \le \alpha$, and $\mathrm{Ch}\{\xi \le \xi_{\inf}(\alpha)\} = \lim_{n\to\infty} \mathrm{Ch}\{\xi \le \xi_{\inf}(\alpha) + 1/n\} \ge \alpha$, which imply that $\mathrm{Ch}\{\xi \le \xi_{\inf}(\alpha)\} = \alpha$ holds. The theorem is proved.

Theorem 7 Let ξ be an ordinary uncertain random variable and $\alpha \in (0,1]$. Then, we have

$$\mathrm{Ch}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha. \tag{10}$$

Proof: Since the continuity of chance distribution, it follows from the definition of the $\alpha-$ optimistic value for each $\alpha \in (0,1]$ that

$$\mathrm{Ch}\{\xi \geq \xi_{\sup}(\alpha)\}=\lim_{n\to\infty}\mathrm{Ch}\{\xi \geq \xi_{\sup}(\alpha)\text{-}1/n\} \geq \alpha$$

and

$$\mathrm{Ch}\{\xi \geq \xi_{\sup}(\alpha)\} = \lim_{n\to\infty}\mathrm{Ch}\{\xi \geq \xi_{\sup}(\alpha)+1/n\} \leq \alpha,$$

which imply that $\mathrm{Ch}\{\xi \geq \xi_{\sup}(\alpha)\} = \alpha$ holds. The theorem is proved.

Theorem 8 Let ξ be an uncertain random variable and $\alpha \in (0,1]$. Then, we have

$$\xi_{\inf}(\alpha) = \Phi^{-1}(\alpha). \tag{11}$$

Proof: It follows from Definition 8 immediately**.**

Theorem 9 Let ξ be an uncertain random variable and $\alpha \in (0,1]$. Then, we have

$$\xi_{\inf}(\alpha) = \xi_{\sup}(1-\alpha). \tag{12}$$

Proof: It follows from Equation (8) that $\mathrm{Ch}\{\xi \geq \xi_{\sup}(1\text{-}\alpha)\}=1\text{-}\alpha$. Thus,

$$\mathrm{Ch}\{\xi \leq \xi_{\sup}(1\text{-}\alpha)\} = 1\text{-}\mathrm{Ch}\{\xi \geq \xi_{\sup}(1\text{-}\alpha)\}=1\text{-}(1\text{-}\alpha) = \alpha$$

Thus, we have $\xi_{\inf}(\alpha) = \xi_{\sup}(1-\alpha)$. The theorem is proved.

Theorem 10 Let ξ be an uncertain random variable and $\alpha \in (0,1]$. Then, we have

$$\xi_{\sup}(\alpha) = \Phi^{-1}(1-\alpha) \quad \text{and} \quad \xi_{\sup}(1\text{-}\alpha)=\Phi^{-1}(\alpha). \tag{13}$$

Proof: It follows from Theorems 8 and 9.

Theorem 11 Let ξ be an ordinary uncertain random variable. Then, we have

$$E[\xi] = \int_0^1 \xi_{\inf}(\alpha)\mathrm{d}\alpha. \tag{14}$$

Proof: Since $\Phi(x)$ is strictly increasing and continuous, we get

$$\begin{aligned}\xi_{\inf}(\alpha) &= \inf\{\mathrm{r}|\mathrm{Ch}\{\xi \leq \mathrm{r}\} \geq \alpha\} \\ &= \inf\{\mathrm{r}|\Phi(\mathrm{r}) \leq \alpha\} \\ &= \Phi^{-1}(\alpha).\end{aligned}$$

Here, we have $E[\xi] = \int_0^1 \xi_{\inf}(\alpha)\mathrm{d}\alpha$. The Theorem is proved.

Theorem 12 Let ξ be an ordinary uncertain random variable. Then, we have

$$E[\xi] = \int_0^1 \xi_{\sup}(\alpha)\mathrm{d}\alpha. \tag{15}$$

Proof: Since $\Phi(x)$ is strictly increasing and continuous, we get

$$\begin{aligned}\xi_{\sup}(\alpha) &= \sup\{\mathrm{r}|\mathrm{Ch}\{\xi \geq \mathrm{r}\} \geq \alpha\} \\ &= \sup\{\mathrm{r}|\Phi(\mathrm{r}) \leq 1\text{-}\alpha\} \\ &= \Phi^{-1}(1-\alpha).\end{aligned}$$

Th us, for all $\alpha \in (0,1),$ we have

$$\begin{aligned}\int_0^1 \xi_{\sup}(\alpha)\mathrm{d}\alpha &= \int_0^1 \Phi^{-1}(1\text{-}\alpha)\mathrm{d}\alpha \\ &= \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = \mathrm{E}[\xi]\end{aligned}$$

The theorem is proved.

Theorem 13 Let ξ be an ordinary uncertain random variable. Then, we have

$$E[\xi] = \frac{1}{2}\int_0^1 \left[\xi_{\inf}(\alpha) + \xi_{\sup}(\alpha)\right]\mathrm{d}\alpha. \tag{16}$$

Proof: It follows directly from Theorem 11 and Theorem 12.

Theorem 14 Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m,$ and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n.$

Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n.$ Then

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m)\, \mathrm{d}\Psi_1(y_1), \mathrm{d}\Psi_2(y_2), \cdots, \mathrm{d}\Psi_m(y_m) \tag{17}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is determined by its inverse uncertainty distribution

$$\begin{aligned}&F^{-1}(y_1, y_2, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_2^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) \\ &F^{-1}(y_1, y_2, \cdots, y_m, (\xi_1)_{\sup}(1-\alpha), \cdots,\end{aligned} \tag{18}$$

$$(\xi_k)_{\sup}(1-\alpha), (\xi_{k+1})_{\sup}(\alpha), \cdots, (\xi_n)_{\sup}(\alpha)). \tag{19}$$

Proof: It follows from Theorem 1 that inverse uncertainty distribution of $F(x; y_1, y_2, \cdots, y_m)$ is determined by

$$\begin{aligned}&F^{-1}(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \\ &\Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)).\end{aligned} \tag{20}$$

According to Theorem 10, we substitute $\Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha)$ with $(\xi_1)_{\sup}(1-\alpha), \cdots, (\xi_k)_{\sup}(1-\alpha)$ and $\Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)$ with $(\xi_{k+1})_{\sup}(\alpha), \cdots, (\xi_n)_{\sup}(\alpha).$ Thus, Formula (19) holds. The theorem is completed.

Theorem 15 Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$.

Assume $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$. Then $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) d\Psi_1(y_1), d\Psi_2(y_2), \cdots, \mathrm{d}\Psi_m(y_m). \tag{21}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is determined by its inverse uncertainty distribution

$$F^{-1}(y_1, y_2, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_2^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) \tag{22}$$

$$\begin{aligned} &F^{-1}(y_1, y_2, \ldots, y_m, (\xi_1)_{\inf}(\alpha), \cdots, (\xi_k)_{\inf}(\alpha), \\ &(\xi_{k+1})_{\inf}(1-\alpha), \cdots, (\xi_n)_{\inf}(1-\alpha)). \end{aligned} \tag{23}$$

Proof: The proof is similar to that of Theorem 14.

4. CONCLUSIONS

In this paper, the concepts of $\alpha-$ pessimistic and $\alpha-$ optimistic values are investigated for uncertain random variables. Moreover, some concepts and notations for uncertain random variables have been extended.

Acknowledgments

The author declare that they have no conflicts of interest.

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