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# ANALYSIS OF ELASTIC LATERAL TORSIONAL BUCKLING OF CANTILEVER I SECTIONS BY THE COMPLEMENTARY FUNCTIONS METHOD

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## *Highlights*

- Lateral torsional buckling response of cantilever I and IPE beams is performed.
- The effects of loading type on the lateral torsional buckling is examined.
- An effective and accurate numerical method is implemented.

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**ABSTRACT:** In this study, an important stability problem, in the design of cantilever I-beams under lateral torsional buckling behavior is theoretically investigated. The elastic lateral torsional buckling behavior of cantilever I beam loaded from shear centers is examined for four different loading types. The governing differential equation is transformed into a set of first-order ordinary differential equations. The Complementary Functions Method (CFM), which is an effective method in solving the first-order differential equation set, is used. Fifth Order Runge-Kutta (RK5) algorithm is used for numerical integrations in CFM, which can transform the boundary value problem into an initial value problem. The obtained results were compared with the existing results in the literature. It has been shown that CFM can be used effectively in the analysis of elastic lateral torsional buckling behavior of I beams.

***Keywords:*** *Complementary functions method, Lateral torsional buckling, Stability*

## 1. INTRODUCTION

The safety of structural elements in terms of lateral torsional buckling is an important issue during the design of structures. In design, safety is evaluated within the framework of limiting the stress values occurring in the structural elements to permissible stresses and the stability of the steady state. To limit the stresses in the sections, the profiles used in steel structures are placed in the structure in a way that the bending moment will be implemented around their strong principal axis. These section profiles, which are considered to be subjected to the bending around their strong principal axes, may twist and buckle around their weak axes at the point with the increased load intensity. In this context, there are many investigations in the literature related to the lateral torsional buckling behavior of beams.

Gupta et al. [1] developed the finite element formulation for the lateral torsional buckling of I beams. They considered the influence of warping deformations and the location of the implemented load in the cross-section in formulations. Sapkás and Kollár [2] examined the stability of thin-walled orthotropic composite beams. They obtained the lateral torsional buckling loads of cantilever beams by considering the influence of shear deformation for several boundary conditions and various types of loadings. Challamel et al. [3] investigated the torsional lateral buckling analysis of cantilever beams with variable cross-section with analytical approaches. They used the finite element approach to verify their results of lateral torsional buckling values obtained for a cantilever beam subjected to a point load to its free end.

The elastic lateral torsional buckling response of cantilever I beams was studied by Özbaşaran [4]. They used the finite difference method in their research. Özbaşaran et al. [5] carried out the critical lateral torsional buckling load of I cantilever beams. They suggested a closed-form solution and verified their results by comparing with those of the ABAQUS. Yilmaz and Kirac [6] studied the lateral torsional buckling response of simply supported IPE and IPN beams. The governing equations were solved for several parameters with the aid of analytical models. They developed an equation that gives the critical lateral torsional buckling loads for various locations of the applied loads in the cross-section of the beam.

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Soltani and Asgarian [7] studied the lateral torsional buckling response of axially functionally graded I beams. The stability of beams with various values of material gradient indices was examined. A differential quadrature method was implemented to carry out the lateral torsional buckling loads for hinged–hinged beams subjected to uniformly distributed loads.

In this study, the critical lateral torsional buckling moments of I-section cantilever beams loaded at the shear center with different types of loadings are obtained via the Complementary Functions Method. The lateral torsional buckling behavior of cantilever beams is investigated for different section profiles and beam lengths.

## 2. GOVERNING EQUATIONS AND PROCEDURE OF THE SOLUTION

A cantilever I beam subjected to a transverse point load to its shear center at the free end is given in Figure 1. The cross-section of the beam is assumed to be symmetric. The governing equation for the lateral torsional buckling of cantilever I beams subjected to a point load at its free end is given by [8] and presented in equation (1).

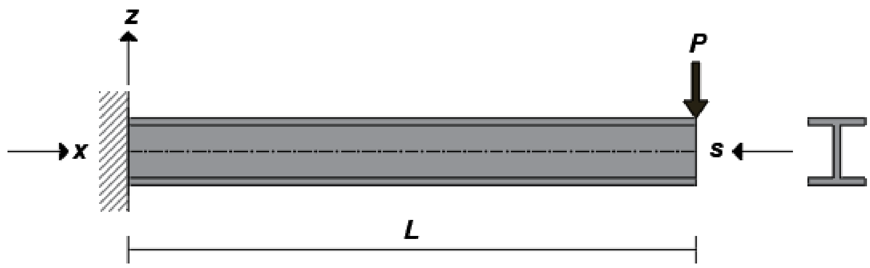


**Figure 1.** A cantilever I beam subjected to a point load

$$C_1 \frac{d^4\phi}{dx^4} - C\frac{d^2\phi}{dx^2} - \frac{P^2}{EI_\eta}(L-x)^2\phi = 0 \tag{1}$$

In this equation; $\phi$ is the angle of twist, $C_1 = EC_w$ shows the warping rigidity, $C = GI_t$ is the torsional rigidity, $P$ stands for the point load, $I_\eta$ is the are moment inertia about the weak axis, $I_t$ is tortional moment of inertia, $C_w$ demonstrates the warping constant, $E$ gives the modulus of elasticity and $G$ shows the shear modulus.

By substituting $s = L - x$ in equation (1) it can be simplified in following form:

$$\frac{d^4\phi}{ds^4} - \frac{C}{C_1}\frac{d^2\phi}{ds^2} - \frac{P^2 s^2}{EI_\eta C_1}\phi = 0 \tag{2}$$

The fourth-order ordinary differential equation (2) will be converted to four first-order ordinary differential equations. For this purpose, the new variables can be determined as in equations (3-5).

$$\phi = t_1 \tag{3}$$

$$\frac{d\phi}{ds} = t_2 \tag{4}$$

$$\frac{d^2\phi}{ds^2} = t_3 \tag{5}$$

$$\frac{d^3\phi}{ds^3} = t_4 \tag{6}$$

By using equation 2 and equations (3-6) the set of governing equations of lateral torsional buckling can be obtained as follows:

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$$\frac{dt_1}{ds}=t_2 \tag{7}$$

$$\frac{dt_2}{ds}=t_3 \tag{8}$$

$$\frac{dt_3}{ds}=t_4 \tag{9}$$

$$\frac{dt_4}{ds}=\frac{C}{C_1}t_3+\frac{P^2s^2}{EI_\eta C_1}t_1 \tag{10}$$

A cantilever I beam subjected to a transverse uniformly distributed load to its shear center is illustrated in Figure 2. The governing equation for the lateral torsional buckling of cantilever I beams subjected to uniformly distributed load is given by [9] and presented in equation (11).

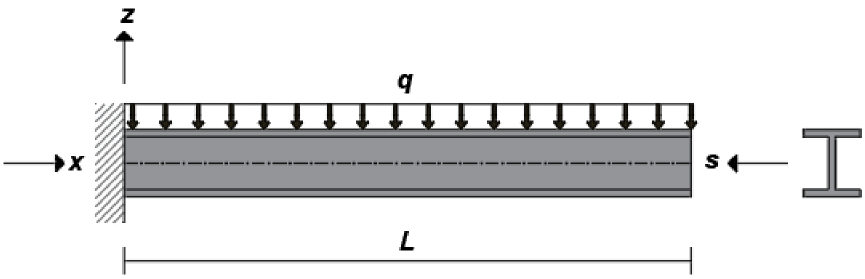


**Figure 2.** A cantilever I beam subjected to uniformly distributed transverse load

$$\frac{d^4\phi}{ds^4}-\frac{C}{C_1}\frac{d^2\phi}{ds^2}-\frac{q^2s^4}{4EI_\eta C_1}\phi=0 \tag{11}$$

In this equation $q$ stands for the magnitude of uniformly distributed load. The variables given in equations (3-6) will be used to convert equation (11) to a set of ordinary canonical differential equations.

$$\frac{dt_1}{ds}=t_2 \tag{12}$$

$$\frac{dt_2}{ds}=t_3 \tag{13}$$

$$\frac{dt_3}{ds}=t_4 \tag{14}$$

$$\frac{dt_4}{ds}=\frac{C}{C_1}t_3+\frac{q^2s^4}{4EI_\eta C_1}t_1 \tag{15}$$

A cantilever I beam subjected to a transverse uniformly distributed load and a point load at its free end is presented in Figure 3. The governing equation for the lateral torsional buckling of this cantilever beam is given in [9] and presented in equation (16). The magnitude of the point is taken as $P=\lambda qL$ .

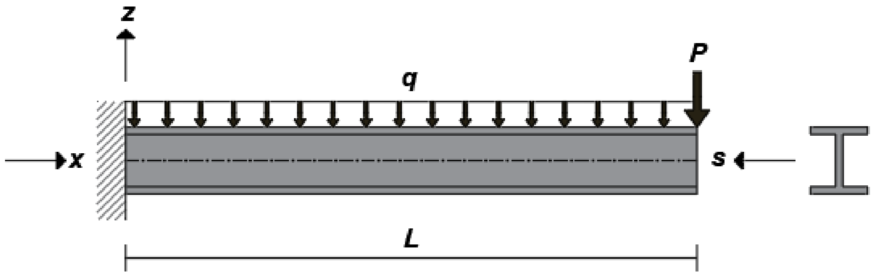


**Figure 3.** An I cantilever beam subjected to a point load and uniformly distributed load

$$\frac{d^4\phi}{ds^4}-\frac{C}{C_1}\frac{d^2\phi}{ds^2}-\frac{q^2s^2(2\lambda L+s)^2}{4EI_\eta C_1}\phi=0 \tag{16}$$

The variables given previously in equations (3-6) will be used to change equation (16) to a set of canonical differential equations. The obtained equations are written as follows:

$$\frac{dt_1}{ds}=t_2 \tag{17}$$

$$\frac{dt_2}{ds}=t_3 \tag{18}$$

$$\frac{dt_3}{ds}=t_4 \tag{19}$$

$$\frac{dt_4}{ds}=\frac{C}{C_1}t_3+\frac{q^2s^2(2\lambda L+s)^2}{4EI_\eta C_1}t_1 \tag{20}$$

A cantilever I beam subjected to a concentrated moment is shown in Figure 4. The governing equation of the torsional lateral buckling response of this beam is presented by [9] and given in equation (21).

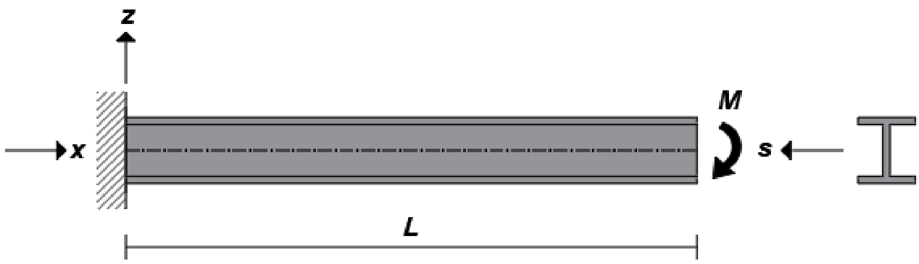


**Figure 4.** A cantilever I beam subjected to a point moment

$$\frac{d^4\phi}{ds^4}-\frac{C}{C_1}\frac{d^2\phi}{ds^2}-\frac{M^2}{EI_\eta C_1}\phi=0 \tag{21}$$

By using the variables given in equations (3-6), equation (11) is converted to a set of ordinary canonical differential equations and presented below (equations (22-25).

$$\frac{dt_1}{ds}=t_2 \tag{22}$$

$$\frac{dt_2}{ds}=t_3 \tag{23}$$

$$\frac{dt_3}{ds} = t_4 \tag{24}$$

$$\frac{dt_4}{ds} = \frac{C}{C_1}t_3 + \frac{M^2}{EI_\eta C_1}t_1 \tag{25}$$

The set of canonical equations obtained from equations (2), equations (11), equations (16) and equations (21) are two-point boundary value problems. The CFM will be implemented to carry out the solution of these equations. The main principle is the implemented method is that it reduces a two-point boundary value problem to an initial value problem. The solution of the set of the ordinary differential equations can be given in equation (26) which is consist of four homogenous and one nonhomogeneous solution [10,11]. In the solution procedure of the initial value problem the Runge – Kutta (5th order) will be used [12]. The CFM is an efficient, accurate and easily applicable method for the numerical solution of the two-point boundary value problems. This method was applied successfully to the solution of various structural mechanics problems [13,14].

$$\phi(s) = \phi_0(s) + a_1\phi_1(s) + a_2\phi_2(s) + a_3\phi_3(s) + a_4\phi_4(s) \tag{26}$$

In this equations, $\phi_0(s)$ is the nonhomogeneous solution, $\phi_1(s), \phi_2(s), \phi_3(s), \phi_4(s)$ are homogenous solutions and $a_1, a_2, a_3, a_4$ are the integrations constants which can be found with the aid of the boundary conditions. The boundary conditions that are required for the solution of the governing equation of the torsional lateral buckling response of the cantilever I beams are listed in Table 1.

**Table 1.** Boundary conditions

| | |
|---|---|
| Free end | $C\frac{d\phi}{ds} - C_1\frac{d^3\phi}{ds^3} = 0$<br>$\frac{d^2\phi}{ds^2} = 0$ |
| Fixed end | $\frac{d\phi}{ds} = 0$<br>$\phi = 0$ |

The governing equation of the torsional lateral buckling of the considered beams is homogenous. Thus, once the integration constants matrix from the homogenous solution of the differential equation is obtained then its determinant can be calculated easily. The set of integrations which makes this determinant zero are the critical lateral torsional buckling loads ($P$, $q$ or $M$). These values can be obtained with the method of Secant with the desired accuracy.

## 3. NUMERICAL EXAMPLES

To validate the accuracy and applicability of the CFM, lateral torsional buckling results of the present paper are compared with those of the available literature. The material and geometric properties of the cantilever I beam are presented in Table 2.

**Table 2.** Geometric and material properties of the I beam

| | |
|---|---|
| $E(MPa)$ | 200000 |
| $G(MPa)$ | 76923 |
| $I_\eta(x10^4\ mm^4)$ | 68.16 |
| $I_t(x10^3\ mm^4)$ | 28.20 |
| $C_w(x10^6\ mm^6)$ | 3958.9 |

The lateral torsional buckling moments for this cantilever I beam are calculated and compared with those of Özbaşaran [4] in Table 3. To solve the governing equation with the CFM, $N=\{5,10,25\}$ collocations points are used.

**Table 3.** Comparison of the lateral torsional buckling moments for I sections (kN.m)

| $L(m)$ | | $P$ | $q$ | $P+q(\lambda=1)$ | $M$ |
|---|---|---|---|---|---|
| 1.5 | $N=5$ | 98.93 | 198.25 | 120.26 | 28.34 |
| | $N=10$ | 98.92 | 198.20 | 120.25 | 28.34 |
| | $N=25$ | 98.92 | 198.20 | 120.25 | 28.34 |
| | Özbaşaran [4] | 98.93 | 198.19 | 120.18 | 28.34 |
| 2 | $N=5$ | 63.97 | 124.77 | 77.33 | 19.20 |
| | $N=10$ | 63.97 | 124.73 | 77.32 | 19.20 |
| | $N=25$ | 63.97 | 124.73 | 77.32 | 19.20 |
| | Özbaşaran [4] | 63.96 | 124.72 | 77.28 | 19.25 |
| 3 | $N=5$ | 35.63 | 66.87 | 42.71 | 11.44 |
| | $N=10$ | 35.62 | 66.84 | 42.70 | 11.44 |
| | $N=25$ | 35.62 | 66.84 | 42.70 | 11.44 |
| | Özbaşaran[4] | 35.61 | 66.83 | 42.66 | 11.47 |
| 4 | $N=5$ | 24.09 | 44.06 | 28.72 | 8.07 |
| | $N=10$ | 24.08 | 44.02 | 28.71 | 8.07 |
| | $N=25$ | 24.08 | 44.02 | 28.71 | 8.07 |
| | Özbaşaran [4] | 24.08 | 44.00 | 28.80 | 8.08 |

When Table 3 is analyzed, it can be obviously seen that the lateral torsional buckling moments obtained via the CFM are in really good agreement with the existing results presented previously by Özbaşaran [4]. It has been perceived that the results listed in Table 3 ensure sufficient sensibility for the used collocation point when two digits after the decimal are taken into account.

In the subsequent numerical applications, the lateral torsional buckling response of cantilever beams with IPE section will be investigated for four different loading types. The cross-sectional properties of the used IPE profiles are given in Table 4.

**Table 4.** Goemetric properties of IPE sections

| Geometric properties | IPE 120 | IPE 180 | IPE 220 | IPE 270 | IPE 330 |
|---|---|---|---|---|---|
| $I_\eta(x10^4\ mm^4)$ | 27.67 | 100.90 | 204.90 | 419.90 | 788.10 |
| $I_t(x10^3\ mm^4)$ | 16.89 | 47.23 | 89.82 | 157.10 | 275.90 |
| $C_w(x10^6\ mm^6)$ | 872 | 7322 | 22310 | 69469 | 196090 |

The lateral torsional buckling loads obtained for cantilever IPE beams are listed in Table 5. In the solution of the present problem of different lengths with the CFM, only 10 collocation points are used.

**Table 5.** Lateral torsional buckling moments obtained for cantilever IPE beams (kNm)

| $L(m)$ | Profil Tipi | $P$ | $q$ | $P+q(\lambda=1)$ | $M$ |
|---|---|---|---|---|---|
| 1.5 | IPE 120 | 38.10 | 72.79 | 45.85 | 11.86 |
| | IPE 180 | 160.06 | 322.15 | 194.76 | 45.51 |
| | IPE 220 | 360.68 | 740.53 | 440.64 | 99.36 |
| | IPE 270 | 823.14 | 1723.99 | 1009.54 | 220.20 |
| | IPE 330 | 1774.78 | 3762.56 | 2181.74 | 466.88 |
| 2 | IPE 120 | 25.41 | 47.25 | 30.40 | 8.28 |
| | IPE 180 | 103.21 | 202.21 | 124.88 | 30.72 |
| | IPE 220 | 229.28 | 459.29 | 278.71 | 65.70 |
| | IPE 270 | 512.36 | 1052.00 | 625.94 | 141.13 |
| | IPE 330 | 1082.96 | 2261.21 | 1327.42 | 290.98 |
| 3 | IPE 120 | 14.87 | 26.68 | 17.65 | 5.12 |
| | IPE 180 | 57.23 | 107.86 | 68.67 | 18.23 |
| | IPE 220 | 124.64 | 240.24 | 150.28 | 38.18 |
| | IPE 270 | 271.92 | 538.80 | 329.79 | 79.37 |
| | IPE 330 | 561.70 | 1137.35 | 684.30 | 158.15 |
| 4 | IPE 120 | 10.41 | 18.26 | 12.29 | 3.69 |
| | IPE 180 | 38.55 | 70.77 | 45.99 | 12.83 |
| | IPE 220 | 82.60 | 154.93 | 99.00 | 26.53 |
| | IPE 270 | 177.03 | 341.26 | 213.46 | 54.23 |
| | IPE 330 | 360.77 | 711.44 | 437.11 | 106.18 |
| 5 | IPE 120 | 7.99 | 13.79 | 9.40 | 2.89 |
| | IPE 180 | 28.79 | 51.83 | 34.20 | 9.86 |
| | IPE 220 | 60.86 | 111.86 | 72.63 | 20.22 |
| | IPE 270 | 128.44 | 242.40 | 154.16 | 40.84 |
| | IPE 330 | 258.82 | 499.60 | 312.16 | 79.09 |
| 6 | IPE 120 | 6.47 | 11.05 | 7.60 | 2.37 |
| | IPE 180 | 22.88 | 40.59 | 27.10 | 8.01 |
| | IPE 220 | 47.87 | 86.59 | 56.93 | 16.30 |
| | IPE 270 | 99.69 | 184.99 | 119.22 | 32.61 |
| | IPE 330 | 198.85 | 377.22 | 238.93 | 62.66 |

For a better interruption of the parametric studies results of the presented approach, graphical forms of the lateral torsional moments are illustrated in Figures (5-6).

From Table 5 and Figures (5-6) it can be understood that the loading type, section properties, and beam length have a significant effect on the lateral torsional buckling behavior of the problems in the hand. By increasing the length of the beam, the lateral torsional buckling moment of the structures decreases. Among the compared loading cases the concentrated bending moment at free end of the cantilever beam is the most critical case. Among the compared sections IPE120 is the weakest section for lateral torsional buckling of cantilever beams.

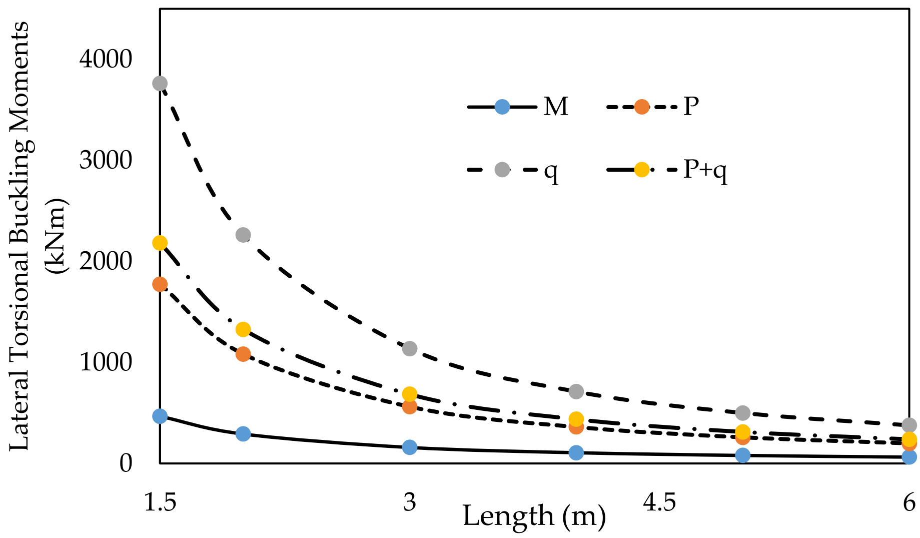


**Figure 5.** Lateral torsional buckling loads of IPE330 for several types of load

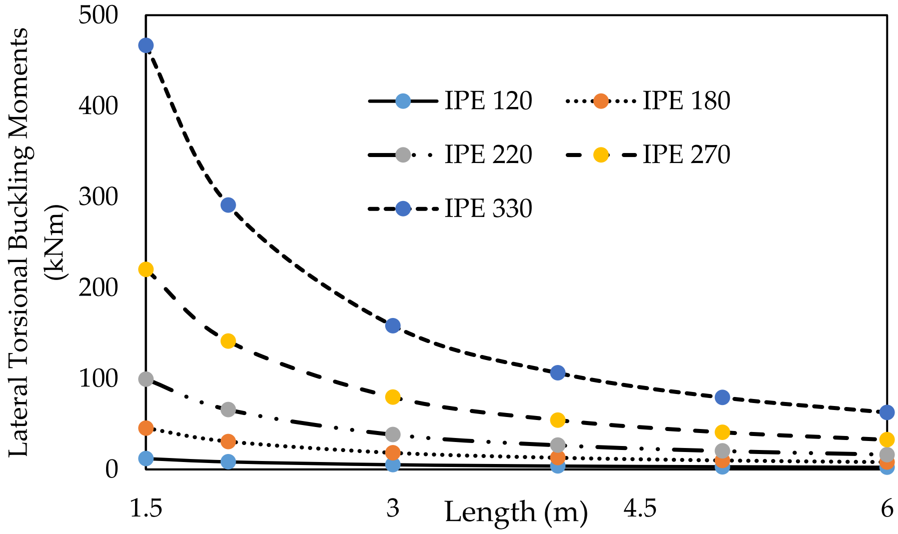


**Figure 6.** Lateral torsional buckling loads of for various cantilever IPE beams subjected to a point moment

## 4. CONCLUSIONS

The differential equations governing the lateral torsional buckling behavior of I and IPE section cantilever beams, loaded at their shear center are solved numerically with the CFM for different loading types. Results are obtained for various values of collocation points and the compared with those of the available literature and excellent agreement is observed. This validation demonstrated the applicability and accuracy of the suggested method for lateral torsional buckling analysis of the problem in the hand. The critical lateral torsional moments are carried out for four different types of loading. It has been observed that type of the loading, length of the beam and type of the cross-section of the cantilever beams have important influences on the critical lateral torsional buckling loads. By increasing the length of the length of beam the critical lateral torsional buckling moment decreases. The IPE330 profile has the highest value capacity among the compared cases. The concentrated moment is the most critical loading among

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the compared loading cases while the uniformly distributed loading case is the safest case.

**Declaration of Ethical Standards**

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

**Credit Authorship Contribution Statement**

All the authors contribute in all stages and all sections of the paper.

**Declaration of Competing Interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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**Data Availability**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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