International Journal of Machine Learning and Cybernetics (2019) 10:1563–1579
https://doi.org/10.1007/s13042-018-0835-4

ORIGINAL ARTICLE

Interactive goal programming algorithm with Taylor series and interval type 2 fuzzy numbers

Hasan Dalman[1] · Mustafa Bayram[1]

Received: 23 April 2017 / Accepted: 31 May 2018 / Published online: 15 June 2018

Abstract
This paper presents an interactive fuzzy goal programming (FGP) approach for solving Multiobjective Nonlinear Programming Problems (MONLPP) with interval type 2 fuzzy numbers (IT2 FNs). The cost and time of the objective functions, and the requirements of each kind of resources are taken to be trapezoidal IT2 FNs. Here, the considered fuzzy problem is first transformed into an equivalent crisp MONLPP, and then the MONLPP is converted into an equivalent multiobjective linear programming problem (MOLPP). By using an algorithm based on Taylor series, this problem is also reduced into a single objective linear programming problem (LPP) which can be easily solved by Maple 2017 optimization toolbox. Finally, the proposed solution procedure is illustrated by a numerical example.

Keywords Fuzzy goals · Taylor series · Interval type 2 fuzzy sets · Multiobjective nonlinear programming · Interactive mechanism

1 Introduction

Most of the real-life problems are frequently represented by multiple and conflicting criteria. Such conditions are usually defined by optimizing multiple objective functions. Furthermore, the parameters are often included imprecise quantities due to various uncontrollable factors when forming real-world problems. In practical mathematical programming problems, a decision maker generally faces a state of uncertainty as well as complexity, due to various unknown factors. In general, it is required to optimize several nonlinear and conflicting objectives simultaneously. But these problems cannot be expressed and solved by conventional techniques due to uncertain information. Since fuzzy quantities are very convenient for modeling these type conditions, different fuzzy numbers are employed in the literature.

Fuzzy programming approach to linear programming with many objectives was investigated by Zimmermann [49]. He developed a fuzzy programming approach to solve the crisp multi-objective linear programming problem.

✉ Hasan Dalman
hsandalman@gmail.com

[1] Department of Computer Engineering, Istanbul Gelisim University, Avcılar, Istanbul, Turkey

In many practical optimization models such as in industrial planning, financial and corporate planning, marketing and media selection, etc., there exist many fuzzy and nonlinear production, planning and scheduling problems. Although intuitionistic fuzzy systems can be considered as an extension of classical fuzzy systems, there are shortages in the mapping. Because the classical fuzzy systems cannot fully describe the uncertainty, intuitionistic fuzzy systems are not equal to the fuzzy systems and have some deficiencies [36]. type-2 fuzzy sets are introduced by Zadeh et al. [47] as the extension of type-1 fuzzy sets. Moreover, type-2 fuzzy sets are designated by two memberships to illustrate more degrees. Since type-2 fuzzy sets have the advantage of modeling uncertain systems more accurately compared with type-1 fuzzy sets, the computational procedures are very difficult when the type-2 fuzzy sets are employed to solve the problems [5, 6]. Mendel et al. presented some definitions and concepts of IT2 fuzzy sets in [31]. Because the use of Interval type 2 (IT2) set instead of type 2 fuzzy sets reduce computational complexity, IT2 fuzzy sets are globally employed to decrease dimensions with remarkable relative illustrations, which are profoundly useful for computation and theoretical studies [32]. Thus, Interval type 2 Fuzzy Numbers (IT2 FNs) are very suitable for modeling real-world problems. However, IT2 fuzzy sets can be observed as a particular illustration of common type-2 fuzzy sets that all the values of secondary membership are equal to 1. So it not only expresses the uncertainty better than type-1 fuzzy sets but also diminishes the complexity. Moreover, IT2 FNs were used by many authors for decision-making problems [4, 6, 22, 38, 40], and many others.

Goal programming (GP) introduced by Charnes et al. [7] in 1955. As an extension of GP, the Fuzzy Goal Programming (FGP) approach first presented by Narasimhan [35]. Tiwari et al. [43] introduced the weighted additive FGP model that incorporates each goal's weight into the objective function, where the weights reveal the relative importance of the fuzzy goals. Mohamed [33] discussed the relationship between GP and fuzzy programming where the highest degree of each of the membership goals is achieved by minimizing over deviation variables. Recently, several researchers presented some novel methods in the field of FGP [21, 26, 39, 45]. Moreover, the FGP approach has been widely applied in many fields with diverse fuzzy numbers such as, bi-level programming [3], structural optimization problem [13], transportation [14], job evaluation [17], linear regression [19], project selection [23], game theory [24, 34], portfolio optimization [28], multi- level programming [25] and so on.

Since the Decision Maker (DM) plays an active role in the decision making process, the use of interactive approaches provides more options to select the best decision for DMs. It is possible that the most appropriate decision will be chosen with his/her common sense judgment [15]. Thus the interactive FGP mechanism ensures integration-oriented, adoption and learning characteristics by analyzing each possibility of a particular field of problems which are joined in logical order adopting an if- then rule [16].

The investigation on modeling and optimization of the MONLPPs with (IT2 FNs) are not only significant in the fuzzy programming theory but also have a great advantage in the application of the real-world practical problems of conflicting nature. Due to the computational complexity of MONLPPs with linear or nonlinear constraints, existing interactive and classical FGP approaches are also insufficient. There are many interesting approaches for solving different nonlinear optimization problems [1, 10–12, 29]. In order to deal with such types of problems, a linearization procedures based on Taylor series in this paper are presented. Thus, efficient solutions for the MONLPPs are obtained by reducing the computational complexity with the help of the linearization procedures.

Briefly, there are several gaps in the literature on MONLPPs.

1. To the best of our experiences, no work has been considered on MONLPPs with trapezoidal IT2 FNs under the linear and/ or nonlinear constraints.
2. There is no investigation involving MONLPPs with IT2 FNs by interactive approaches based on linearization procedures in the literature.
3. Conventional fuzzy goal programming approaches for the MONLPPs may not generate feasible solutions and/ or efficient solutions in all situations. Sometimes it is difficult for the DM to reach the desired levels using conventional approaches.

Therefore an interactive fuzzy goal programming (FGP) approach based on Taylor series is presented to addresses these shortcomings by achieving the highest degree of membership functions for MONLPPs. After the MONLPP with trapezoidal IT2 FNs is modeled, then the expected value function of trapezoidal IT2 FNs is applied to convert the IT2 fuzzy model into its crisp equivalent. Then aspiration level and the tolerance interval of each of objective function is determined by getting individual optimal solutions. Thereby the feasible region for the given problem is redefined by taking the upper and lower limits of decision variables. After these procedures, nonlinear membership function associated with each nonlinear objective is defined. Thus nonlinear membership functions are transformed into linear functions by using Taylor series around its own solution which is obtained by maximizing membership functions under redefined linear constraints. In this way, the problem is reduced to a single objective Linear Programming Problem (LPP) by using a FGP model, and then interactive solution procedures of a FGP model are proposed to obtain optimal solutions. Finally, numerical examples are given to demonstrate the feasibility of the presented solution procedures.

The paper is constructed as follows: Sect. 2 deals with some definitions and arithmetic operations on IT2 FNs. Section 3 deals with problem formulation and its solution procedures. In Sect. 4, numerical examples are given to illustrate the methodology. Finally, we concluded in Sect. 5.

2 Preliminaries

2.1 Interval type-2 fuzzy set

Definition 1 (Mendel et. al. [31]) Let $\tilde{A}$ be a type-2 fuzzy set, then $\tilde{A}$is defined as

$$\tilde{A} = \left\{ (x, \mu), \mu_{\tilde{A}}(x, \mu) \middle| \forall x \in X, \forall \mu \in J_x \subseteq [0, 1], 0 \le \mu_{\tilde{A}}(x, \mu) \le 1 \right\},$$

where J_x is the primary membership function of $x$in $[0, 1]$and $u$are the primary membership values. $X$is the universe of discourse and $\mu_{\tilde{A}}(x, \mu)$denotes the membership function of $\tilde{A}$. $\tilde{A}$ can be defined as $\tilde{A} = \int_{x\in X} \int_{\mu \in J_x} \mu_{\tilde{A}}(x, \mu) / (x, \mu)$,

$\mu \in J_x \subseteq [0,1]$,where $\iint$ denotes the union over all admissible x and u.

Definition 2 (Mendel et. al. [31]) If all $\mu_{\tilde{A}}(x,\mu) = 1$, then $\tilde{A}$called an IT2 fuzzy set i.e.$\tilde{A} = \int_{x\in X}\int_{\mu\in J_x} {}^{1}/(x,\mu)$, $\mu \in J_x \subseteq [0,1]$.

Uncertainty in the first memberships of a type-2 fuzzy set $\tilde{A}$consists of a bounded region that we call the footprint of uncertainty. It is the union of all first memberships.

The footprint of uncertainty is characterized by the upper membership function and the lower membership function, and are denoted by $\bar{\mu}_{\tilde{A}}$and $\underline{\mu}_{\tilde{A}}$ (Mendel et. al. [31]).

Definition 3 An IT2 FN is called a trapezoidal IT2 FN where the upper membership function and the lower membership function are both trapezoidal fuzzy numbers, i.e.,

$$A = \left(\bar{A}, \underline{A}\right) = \left(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4; H_1\left(\bar{A}\right), H_2\left(\bar{A}\right)\right), \left(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4; H_1\left(\underline{A}\right), H_2\left(\underline{A}\right)\right) \tag{1}$$

where $H_j\left(\underline{A}\right)$ and $H_j(\bar{A})$ denote membership values of the corresponding elements $\underline{a}_{j+1}$ and $\bar{a}_{j+1}$respectively. Further, Li et. al [27] defined the arithmetic operations of interval type-2 Fuzzy Set (See for detail, Li et. al [27]).

2.2 Defuzzification of trapezoidal interval type-2 fuzzy numbers

Let us consider a trapezoidal IT2 FNAcharacterized by Eq. (1). The expected value of A is determined as follows (Hu et. al [20]):

$$f(A) = \frac{1}{2}\left(\frac{1}{4}\sum_{i=1}^{4}\left(\underline{a}_i + \bar{a}_i\right)\right) \times \frac{1}{4}\left(\sum_{i=1}^{2}\left(H_i\left(\underline{A}\right) + H_i(\bar{A})\right)\right) \tag{2}$$

Assuming that A_1 and A_2 are two trapezoidal IT2 FNs, then we get $A_1 > A_2$ if and only if $f\left(A_1\right) > f\left(A_2\right)$.

When $\bar{a}_i = \underline{a}_i, (i = 1,2,3,4)$ and $H_1\left(\underline{A}\right) = H_2\left(\underline{A}\right) = H_1\left(\bar{A}\right) = H_2\left(\bar{A}\right) = 1$, the trapezoidal IT2 FN reduces to trapezoidal fuzzy number, just as $\tilde{A} = \left(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4\right)$ Then, the expected value of $\tilde{A}$is

$$f(\tilde{A}) = \left(\underline{a}_1 + \underline{a}_2 + \underline{a}_3 + \underline{a}_4/4\right)$$

3 Problem formulation

In real-world decision-making problems such as in production, planning, scheduling, etc. the existing quantity of resources as well as the production quantity or the demand quantity or the target over a period might be imprecise and possess various types of fuzziness due to many factors such as market price, existence of men power, perception with the operators, weather, rain, transportation, traffic, etc. Such types of models can be illustrated more practical by human decision process if its parameters are estimated to be imprecise in nature [41, 42]. Because type 1 and intuitionistic fuzzy systems cannot fully describe the uncertainty and use of IT2 fuzzy sets instead of type 2 fuzzy sets reduce computational complexity, IT2 FNs are very suitable for modeling real-world problems.

A traditional multiobjective nonlinear programming problem (MONLPP) can be modeled as:

$$\begin{aligned} & Max f_k(x),\ 1 \le k \le l' \\ & Min f_k(x),\ l' + 1 \le k \le l \\ & s.t. \begin{cases} g_j(x) \le b_j,\ 1 \le j \le m' \\ g_j(x) \ge b_j,\ m' + 1 \le j \le m'' \\ g_j(x) = b_j,\ m'' + 1 \le j \le m \\ x \ge 0 \end{cases} \end{aligned} \tag{3}$$

where $f_k(x),\ 1 \le k \le l$ and $g_j(x),\ 1 \le j \le m$ are the real valued linear and/or nonlinear functions and x is n-dimensional decision variable vector.

Assuming that the objective functions $\tilde{f}_k(x)$ and the resource constraint $\tilde{g}_j(x)$ are nonlinear with estimated coefficient parameters which are in terms of trapezoidal IT2 FNs. Then a MONLPP with IT2 FNs can be formulated as:

$$\begin{aligned} & Max \tilde{f}_k(x),\ 1 \le k \le l' \\ & Min \tilde{f}_k(x),\ l' + 1 \le k \le l \\ & s.t. \begin{cases} \tilde{g}_j(x) \le \tilde{b}_j,\ 1 \le j \le m' \\ \tilde{g}_j(x) \ge \tilde{b}_j,\ m' + 1 \le j \le m'' \\ \tilde{g}_j(x) = \tilde{b}_j,\ m'' + 1 \le j \le m \\ x \ge 0 \end{cases} \end{aligned} \tag{4}$$

where $\tilde{f}_k(x) = \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l},\ 1 \le k \le l$ and $\tilde{g}_j(x) = \sum_{r=1}^{r_j} \tilde{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l}, 1 \le j \le m.$ $\tilde{a}_{jr}$ and $\tilde{c}_{ki}$are estimated coefficient parameters with IT2 FNs. $x$is n- dimensional decision variable vector $x = \left(x_1, x_2, \ldots x_n\right)$.$\tilde{b}_j,\ 1 \le j \le m$ is IT2 fuzzy available resource vector.

Further assuming that $\tilde{a}_{jr}$, $1 \le j \le m$ is the required fuzzy requirements to produce each product; $\tilde{c}_{ki}$are estimated coefficient parameters of each unit cost, $x$is the planned production quantity for each product; α and β are real numbers, respectively; $\tilde{b}_{j\,(1\le j\le m')}$ is the estimated maximum amount of available resources with some enhancement which is acceptable by the DM; $\tilde{b}_{j\,(m'+1\le j\le m'')}$ is the estimated minimum planned amount of production with some tolerances which is acceptable by the DM; $\tilde{b}_{j\,(m''+1\le j\le m)}$ is the estimated amount of different resources with some errors which is allowable by the DM.

Thus, model (4) can be rewritten as follows:

$$
\begin{aligned}
&Max\,\tilde{f}_k(x) = \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l},\ 1 \le k \le l' \\
&Min\,\tilde{f}_k(x) = \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l},\ l'+1 \le k \le l \\
&s.t.\begin{cases}
\sum_{r=1}^{r_j} \tilde{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \le \tilde{b}_j,\ 1 \le j \le m' \\
\sum_{r=1}^{r_j} \tilde{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \ge \tilde{b}_j,\ m'+1 \le j \le m'' \\
\sum_{r=1}^{r_j} \tilde{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \tilde{b}_j,\ m''+1 \le j \le m \\
x_l \ge 0,\ 1 \le l \le n
\end{cases}
\end{aligned}
\tag{5}
$$

Employing the expected value function (2), problem (5) is further transformed into an equivalent crisp MONLPP as:

$$
\begin{aligned}
&Max\,f_k(x) \cong \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l},\ 1 \le k \le l' \\
&Min\,f_k(x) \cong \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l},\ l'+1 \le k \le l \\
&s.t.\begin{cases}
g_j(x) \cong \sum_{r=1}^{r_j} \widehat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \le \widehat{b}_j,\ 1 \le j \le m' \\
g_j(x) \cong \sum_{r=1}^{r_j} \widehat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \ge \widehat{b}_j,\ m'+1 \le j \le m'' \\
g_j(x) \cong \sum_{r=1}^{r_j} \widehat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \widehat{b}_j,\ m''+1 \le j \le m \\
x_l \ge 0,\ 1 \le l \le n
\end{cases}
\end{aligned}
\tag{6}
$$

where $\widehat{c}_{ki}$, $\widehat{a}_{jr}$ and $\widehat{b}_j$ are the expected functional values, i.e. $\widehat{c}_{ki} = f\left(\tilde{c}^I_{ki}\right)$, $1 \le k \le l$,$\widehat{a}_{jr} = f\left(\tilde{a}^I_{jr}\right)$, $1 \le j \le m$,respectively.

Theorem 1 *An efficient solution for problem* (6) *is efficient for problem* (5).

Proof Let $x = \left(x_1, x_2, \ldots x_n\right)$ be an efficient solution for the crisp problem (6). Therefore $x$is feasible for problem (6), i.e., the following hold.

$$
\begin{aligned}
&\textstyle\sum_{r=1}^{r_j} \widehat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \le \widehat{b}_j,\ 1 \le j \le m' \\
&\sum_{r=1}^{r_j} \widehat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \ge \widehat{b}_j,\ m'+1 \le j \le m'' \\
&\sum_{r=1}^{r_j} \widehat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \widehat{b}_j,\ m''+1 \le j \le m \\
&x_l \ge 0,\ 1 \le l \le n
\end{aligned}
$$

Since the expected value function$f$ is linear,

$$
\begin{aligned}
&\sum_{r=1}^{r_j} f\left(\tilde{a}_{jr}\right) \prod_{l=1}^{n} x_l^{\beta_l} \le f\left(\tilde{b}_j\right),\ 1 \le j \le m' \\
&\sum_{r=1}^{r_j} f\left(\tilde{a}_{jr}\right) \prod_{l=1}^{n} x_l^{\beta_l} \ge f\left(\tilde{b}_j\right),\ m'+1 \le j \le m'' \\
&\sum_{r=1}^{r_j} f\left(\tilde{a}_{jr}\right) \prod_{l=1}^{n} x_l^{\beta_l} = f\left(\tilde{b}_j\right),\ m''+1 \le j \le m \\
&x_l \ge 0,\ 1 \le l \le n
\end{aligned}
$$

which indicate that

$$
\begin{aligned}
&\sum_{r=1}^{r_j} \tilde{a}^I_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \le \tilde{b}^I_j,\ 1 \le j \le m' \\
&\sum_{r=1}^{r_j} \tilde{a}^I_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \ge \tilde{b}^I_j,\ m'+1 \le j \le m'' \\
&\sum_{r=1}^{r_j} \tilde{a}^I_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \tilde{b}^I_j,\ m''+1 \le j \le m
\end{aligned}
$$

Consequently, x is a feasible solution for problem (5).

On the other hand; since $x$is a feasible solution for problem (6), there does not exist any $\tilde{x} = \left(\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n\right)$ such that $f_k(\tilde{x}) \ge f_k(x)$, $1 \le k \le l'$ and $f_k(\tilde{x}) > f_k(x)$ for at least one index k and $f_k(\tilde{x}) \le f_k(x)$, $1 \le k \le l$ and $f_k(\tilde{x}) < f_k(x)$ at least one index k. So we have no $\tilde{x} = \left(\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n\right)$ such that $Max\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} \ge Max\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$, $1 \le k \le l'$ and $Max\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} > Max\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$ for at least one index k. Additionally, $Min\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} \le Min\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$, $l'+1 \le k \le l$ and $Min\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} < Min\ \sum_{i=1}^{l_i} \widehat{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$ for at least one index k.

Since the expected value function f is linear, we have no $\tilde{x}$ such that $Max \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} \geq Max \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$, $1 \leq k \leq l'$ and $Max \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} > Max \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$ for at least one index k. Additionally, $Min \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} \leq Min \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$, $l'+1 \leq k \leq l$ and $Min \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} < Min \sum_{i=1}^{l_i} \tilde{c}_{ki} \prod_{l=1}^{n} \tilde{x}_l^{\alpha_l}$ for at least one index k.

Therefore x is efficient solution of problem (5).

3.1 Goal programming

This method was first presented by Charnes and Cooper [8]. The goal programming is to minimize the distance between objectives ($f_k = (f_1, f_2, ...f_l)$) and aspiration levels ($\bar{f}_k = (\bar{f}_1, \bar{f}_2, ...\bar{f}_l)$), which are defined by the decision maker. Thus, positive and negative deviational variables can be defined as follows:

$$d_k^+ = Max\left(0, \bar{f}_k - f_k\right) = \frac{1}{2}\left[\bar{f}_k - f_k + \left|\bar{f}_k - f_k\right|\right], \quad 1 \leq k \leq l,$$

$$d_k^- = Max\left(0, f_k - \bar{f}_k\right) = \frac{1}{2}\left[f_k - \bar{f}_k + \left|f_k - \bar{f}_k\right|\right], \quad 1 \leq k \leq l.$$

Then, minimizing the distance between $f_k$and $\bar{f}_k$ give rise to minimizing d_k^+ when$f_k \leq \bar{f}_k$is needed in a minimization problem. On the other hand, minimizing the distance between $f_k$and $\bar{f}_k$ give rise to minimizing d_k^- when$f_k \geq \bar{f}_k$is needed in a maximization problem [33]. In this condition, by utilizing the min–max form of goal programming, problem (6) turns into the following model:

$$Min \quad g\left\{d_k^-, d_k^+\right\}$$
$$s.t. \begin{cases} f_k(x) + d_k^- - d_k^+ = \bar{f}_k, \quad 1 \leq k \leq l' \\ f_k(x) + d_k^- - d_k^+ = \bar{f}_k, \quad l'+1 \leq k \leq l \\ \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \leq \hat{b}_j, \ 1 \leq j \leq m' \\ \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \geq \hat{b}_j, \ m'+1 \leq j \leq m'' \\ \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \hat{b}_j, \ m''+1 \leq j \leq m \\ d_k^-, \ d_k^+ \geq 0 \quad d_k^- \times d_k^+ = 0, \ 1 \leq k \leq l \\ x_l \geq 0, \quad 1 \leq l \leq n \end{cases} \tag{7}$$

where $g\left\{d_k^-, d_k^+\right\} = d_k^-$ in the event of maximizing f_k, $g\left\{d_k^-, d_k^+\right\} = d_k^+$ in the event of minimizing f_k. $\bar{f}_k$, $1 \leq k \leq l$is the aspiration level of each objective. d_k^-, $d_k^+ \geq 0$ are the negative and positive deviations from the aspired levels, respectively.

The model (7) is further transformed to the following programming problem.

$$Min \ \phi$$
$$s.t. \begin{cases} \phi \geq g\left\{d_k^-, d_k^+\right\} \\ f_k(x) + d_k^- - d_k^+ = \bar{f}_k, \quad 1 \leq k \leq l' \\ f_k(x) + d_k^- - d_k^+ = \bar{f}_k, \quad l'+1 \leq k \leq l \\ \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \leq \hat{b}_j, \ 1 \leq j \leq m' \\ \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \geq \hat{b}_j, \ m'+1 \leq j \leq m'' \\ \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \hat{b}_j, \ m''+1 \leq j \leq m \\ d_k^-, \ d_k^+ \geq 0 \quad d_k^- \times d_k^+ = 0, \ 1 \leq k \leq l \\ x_l \geq 0, \quad 1 \leq l \leq n \end{cases} \tag{8}$$

where $\bar{f}_k$, $1 \leq k \leq l$represents the aspiration level for each objective function.

3.2 Construction of fuzzy multiobjective nonlinear programming model

In a multiobjective programming, if an imprecise aspiration level is injected to each of the objectives, then these fuzzy objectives are expressed as fuzzy goals. Let s_k be the aspiration level assigned to the k^{th} objective $f_k(x)$. Then the fuzzy goals are $f_k(x) \gtrsim s_k$ for the maximization type of objective and $f_k(x) \lesssim s_k$ for the minimization type of objective where $\gtrsim$ and $\lesssim$ represent the fuzzified inequalities. Therefore, the fuzzy multiobjective nonlinear goal programming problem for MNOLPP (6) can be formulated as follows:

$$f_k(x) = \sum_{i=1}^{l_i} \hat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} \gtrsim s_k, 1 \leq k \leq l'$$
$$f_k(x) = \sum_{i=1}^{l_i} \hat{c}_{ki} \prod_{l=1}^{n} x_l^{\alpha_l} \lesssim s_k, \ l'+1 \leq k \leq l$$
$$s.t. \begin{cases} g_j(x) \cong \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \leq \hat{b}_j, \ 1 \leq j \leq m' \\ g_j(x) \cong \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} \geq \hat{b}_j, \ m'+1 \leq j \leq m'' \\ g_j(x) \cong \sum_{r=1}^{r_j} \hat{a}_{jr} \prod_{l=1}^{n} x_l^{\beta_l} = \hat{b}_j, \ m''+1 \leq j \leq m \\ x_l \geq 0, \quad 1 \leq l \leq n \end{cases} \tag{9}$$

Now, consider the k^{th} fuzzy goal $f_k(x) \gtrsim s_k,\ 1 \le k \le l'$ for problem (9). Its membership function can be defined as follows:

$$\mu_k\big(f_k(x)\big)_{1\le k\le l'.} \cong \begin{cases} 1 & f_k(x) \ge s_k \\ \dfrac{f_k(x)-l_k}{\left(s_k-l_k\right)}, & l_k \le f_k(x) \le s_k \\ 0 & l_k \ge f_k(x) \end{cases} \tag{10}$$

where $l_k$is the lower tolerance limit for the k^{th}fuzzy goal and $\left(l_k, s_k\right)$is the tolerant interval which is subjectively selected, respectively. Furthermore, the tolerant interval for $f_k(x) \gtrsim s_k,\ 1 \le k \le l'$ are determined as follows:

$$s_k = Max\left\{f_k(x), x \in X\right\} \text{ and } l_k = Min\left\{f_k(x), x \in X\right\}, 1 \le k \le l'. \tag{11}$$

Similarly, consider the k^{th} fuzzy goal of $f_k(x) \lesssim s_k$, $l'+1 \le k \le l$. Its membership function can be defined as follows:

$$\mu_k\big(f_k(x)\big)_{l'+1\le k\le l.} = \begin{cases} 1 & f_k(x) \le l_k \\ \dfrac{s_k-f_k(x)}{s_k-l_k}, & l_k \le f_k(x) \le s_k \\ 0 & l_k \le f_k(x) \end{cases} \tag{12}$$

where $s_k$is the upper tolerance limit for the k^{th} fuzzy goal and $\left(l_k, s_k\right)$ the tolerant interval which is subjectively selected, respectively. The tolerant interval for $f_k(x) \lesssim s_k,\ l'+1 \le k \le l.$ are determined as follows:

$$s_k = Max\left\{f_k(x), x \in X\right\} \text{ and } l_k = Min\left\{f_k(x), x \in X\right\}, l'+1 \le k \le l. \tag{13}$$

3.3 Linearization nonlinear membership and constraint functions using the Taylor series

Taylor series is an expansion of a function into an infinite series of a variable x or into a finite series with a remainder (error) term [2]. The coefficients of the expansion include the consecutive derivatives of the function and also this function has a n^{th} derivative in the interval of expansion.

Let $a \le \tau \le b$ be the interval of expansion. Then the remainder (error) term in Lagrangian form is given as: $p_n = \left(\frac{x-a}{n!}\right)^n f^{(n)}(\tau)$ where a is the reference point.

$f^{(n)}(\tau)$ represents n^{th}derivative around a. When $\lim_{n\to\infty} p_n = 0$, the expanding function is obtained as $f(x) = \sum_{n=0}^{\infty} \left(\frac{x-a}{n!}\right)^n f^{(n)}(a)$.

For $n = 1$, the expanding function becomes $f(x) = f(a) + f'(a)(x-a)$ This is called the first degree Taylor Polynomial of $f(x)$.

Therefore MONLPP (6) will transform into an equivalent multiobjective linear programming problem (MOLPP) using Taylor series approach. Taylor series approach has been applied to nonlinear programming problems in literatures [9, 17, 37, 44]. Because it is very difficult to solve nonlinear problems, this approximation reduces the computational burden in the problem. But these methods may not always be enough to obtain efficient solutions alone. Moreover, these methods are not applicable to nonlinear constrained problems. In order to solve these types of problems and obtain effective results, the nonlinear constraints of problems are first reduced to linear inequalities with the help of decision variables.

Note that the feasible region for a programming problem is the whole set of alternatives for the decision variables over which the objective function is to be optimized.

These linear inequalities are used to optimize the nonlinear membership and constraint functions, and provide the best starting points for using a Taylor series approach.

The suggested solution procedure can be continued as follows:

- (Assuming that the feasible region is nonempty) solve MONLPP (6) as a single objective programming problem, considering each time only one objective as the objective function and ignoring all others.
- Let the solutions obtained be $x_l = \left(x_1, x_2, \ldots, x_n\right)$. Compute the best and worst values of each objective function at each solution $x_l = \left(x_1, x_2, \ldots, x_n\right)$.
- By employing solutions $x_l = \left(x_1, x_2, \ldots, x_n\right)$, reconstruct the feasible region as the limit of each of decision variables, i.e., find $\underset{-}{x}_l \le x_l \le \bar{x}_l$ for each decision variable using $\underset{-}{x}_l = Min\left\{x_l\right\} \in x_1$ and $\bar{x}_l = Max\left\{x_l\right\} \in x_l$.
- Determine tolerant intervals, and then construct the membership functions corresponding to each objective function as defined in (10) and (12).
- Since $\mu_k\big(f_k(x)\big),\ 1 \le k \le l$ means the satisfaction of the decision makers with the result, this means that the degree of satisfaction obtained aims to achieve the best value of function $f_k(x),\ 1 \le k \le l$. Therefore, find $\tilde{x}_l^* = \left(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\right)$, which is the solution that is employed to maximize the k^{th} nonlinear membership function $\mu_k\big(f_k(x)\big)$ associated with k^{th} nonlinear objective under the linear inequalities, i.e.,

$$Max\ \mu_k\big(f_k(x)\big),\ 1 \le k \le l$$

$$s.t. \begin{cases} \underset{-}{x}_l \le x_l \le \bar{x}_l, & 1 \le l \le n \\ \mu_k\big(f_k(x)\big), & 1 \le k \le l \\ \lambda \in [0,1] \\ x_l \ge 0, & 1 \le l \le n \end{cases}$$

where $\mu_k\big(f_k(x)\big)$ is the k^{th} nonlinear membership function associated with k^{th} nonlinear objective.

- Because Taylor series approach generally provides a relatively good approximation to a differentiable function but only around a given point, and not over the entire domain. Then, transform nonlinear membership and constraint functions by using the first degree Taylor polynomial approach around the solution $\tilde{x}_l^* = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$, as follows:

$$\begin{aligned} \tilde{\mu}_k\big(f_k(x)\big)_{1\le k\le l} &\cong \left[\frac{\mu_k\big(f_k(\tilde{x}_l^*)\big)}{\partial x_1}\bigg|_{\tilde{x}_l^*}(x_1-\tilde{x}_1) + \frac{\mu_k\big(f_k(\tilde{x}_l^*)\big)}{\partial x_2}\bigg|_{\tilde{x}_l^*}(x_2-\tilde{x}_2) + \ldots + \frac{\mu_k\big(f_k(\tilde{x}_l^*)\big)}{\partial x_n}\bigg|_{\tilde{x}_l^*}(x_n-\tilde{x}_n) \right] \\ \tilde{g}_j(x)_{1\le j\le m} &\cong \left[\frac{g_j(\tilde{x}_l^*)}{\partial x_1}\bigg|_{\tilde{x}_l^*}(x_1-\tilde{x}_1) + \frac{g_j(\tilde{x}_l^*)}{\partial x_2}\bigg|_{\tilde{x}_l^*}(x_2-\tilde{x}_2) + \ldots + \frac{g_j(\tilde{x}_l^*)}{\partial x_n}\bigg|_{\tilde{x}_l^*}(x_n-\tilde{x}_n) \right] \end{aligned} \tag{14}$$

In (14), $\tilde{\mu}_k\big(f_k(x)\big)_{1\le k\le l}$ are linear membership functions which are equivalent to nonlinear membership functions associated with each objective. Also, $\tilde{g}_j(x)_{1\le j\le m}$ are linear constraints associated with each objective.

Thus linear membership $\tilde{\mu}_k\big(f_k(x)\big)_{1\le k\le l}$ and constraint functions $\tilde{g}_j(x)_{1\le j\le m}$ approximate the nonlinear functions $\mu_k\big(f_k(x)\big)$ and $g_j(x)$ around the solution $\tilde{x}_l^* = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$.

It should be noted that if there are more than one nonlinear membership functions, the constraint functions can be expanded around the solution of any membership function.

Hence, applying the Max–Min form presented by Zadeh with the membership functions represented in (10) and (12), the fuzzy programming problem corresponding to problem (6) is formulated as follows:

$$Max\ \lambda$$

$$s.t.\begin{cases} \lambda \le \tilde{\mu}_k\big(f_k(x)\big),\ 1\le k\le l' \\ \lambda \le \tilde{\mu}_k\big(f_k(x)\big),\ l'+1\le k\le l \\ \tilde{g}_j(x)\le 0,\ 1\le j\le m' \\ \tilde{g}_j(x)\ge 0,\ m'+1\le j\le m'' \\ \tilde{g}_j(x)= 0,\ m''+1\le j\le m \\ \lambda\in[0,1] \\ x_l\ge 0,\ 1\le l\le n \end{cases} \tag{15}$$

where λ is control parameter. $\tilde{\mu}_k\big(f_k(x)\big)$ for each $1\le k\le l$ is linear membership function. $\tilde{g}_j(x)$ for each $1\le j\le m$ is the linear constraint functions.

3.4 A fuzzy goal programming model to multiobjective linear programming problem

The FGP approach was originally introduced by Zimmermann [49] in 1978. He employed the concept of membership functions. Tiwari et al. [43] proposed a weighted additive model that associates each goal's weight with the objective function, where weights show the relative importance of the fuzzy goals. Mohamed [33] suggested a FGP approach, which is introduced in the general form of FGP model. In [48], Mohamed's approach used to present a FGP approach for solving multiobjective programming problems and then Gupta and Bhattacharjee [18] formulated two FGP approaches for solving multiobjective programming problems. But these approaches generally do not yield effective results. They do not offer alternative solutions to make effective decisions. In this section, an approach is proposed to overcome these deficiencies.

According to paper [33], the highest degree of membership function is always 1 and therefore the nonlinear membership functions in (10) and (12) can be constructed as the following nonlinear membership goals;

$$\frac{f_k(x)-l_k}{(s_k-l_k)} + d_k^- - d_k^+ = 1,\ \ 1\le k\le l' \text{ (for maximizing objective)},$$

$$\frac{s_k-f_k(x)}{(s_k-l_k)} + d_k^- - d_k^+ = 1,\ \ l'+1\ \le k\le l \text{ (for minimizing objective)},$$

Where $d_k^- \times\ d_k^+ = 0$ and $d_k^- \ge 0$ and $d_k^+ \ge 0$ stand for the negative and positive deviations from aspired levels, respectively. (l_k, s_k) is the tolerant interval.

After membership functions (10) and (12) are linearized using the first degree Taylor polynomial (14), the above nonlinear membership goal functions can be reconstructed as the following linear membership goals.

$$\tilde{\mu}_k\big(f_k(x)\big) + d_k^- - d_k^+ = 1,\ \ 1\le k\le l' \tag{16}$$

$$\tilde{\mu}_k\big(f_k(x)\big) + d_k^- - d_k^+ = 1,\ \ l'+1\ \le k\le l \tag{17}$$

By applying model (8) of goal programming model to the fuzzy model (9), we obtain the following fuzzy goal programming model.

$$Min\ \beta$$

$$s.t.\begin{cases} \beta \geq d_k^-,\ 1 \leq k \leq l \\ \tilde{\mu}_k\big(f_k(x)\big) + d_k^- - d_k^+ = 1,\ 1 \leq k \leq l' \\ \tilde{\mu}_k\big(f_k(x)\big) + d_k^- - d_k^+ = 1,\ l'+1 \leq k \leq l \\ \tilde{g}_j(x) \leq 0,\ 1 \leq j \leq m' \\ \tilde{g}_j(x) \geq 0,\ m'+1 \leq j \leq m'' \\ \tilde{g}_j(x) = 0,\ m''+1 \leq j \leq m \\ d_k^-, d_k^+ \geq 0,\ d_k^- \times d_k^+ \geq 0,\quad 1 \leq k \leq l, \\ x_l \geq 0,\ \beta \in [0,1] \end{cases} \tag{18}$$

where d_k^- and d_k^+ represent the negative and positive deviations from the aspired levels, respectively.

Theorem 2 *The fuzzy model* (18) *is equivalent to model* (15).

Proof We rewrite the model (15) as:

$$Min\ 1 - \lambda$$

$$s.t.\begin{cases} 1-\lambda \geq 1 - \tilde{\mu}_k\big(f_k(x)\big),\ 1 \leq k \leq l' \\ 1-\lambda \geq 1 - \tilde{\mu}_k\big(f_k(x)\big),\ l'+1 \leq k \leq l \\ \tilde{g}_j(x) \leq 0,\ 1 \leq j \leq m' \\ \tilde{g}_j(x) \geq 0,\ m'+1 \leq j \leq m'' \\ \tilde{g}_j(x) = 0,\ m''+1 \leq j \leq m \\ \lambda \in [0,1] \\ x_l \geq 0,\ 1 \leq l \leq n \end{cases} \tag{19}$$

Since $\lambda \in [0,1]$ is a control variables of the membership functions, $\lambda \leq 1$ which shows that $1-\lambda \geq 0$. Let $\beta = 1-\lambda$, and then model (19) can be converted to the following fuzzy programming model.

$$Min\ \beta$$

$$s.t.\begin{cases} \beta \geq 1 - \tilde{\mu}_k\big(f_k(x)\big),\ 1 \leq k \leq l' \\ \beta \geq 1 - \tilde{\mu}_k\big(f_k(x)\big),\ l'+1 \leq k \leq l \\ \tilde{g}_j(x) \leq 0,\ 1 \leq j \leq m' \\ \tilde{g}_j(x) \geq 0,\ m'+1 \leq j \leq m'' \\ \tilde{g}_j(x) = 0,\ m''+1 \leq j \leq m \\ \beta \in [0,1] \\ x_l \geq 0,\ 1 \leq l \leq n \end{cases} \tag{20}$$

To formulate the above fuzzy problem as a FGP model, let us define the negative and positive deviational variables;

$d_k^- = Max\{0, 1 - \tilde{\mu}_k(f_k(x))\}$ for $1 \leq k \leq l'$ and $l'+1 \leq k \leq l$.

$d_k^+ = Max\{0, \tilde{\mu}_k(f_k(x)) - 1\}$ for $1 \leq k \leq l'$ and $l'+1 \leq k \leq l$.

Thus from model (20), we obtain $\beta \geq d_k^-$ where $\tilde{\mu}_k(f_k(x)) + d_k^- - d_k^+ = 1$for each $1 \leq k \leq l$.

In this case, model (20) can be reconstructed as:

$$Min\ \beta$$

$$s.t.\begin{cases} \tilde{\mu}_k\big(f_k(x)\big) + d_k^- - d_k^+ = 1,\ 1 \leq k \leq l' \\ \tilde{\mu}_k\big(f_k(x)\big) + d_k^- - d_k^+ = 1,\ l'+1 \leq k \leq l \\ \beta \geq d_k^-,\quad 1 \leq k \leq l \\ \tilde{g}_j(x) \leq 0,\ 1 \leq j \leq m' \\ \tilde{g}_j(x) \geq 0,\ m'+1 \leq j \leq m'' \\ \tilde{g}_j(x) = 0,\ m''+1 \leq j \leq m \\ d_k^-, d_k^+ \geq 0,\ d_k^- \times d_k^+ \geq 0,\quad 1 \leq k \leq l \\ \beta \in [0,1] \\ x_l \geq 0,\ 1 \leq l \leq n \end{cases} \tag{21}$$

Thus the proof is completed.

Since any positive deviation from 1 shows the full achievement of the membership value, it is sufficient to minimize the negative deviation variable from 1. Therefore the positive deviational variables in the linear membership goals are unnecessary [18]. Thus membership goals (16) and (17) are reconstructed as:

$$\tilde{\mu}_k\big(f_k(x)\big) + d_k^- = 1,\quad 1 \leq k \leq l' \tag{22}$$

$$\tilde{\mu}_k\big(f_k(x)\big) + d_k^- = 1,\quad l'+1 \leq k \leq l \tag{23}$$

Where $d_k^-(\geq 0)$ for each $1 \leq k \leq l$ represents the negative deviations from aspired levels.

Consequently, the linear FGP model (21) can be reconstructed as:

$$Min\ \beta$$

$$s.t.\begin{cases} \tilde{\mu}_k\big(f_k(x)\big) + d_k^- = 1,\ 1 \leq k \leq l' \\ \tilde{\mu}_k\big(f_k(x)\big) + d_k^- = 1,\ l'+1 \leq k \leq l \\ \beta \geq d_k^-,\ 1 \leq k \leq l \\ \tilde{g}_j(x) \leq 0,\ 1 \leq j \leq m' \\ \tilde{g}_j(x) \geq 0,\ m'+1 \leq j \leq m'' \\ \tilde{g}_j(x) = 0,\ m''+1 \leq j \leq m \\ d_k^- \geq 0,\quad 1 \leq k \leq l \\ \beta \in [0,1] \\ x_l \geq 0, 1 \leq l \leq n \end{cases} \tag{24}$$

where $d_k^- \geq 0, 1 \leq k \leq l$ represents the negative deviations from aspired levels. $\tilde{g}_j(x)$ for each $1 \leq j \leq m$ is the linear constraints. $\tilde{\mu}_k\left(f_k(x)\right), 1 \le k \le l$ are linear membership functions.

Theorem 3 *If $\tilde{x} \in X$ is an optimal solution of FGP problem* (24), *then $\tilde{x}$ is an efficient solution to MONLPP* (6).

Proof Assume that $\tilde{x}$ is not an efficient solution of problem (6). So, there exist another solution $x \in X$ such that $\tilde{\mu}_k\left(f_k(x)\right) \ge \mu_k\left(f_k(\tilde{x})\right)$ for all $1 \le k \le l$ and $\tilde{\mu}_l\left(f_l(x)\right) > \tilde{\mu}_l\left(f_l(\tilde{x})\right)$ with at least one index l.Finally, $\sum_{k=1}^{l} \tilde{\mu}_k\left(f_k(x)\right) \ge \sum_{k=1}^{l} \tilde{\mu}_k\left(f_k(\tilde{x})\right)$ and $\tilde{x}$ is not an optimal solution to the problem, a contradiction that complete the proof.

Theorem 4 *Let $\tilde{x} \in X$ be A fuzzy efficient solution of problem* (6). *Then $\tilde{x} \in X$ is a Pareto optimal solution to MONLPP* (5).

Proof From proof of theorem 3, fuzzy efficiency of $\tilde{x}$ to MONLPP (6) indicates that there does not exist a solution $x \in X$ such that $\tilde{\mu}_k\left(f_k(x)\right) \ge \mu_k\left(f_k(\tilde{x})\right)$ for all $1 \le k \le l'$ and $\tilde{\mu}_l\left(f_l(x)\right) > \tilde{\mu}_l\left(f_l(\tilde{x})\right)$ with at least one index l. Actually, it must be $\tilde{\mu}_k\left(f_k(\tilde{x})\right) \ge \mu_k\left(f_k(x)\right)$ is equivalent to say that $f_k(x) \le f_k(\tilde{x})$ for all $1 \le k \le l'$ and $f_k(x) \ge f_k(\tilde{x})$ for all $l' + 1 \le k \le l$ which is seems from the illustration of membership functions in Eqs. (10) and (12). Since there is not a solution that conflicts the fuzzy efficiency of $\tilde{x}$ to problem (6) and then there is not a solution that conflicts the fuzzy efficiency of $\tilde{x}$ to problem (5). So the theorem proved.

Hence, solving the FGP problem (24), the Pareto optimal solution of the MNLOPP with IT2 FNs is found.

3.5 Interactive fuzzy goal programming approaches based on Taylor series for MNLOPP with IT2 FNs

By the aid of the interactive algorithm, interactive fuzzy decision making approaches have been widely studied to develop the flexibility and robustness of multiobjective decision making techniques. They give learning scheme concerning the system, while the DM can learn to identify best solutions and the corresponding importance of component in the system [30, 46].

The principal interest of interactive procedures is that the DM controls the search way through the solution procedure. Hereby the efficient solution is obtained with his/her preferences. Thus, interactive fuzzy goal programming approaches in this paper is presented to achieve the highest degree for the membership functions.

The complete suggested solution procedures can be summarized as follows.

Step 1 Construct the mathematical model of MONLPP with IT2 FNs (5).

Step 2 By using expected value function (2), obtain the corresponding crisp MONPP (6).

Step 3 Solve MONLPP (6) as a single objective problem, considering each time only one objective as the objective function and ignoring all others. Find the solutions.

Step 4 Compute the best and worst bounds of each of objective function

Step 5 Reduce the nonlinear constrained region to the linear inequalities using the limit of each of decision variable.

Step 6 Determine tolerant intervals for each objective.

Step 7 Then construct the nonlinear membership functions as defined in (10) and (12), respectively.

Step 8 Determine $\tilde{x}_l^* = \left(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\right)$, which is the solution that is employed to maximize the k^{th} nonlinear membership function associated with k^{th} nonlinear objective under the obtained feasible region. Then linearize each nonlinear membership and constraint functions using Taylor polynomial approach (14) at $\tilde{x}_l^* = \left(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\right)$.

Step 9 Define the linear membership goals (22) and (23), respectively.

Step 10 Construct the linear FGP model (24), then solve it to obtain the candidate optimal solution of MONLPP with IT2 FNs (5).

Step 11 If the decision-maker is satisfied by the current solution in Step 10, go to Step 12, else go to Step 13.

Step 12 The current solution is the optimal solution for the MONPP with IT2 FNs.

Step 13 Hold the candidate solution to linearize the nonlinear membership functions, and compare the lower (upper) tolerance limit of each objective with the new value of the objective function. If the new value is higher (lower) than the lower tolerance limit, take this as a new lower (upper) tolerance limit. If else, hold the old one as is and then go to step (7).

It has pointed out that the algorithm finishes (at Step 11) if the decision maker admits the obtained solution as the optimal solution; or if there is no notable change in the objective and membership function values after further changes; or if the modification of the tolerant intervals produce an infeasible solution.

4 Numerical examples

Example 1 A manufacturing factory is going to produce three kinds of products A, B and C in a specified period (say one month). The production of each of product require three kinds raw materials R_1,R_2 and R_3. Thus, to produce each unit of A, the requirements of R_1,R_2 and R_3 are around

3, 4 and 3 units, respectively. To produce each unit B, the respective requirements of R_1,R_2 and R_3 are around 4, 3 and 3 units and that for that of product C are around 3, 4 and 3 units, respectively. The required existing resource R_1 and R_2 are around 80 and 70 units, respectively. But there are about 20 and 10 units additional safety store for emergency use which are administrated by the manager. For better quality of the products, at least 60 units of resource R_3 has to be employed with tolerance about 10 units can be allowed by governance. In addition, the conjectural time requirements in producing each unit of products $\tilde{t}_1$,$\tilde{t}_2$ and $\tilde{t}_3$ are 3, 3 and 4 h, respectively. Let the planned production quantities of A, B and C be $\left(x_1x_2, x_1^2, x_1\right)$, $\left(x_2, x_1x_2, x_2\right)$ and $\left(x_3^2, x_3, x_3\right)$, respectively. Moreover, assume that the unit cost and sale's price of A, B and C are $UC_1 = \tilde{c}_1$, $UC_2 = \tilde{c}_2$,$UC_3 = \tilde{c}_3$, $US_1 = \tilde{s}_1/x_1^{1/a_1}$, $US_2 = \tilde{s}_2/x_2^{1/a_2}$ and $US_3 = \tilde{s}_3/x_1^{1/a_3}$ where

$a_1 = 2; a_2 = 2; a_3 = 3$ are real numbers. The decision maker expects to maximize whole profit and minimize the integral time requirement.

respectively. Also, let $\tilde{s}_1$ = ((20,22,24,27;0.95,0.98),(21,23,25, 26;0.97,0.99)), $\tilde{s}_2$ = ((21,23,24,28; 0.94,0.99), (22,23,25,26; 0.95,0.97)), $\tilde{s}_3$ = ((22,23,24,26;0.94, 0.97), (22,24,25,26; 0.95,0.97)); $\tilde{c}_1$ = ((1,3,3,4;0.90,0.91), (1,2,4,5;0.92,0.93)), $\tilde{c}_2$ = ((2,3,5,5;0.91,0.94), ((2,3,6,8;0.93, 0.95)), $\tilde{c}_3$ = ((2,4,4, 5;0.90,0.91),(3,4,5,5;0.92,0.93)).

(Step 1): Then the mathematical model of MONLPP with IT2 FNs can be formulated as follows, based on model (5):

$$
\begin{aligned}
&Max f_2(x) = \tilde{s}_1 x_1^{1-1/a_1} - \tilde{c}_1 x_1 + \tilde{s}_2 x_2^{1-1/a_2} - \tilde{c}_2 x_2 + \tilde{s}_3 x_3^{1-1/a_3} - \tilde{c}_3 x_3 \\
&Min f_2(x) = \tilde{3}x_1 + \tilde{3}x_2 + \tilde{4}x_3, \\
&s.t.\begin{cases} \tilde{3}x_1x_2 + \tilde{4}x_2 + \tilde{3}_{13}x_3^2 \le \tilde{b}_1, \\ \tilde{4}x_1^2 + \tilde{3}x_1x_2 + \tilde{4}x_3 \le \tilde{b}_2, \\ \tilde{3}x_1 + \tilde{3}x_2 + \tilde{3}x_3 \ge \tilde{b}_3 \\ x_1, x_2, x_3 \ge 0 \end{cases}
\end{aligned} \tag{25}
$$

(Step 2): By applying the expected value function (2) to problem (25), the equivalent deterministic model based on model (6) can be formulated as follows:

$$
\begin{aligned}
&Max\ f_1(x) = 22.854x_1^{(1/2)} \text{-} 2.631x_1 + 23.100x_2^{(1/2)} - 3.963x_2 + 22.980x_3^{(2/3)} \text{-} 3.660x_3 \\
&Min\ f_2(x) = 2.753x_1 + 3.609x_2 + 3.889x_3, \\
&s.t.\begin{cases} g_1(x) = 3.889x_1x_2 + 4.123x_2 + 3.660x_3^2 \le 87.364, \\ g_2(x) = 4.123x_1^2 + 3.889x_1x_2 + 3.889x_3 \le 75.369, \\ g_3(x) = 3.660x_1 + 3.889x_2 + 3.660x_3 \ge 59.259 \\ x_1, x_2, x_3 \ge 0 \end{cases}
\end{aligned} \tag{26}
$$

Let us assume that all the imprecise parameters estimated by the decision maker to be the trapezoidal IT2 FNs. To produce each unit of A, B and C, the requirements of R_1, R_2 and $R_3$are estimated as: $\tilde{3}$ = ((3, 3, 4, 5; 0.90, 0.91), (4, 4, 5, 6; 0. 92,0. 93)); $\tilde{4}$ = ((3,5,5,7;0.90,0.98), (2,4,4,5;0 .92,0.97)); $\tilde{3}$ = ((2,4,4,5;0.90,0.91), (3,4,5,5;0.92,0.93)); $\tilde{4}$ = ((3, 5, 5, 7; 0.90, 0.98),(2, 4, 4, 5; 0.92, 0.97))$\tilde{3}$ = ((3,3,4, 5;0.90,0.91),(4,4,5,6,0.92,0.93)); $\tilde{3}$ = ((3,3,4,5;0.90,0.91), (4,4,5,6;0.92,0.93)); $\tilde{3}$ = ((2, 4, 4, 5; 0.90, 0.91), (3, 4, 5, 5; 0.92, 0.93)); $\tilde{4}$ = ((3,3,4,5;0.90,0.91), (4,4,5,6;0.92,0.93)) ; $\tilde{3}$ = ((2,4,4,5;0.90,0.91),(3,4,5,5;0.92,0.93)), respectively. The required existing resources for R_1,R_2 and R_3 are estimated as:$\tilde{90}$ = ((80,95,70,90;0.96,0.99)(90,80,100,110;0.97,0.99)), $\tilde{70}$ = ((90,50,70,70;0.95,0.98), (90,80,80,90;0.97,0.99)); $\tilde{60}$ = ((50,60,60,70;0.95,0.99), (50,60,60,70;0.94,0.99)),respectively. The conjectural time requirements in producing each unit of products $\tilde{t}_1$,$\tilde{t}_2$ and $\tilde{t}_3$are estimated as: $\tilde{3}$ = ((2,3,4,5;0.95,0.99), (1,2,3,3;0.92,0.97)), $\tilde{3}$ = ((3,4,5,6;0.96,0.98), (1,2,3,3;0.95, 0.96)) ; $\tilde{4}$ = ((3,3,4,5;0.90,0.91), (4,4,5,6; 0.92,0.93)),

(Step 3): then, solving $f_1(x)$ and $f_2(x)$ as a single objective nonlinear programming problem under the given constraints using Maple 2017 nonlinear optimization toolbox, the solution of each of objective is shown in Table 1.

(Step 4): The objective function values obtained are as follows: $Max\ f_1(x) = 76.168$, $Min\ f_1(x) = 29.785$, $Max\ f_2(x) = 77.653$, $Min\ f_2(x) = 54.699$.

Based on (11)–(13), the best and worst bounds for each objective are determined as follows:$29.785 \le f_1 \le 76.168$and $54.699 \le f_2 \le 77.653$, respectively.

(Step 5): Since the feasible set is the set of all points that are possible solutions, the nonlinear constrained region of (26) is reduced to the following linear inequalities by employing the individual optimal solutions.

$\underline{x}_1 = Min\{x_1\} = 0$ and $\bar{x}_1 = Max\{x_1\} = 0.457 \Rightarrow 0 \le x_1 \le 0.457$,

$\underline{x}_2 = Min\{x_2\} = 12.541$ and $\bar{x}_2 = Max\{x_2\} = 20.862 \Rightarrow 12.344 \le x_2 \le 20.862$,

Table 1 The individual optimal solutions for each objective

x	$Max\ f_1(x)$	$Min\ f_1(x)$	$Max\ f_2(x)$	$Min\ f_2(x)$
x_1	0.197	0	0	0.457
x_2	12.541	15.238	20.862	14.808
x_3	2.669	0	0.607	0

$\underset{-3}{x} = Min\{x_3\} = 0$ and $\bar{x}_3 = Max\{x_3\} = 2.669 \Rightarrow 0 \leq x_3 \leq 2.667$.

(Step 6) The corresponding tolerant interval of each of objective function is: $(29.785, 76.168)$ and $(54.699, 77.653)$, respectively. Also, the lower tolerance limit for $f_1(x)$ is $l_1 = 29.785$, and the upper tolerance limit for $f_2(x)$ is $s_2 = 77.653$.

(Step 7) So, the membership functions are constructed as follows, based on (10) and (12):

$$\mu_1(f_1(x)) \cong \begin{cases} 1 & f_1(x) \geq 76.168, \\ \dfrac{f_1(x) - 29.785}{(76.168 - 29.785)}, & 29.785 \leq f_1(x) \leq 76.168, \\ 0 & 29.785 \geq f_1(x) \end{cases}$$

$$\mu_1(f_1(x)) = 0.493\sqrt{x_1} - 0.057x_1 + 0.498\sqrt{x_2} - 0.085x_2 + 0.495x_3^{(2/3)} - 0.079x_3 - 0.642 \quad (27)$$

$$\mu_2(f_2(x)) = \begin{cases} 1 & f_2(x) \leq 54.699, \\ \dfrac{77.653 - f_2(x)}{77.653 - 54.699}, & 54.699 \leq f_2(x) \leq 77.653 \\ 0 & 77.653 \leq f_2(x) \end{cases}$$

$$\mu_2(f_2(x)) = -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 \quad (28)$$

In order to apply Taylor polynomial approach (14), we need to determine an initial feasible point in the feasible region (26). Thus nonlinear membership function (27) under the linear inequalities is maximized as follows:

$$Max\ \mu_1(f_1(x)) = 0.493\sqrt{x_1} - 0.057x_1 + 0.498\sqrt{x_2} - 0.085x_2 + 0.495x_3^{(2/3)} - 0.079x_3 - 0.642$$

$$s.t. \begin{cases} 0 \leq x_1 \leq 0.457, \\ 12.541 \leq x_2 \leq 20.862, \\ 0 \leq x_3 \leq 2.669, \\ \mu_1(f_1(x)) \leq 1 \\ x_1, x_2, x_3 \geq 0 \end{cases} \quad (29)$$

This problem is solved by using Maple 2017 nonlinear optimization toolbox and the following result is obtained.

$\tilde{x}^* = (\tilde{x}_1 = 0.457,\ \tilde{x}_2 = 12.541, \tilde{x}_3 = 2.078),\ \mu_1(f_1(x)) = 1.$

On the other hand, since function (28) is a linear function, it remains unchanged.

(Step 8) Applying Taylor polynomial approach (14) to function (27) around its solution $\tilde{x}^* = (\tilde{x}_1 = 0.457, \tilde{x}_2 = 12.541, \tilde{x}_3 = 2.078), \mu_1(f_1x) = 1$ an equivalent linear membership function to nonlinear membership function (27) is obtained as follows:

$$\tilde{\mu}_1(f_1(x)) \cong \left[\frac{\mu_1(f_1(\tilde{x}^*))}{\partial x_1}\bigg|_{\tilde{x}^*}(x_1 - 0.457) + \frac{\mu_1(f_1(\tilde{x}^*))}{\partial x_2}\bigg|_{\tilde{x}^*}(x_2 - 12.541) + \frac{\mu_1(f_1(\tilde{x}^*))}{\partial x_3}\bigg|_{\tilde{x}^*}(x_3 - 2.078) \right] = 0.308x_1 - 0.015x_2 + 0.180x_3 + 0.675 \quad (30)$$

And the constrained region of (26) can be rewritten as follows:

$$s.t. \begin{cases} g_1(x) = 3.889x_1x_2 + 4.123x_2 + 3.660x_3^2 - 87.364 \leq 0, \\ g_2(x) = 4.123x_1^2 + 3.889x_1x_2 + 3.889x_3 - 75.369 \leq 0, \\ g_3(x) = 3.660x_1 + 3.889x_2 + 3.660x_3 - 59.259 \geq 0, \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Also, the nonlinear constraint functions given above are linearized at $\tilde{x}^* = (\tilde{x}_1 = 0.457,\ \tilde{x}_2 = 12.541, \tilde{x}_3 = 2.078)$ as follows:

$$\tilde{g}_j(x)_{j=1,2,3} \cong \left[\frac{g_j(\tilde{x}^*)}{\partial x_1}\bigg|_{\tilde{x}^*}(x_1 - 0.457) + \frac{g_j(\tilde{x}^*)}{\partial x_2}\bigg|_{\tilde{x}^*}(x_2 - 12.541) + \frac{g_j(\tilde{x}^*)}{\partial x_3}\bigg|_{\tilde{x}^*}(x_3 - 2.078) \right]$$

Then the linear constraint functions obtained are as follows:

$$\tilde{g}_1(x) \cong 48.772\text{x}_1 + 5.900\text{x}_2 + 15.212\text{x}_3 - 125.456 \leq 0; \quad (31)$$

However, $g_3(x)$ is a linear function and then it remains unchanged.

(Step 9): Based on (22) and (23), the linear membership goals is defined as follows:

$$0.308x_1 - 0.015x_2 + 0.180x_3 + 0.675 + d_1^- = 1,\\ -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 + d_2^- = 1 \quad (32)$$

where $d_1^- \geq 0$ and $d_2^- \geq 0$ represent the negative deviations from the aspired levels, respectively.

(Step 10): Based on model (24), the equivalent linear FGP is constructed as follows:

$$Min\ \beta$$

$$s.t.\begin{cases} 0.308x_1 - 0.015x_2 + 0.180x_3 + 0.675 + d_1^- = 1 \\ -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 + d_2^- = 1 \\ 48.772\mathrm{x}_1 + 5.900\mathrm{x}_2 + 15.212\mathrm{x}_3 - 125.456 \leq 0 \\ 52.540\mathrm{x}_1 + 1.777\mathrm{x}_2 + 3.889\mathrm{x}_3 - 98.515 \leq 0 \\ 3.660x_1 + 3.889x_2 + 3.660x_3 - 59.259 \geq 0 \\ \beta \geq d_1^-, \beta \geq d_2^- \\ \beta \in [0, 1] \\ d_1^-,\ d_2^- \geq 0 \end{cases} \quad (33)$$

Since the problem given above is a single objective linear programming problem, it is solved by Maple 2017, and then the solutions obtained are as follows:

$$\beta = d_1^- = d_2^- = 0.045, \quad x = \left(x_1 = 0.374,\ x_2 = 12.998,\ x_3 = 2.006\right).$$

The membership values in (27) and (28) are obtained as: $\mu_1(f_1) = 0.953$, $\mu_2(f_2) = 0.955$. Also, value of each of the objective function is $f_1 = 73.977$ and $f_2 = 55.739$, respectively.

(Step 11) Let us suppose the DM does not accept this solution and desires more profit and then go to Step 13. The new lower tolerance limit in the first objective becomes 73.977. The upper tolerance limit in the second objective function remains unchanged since the time is very close to the desired levels.

Thus the upper and lower limits of each objective function can be written as follows: $73.977 \leq f_1 \leq 326.279$; and $20.267 \leq f_2 \leq 34.416$. Keeps the current solution $x = \left(x_1 = 0.374,\ x_2 = 12.998,\ x_3 = 2.006\right)$ to linearize the membership functions, and then returns to Step 7.

So, the membership functions in (27) are redefined as follows:

$$\mu_1\left(f_1(x)\right) \cong \begin{cases} 1 & f_1(x) \geq 76.168, \\ \dfrac{f_1(x) - 73.977}{(76.168 - 73.977)}, & 73.977 \leq f_1(x) \leq 76.168, \\ 0 & 73.977 \geq f_1(x) \end{cases} \quad (34)$$

Since the tolerance limits have not changed, the previous membership function (28) is taken as:

$$\mu_2\left(f_2(x)\right) = -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 \quad (35)$$

(Step 8) Applying Taylor polynomial approach (14) to nonlinear membership function (34) around its solution $x = \left(x_1 = 0.374,\ x_2 = 12.998,\ x_3 = 2.006\right)$, an equivalent linear membership function to nonlinear function (27) is obtained as follows:

$$\tilde{\mu}_1\left(f_1(x)\right) \cong \left[\left.\frac{\mu_1\left(f_1\left(\tilde{x}^*\right)\right)}{\partial x_1}\right|_{\tilde{x}^*} \left(x_1 - 0.374\right) + \left.\frac{\mu_1\left(f_1\left(\tilde{x}^*\right)\right)}{\partial x_2}\right|_{\tilde{x}^*} \left(x_2 - 12.998\right) + \left.\frac{\mu_1\left(f_1\left(\tilde{x}^*\right)\right)}{\partial x_3}\right|_{\tilde{x}^*} \left(x_3 - 2.006\right) \right]$$
$$= 7.362x_1 - 0.348x_2 + 3.895x_3 - 6.039 \quad (36)$$

Also, the nonlinear constraint functions of (26) are linearized around $x = \left(x_1 = 0.374,\ x_2 = 12.998,\ x_3 = 2.006\right)$, as follows:

$$\tilde{g}_j(x)_{j=1,2} \cong \left[\left.\frac{g_j\left(\tilde{x}^*\right)}{\partial x_1}\right|_{\tilde{x}^*} \left(x_1 - 0.457\right)374 + \left.\frac{g_j\left(\tilde{x}^*\right)}{\partial x_2}\right|_{\tilde{x}^*} \left(x_2 - 12.998\right) + \left.\frac{g_j\left(\tilde{x}^*\right)}{\partial x_3}\right|_{\tilde{x}^*} \left(x_3 - 2.006\right) \right]$$

Thus, the linear constraint functions obtained are as follows:

$$\tilde{g}_1(x) \cong 50.548\mathrm{x}_1 + 5.579\mathrm{x}_2 + 14.682\mathrm{x}_3 - 121.011 \leq 0; \quad (37)$$

However, $g_3(x)$ is a linear function, it remains unchanged.

(Step 9): Thus the linear membership goals are redefined as:

$$7.362x_1 - 0.348x_2 + 3.895x_3 - 6.039 + d_1^- = 1,\\ -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 + d_2^- = 1 \quad (38)$$

where $d_1^- \geq 0$ and $d_2^- \geq 0$ represent the negative deviations from the aspired levels, respectively.

(Step 10): Then the linear FGP (33) is updated as follows:

$$Min\ \beta$$

$$s.t.\begin{cases} 7.362x_1 - 0.348x_2 + 3.895x_3 - 6.039 + d_1^- = 1 \\ -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 + d_2^- = 1 \\ 50.548\mathrm{x}_1 + 5.579\mathrm{x}_2 + 14.682\mathrm{x}_3 - 121.011 \leq 0 \\ 53.635\mathrm{x}_1 + 1.456\mathrm{x}_2 + 3.889\mathrm{x}_3 - 94.871 \leq 0 \\ 3.660x_1 + 3.889x_2 + 3.660x_3 - 59.259 \geq 0 \\ \beta \geq d_1^-, \beta \geq d_2^- \\ \beta \in [0, 1] \\ d_1^-,\ d_2^- \geq 0 \end{cases} \quad (39)$$

Since the problem is a single objective linear optimization problem, then it is solved by Maple 2017, and then the solutions obtained are as follows:

$$x = (x_1 = 0.304,\ \ x_2 = 12.733,\ \ x_3 = 2.358).$$

By putting the obtained solution $x = (x_1 = 0.304,$ $x_2 = 12.733,\ x_3 = 2.358)$ in (27) and (28), the original membership values are obtained as: $\mu_1(f_1) = 0.993$, $\mu_2(f_2) = 0.945$. Also, value of each of the objective function is $f_1 = 75.847$ and $f_2 = 55.958$, respectively.

If these results are unacceptable for the DM at Step 11, the principal modifications are implemented in Step 13, and then similar procedures are repeated from Step 7 to Step 10. Hence the following results achieved are as follows:

$$\beta = d_1^- = d_2^- = 0.060,\quad x = (x_1 = 0.249,\ \ x_2 = 12.637,\ \ x_3 = 2.514).$$

The original membership values in (27) and (28) are obtained as: $\mu_1(f_1) = 0.998$, $\mu_2(f_2) = 0.940$. Also, value of each of the objective function is $f_1 = 76.072$and $f_2 = 56.071$, respectively.

Repeating the same procedures once again, the following results are obtained.

$$\beta = d_1^- = d_2^- = 0.062,\quad x = (x_1 = 0.223,\ \ x_2 = 12.590,\ \ x_3 = 2.590).$$

The original membership values in (27) and (28) are obtained as:$\mu_1(f_1) = 0.999$, $\mu_2(f_2) = 0.938$. Also, value of each of the objective function is $f_1 = 76.141$ and $f_2 = 56.125$, respectively.

Let us suppose the DM accepts this solution as the optimal solution of problem (27), and thus the algorithm is terminated at Step 12.

The membership and objective function values obtained by the presented FGP approach are given in Figs. 1 and 2.

The results obtained above show that the presented interactive FGP approach converged toward the desired levels by updating the upper and lower tolerance limit. Thanks to the proposed solution procedure, the decision maker analyzes the results in each iteration and obtains the best decision.

In order to further demonstrate the performance of the recommended linearization procedures, the above numerical example has also been solved as a nonlinear programming problem by using different FGP approaches. Then these FGP approaches are converted to linear form the presented linearization method. Solving these models using Maple 2017, results of nonlinear models and their linear forms were compared. Comparison of results is shown in Tables 2 and 3. Also, the general models for these approaches are stated as shown below.

The FGP approach which is proposed by Mohamed [33]:

$$Min\left(\sum_{k=1}^{l} w_k d_k^-\right)$$
$$s.t.\begin{cases} \mu_k(f_k(x)) + d_k^- - d_k^+ = 1,\ 1 \le k \le l' \\ \mu_k(f_k(x)) + d_k^- - d_k^+ = 1,\ l'+1 \le k \le l \\ \text{constraints of model (6)} \\ d_k^-, d_k^+ \ge 0,\ \ 1 \le k \le l \\ d_k^- \times d_k^+ \ge 0 \\ x_l \ge 0, 1 \le l \le n \end{cases} \tag{40}$$

where d_k^- and d_k^+ is the negative and positive deviation from aspired levels, respectively. w_k, $1 \le k \le l$ is the relative importance of achieving the aspired levels of the fuzzy goals, which is considered as $w_k = \frac{1}{s_k - L_K}, 1 \le k \le l.$

A FGP approach which is proposed by Gupta and Bhattacharjee [18]:

$$Max\ \lambda$$
$$s.t.\begin{cases} w_k\lambda \le \mu_k(f_k(x)),\ 1 \le k \le l' \\ w_k\lambda \le \mu_k(f_k(x)),\ l'+1 \le k \le l \\ \lambda + d_k^- = 1,\ \ 1 \le k \le l \\ \text{constraints of model (6)} \\ d_k^- \ge 0,\ \ 1 \le k \le l \\ x_l \ge 0, 1 \le l \le n \\ \lambda \in [0,1] \end{cases} \tag{41}$$

where $w_k = \frac{1}{s_k - l_k}, 1 \le k \le l.$

Taking the membership functions defined in (27) and (28), the corresponding FGP model of (40) for problem (26) is formulated as:

$$Min\ \left(\sum_{k=1}^{2} w_k d_k^-\right)$$
$$\begin{cases} \mu_1(f_1(x)) + d_1^- + d_1^+ = 1, \\ \mu_2(f_2(x)) + d_2^- + d_2^+ = 1, \\ 3.889x_1x_2 + 4.123x_2 + 3.660x_3^2 \le 87.364 \\ 4.123x_1^2 + 3.889x_1x_2 + 3.889x_3 \le 75.369 \\ 3.660x_1 + 3.889x_2 + 3.660x_3 \ge 59.259 \\ d_k^-, d_k^+ \ge 0,\ \ 1 \le k \le 2 \\ d_k^- \times d_k^+ \ge 0, 1 \le k \le 2 \\ x_1, x_2, x_3 \ge 0 \end{cases} \tag{42}$$

where $w_1 = 0.22$ and $w_1 = 0.44$.

It is solved as a nonlinear programming problem by Maple 2017 and the results are obtained as:

objective value = 0.0027

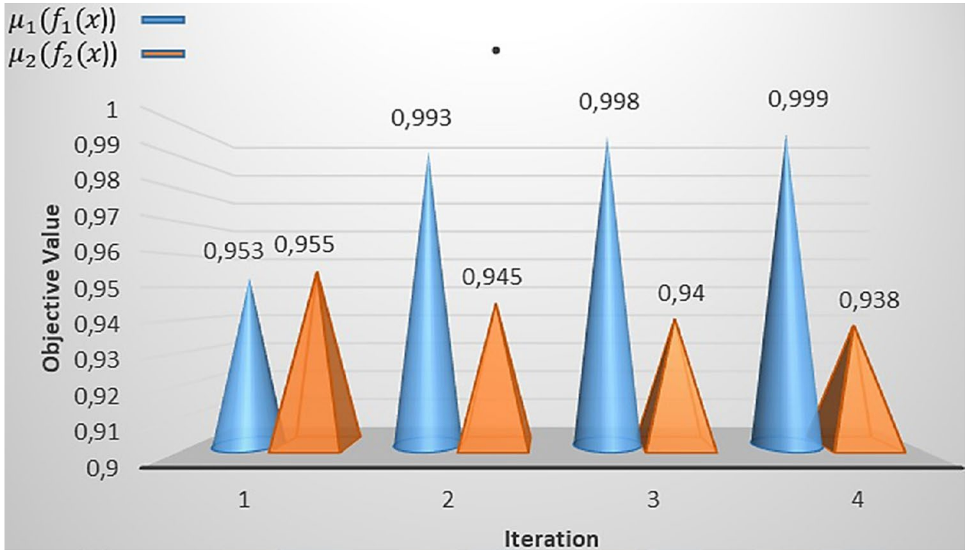

Fig. 1 Change of the membership values obtained by the proposed FGP approach

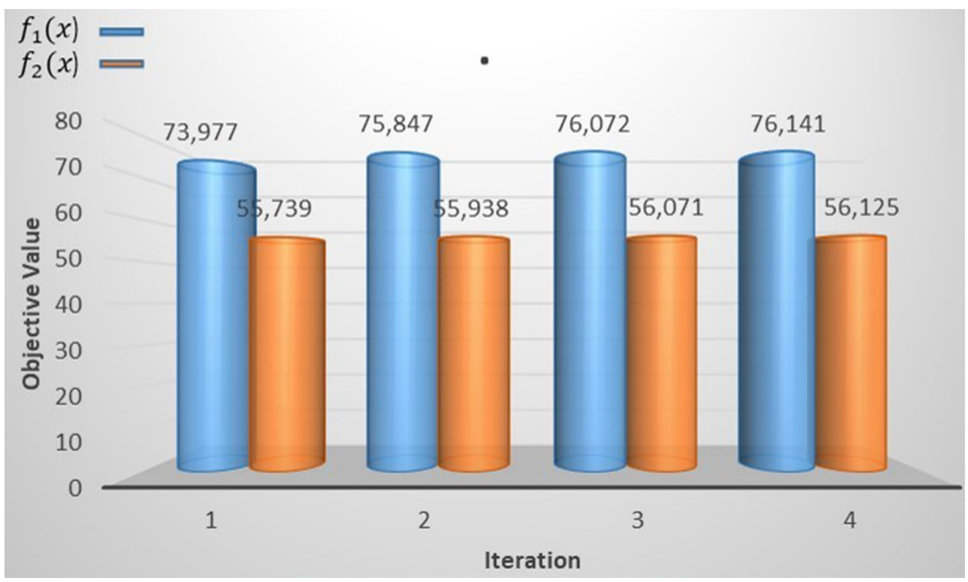

Fig. 2 Change of the objective values obtained by the proposed FGP approach

$$d_1^- = 0.008,\ d_2^- = 0.057,\ d_2^+ = 0,\ d_2^+ = 0,$$

$$x = \left(x_1 = 0.276,\ x_2 = 12.706,\ x_3 = 2.414\right),$$

The membership values in (27) and (28) are obtained as: $\mu_1(f_1) = 0.992$, $\mu_2(f_2) = 0.943$. Also, value of each of the objective function is $f_1 = 75.789$ and $f_2 = 56.003$, respectively.

Taking the linear membership functions and the linear constraint function as defined in (28), (30) and (31), the corresponding linear FGP model of (40) for problem (26) is formulated as follows:

$$Min\left(\sum_{k=1}^{2} w_k d_k^-\right)$$
$$s.t.\begin{cases} 0.308x_1 - 0.015x_2 + 0.180x_3 + 0.675 + d_1^- - d_1^+ = 1 \\ -0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 + d_2^- - d_2^+ = 1 \\ 48.772\text{x}_1 + 5.900\text{x}_2 + 15.212\text{x}_3 - 125.456 \le 0 \\ 52.540\text{x}_1 + 1.777\text{x}_2 + 3.889\text{x}_3 - 98.515 \le 0 \\ 3.660x_1 + 3.889x_2 + 3.660x_3 - 59.259 \ge 0 \\ d_k^-, d_k^+ \ge 0,\quad 1 \le k \le 2 \\ d_k^- \times d_k^+ \ge 0, 1 \le k \le 2 \\ x_1, x_2, x_3 \ge 0 \end{cases} \tag{43}$$

Table 2 Comparison of the solutions obtained from nonlinear FGP models

	Mohammed's approach (40)	Gupta and Bhattacharjee's (41)
f_1	75.789	67.029
f_2	56.003	56.512
$\mu_1(f_1)$	0.992	0.803
$\mu_2(f_2)$	0.943	0.921

Table 3 Comparison of results based on linearization procedures

	The presented approach	Mohammed's approach (40)	Gupta and Bhattacharjee's (41)
f_1	76.141	75.771	73.914
f_2	56.125	55.978	56.805
$\mu_1(f_1)$	0.999	0.992	0.951
$\mu_2(f_2)$	0.938	0.944	0.908

where $w_1 = 0.22$ and $w_1 = 0.44$.

It is solved as a single objective linear programming problem by Maple 2017 and the results are obtained as: objective value = 0.0024,

$$d_1^- = 0,\ d_2^- = 0.0.056,\ d_1^+ = 0,\ d_2^+ = 0,$$

$$x = (x_1 = 0.291\ x_2 = 12.727,\ \ x_3 = 2.377),$$

The membership values in (27) and (28) are obtained as: $\mu_1(f_1) = 0.992$, $\mu_2(f_2) = 0.944$. Also, value of each of the objective function is $f_1 = 75.771$ and $f_2 = 55.978$ respectively.

The membership values obtained from problems (42) and (43) show that the solution procedure proposed in this study gives better results for the DMs.

By putting the nonlinear membership functions (27), (28) and constraints of (26) in model (41), the following nonlinear FGP problem is obtained.

$$Max\ \lambda$$

$$s.t.\begin{cases} w_1\lambda \le \mu_1(f_1(x)) \\ w_2\lambda \le \mu_2(f_2(x)) \\ \lambda + d_1^- = 1,\ \ \lambda + d_2^- = 1 \\ 3.889x_1x_2 + 4.123x_2 + 3.660x_3^2 \le 87.364 \\ 4.123x_1^2 + 3.889x_1x_2 + 3.889x_3 \le 75.369 \\ 3.660x_1 + 3.889x_2 + 3.660x_3 \ge 59.259 \\ d_1^-, d_2^- \ge 0 \\ x_1, x_2, x_3 \ge 0 \\ \lambda \in [0, 1] \end{cases} \tag{44}$$

where $w_1 = 0.22$ and $w_1 = 0.44$.

Solving the above problem by Maple 2017, the solutions obtained are as follows:

$$\lambda = 1,\ d_1^- =,\ d_2^- = 0,$$

$$x = (x_1 = 0.324\ \ x_2 = 13.883,\ \ x_3 = 1.418),$$

$$\mu_1(f_1) = 0.803,\ \mu_2(f_2) = 0.921,$$

$$f_2 = 56.512.$$

Similarly, by taking linear membership functions (28)–(30) and linear constraints (31), problem (44) is reduced to the following linear FGP problem:

$$Max\ \lambda$$

$$s.t.\begin{cases} w_1\lambda \le 0.308x_1 - 0.015x_2 + 0.180x_3 + 0.675 \\ w_2\lambda \le 0.120x_1 - 0.157x_2 + 0.169x_3 + 3.383 \\ \lambda + d_1^- = 1,\ \ \lambda + d_2^- = 1 \\ 48.772x_1 + 5.900x_2 + 15.212x_3 - 125.456 \le 0 \\ 52.540x_1 + 1.777x_2 + 3.889x_3 - 98.515 \le 0 \\ 3.660x_1 + 3.889x_2 + 3.660x_3 - 59.259 \ge 0 \\ d_1^-, d_2^- \ge 0,\ \ x_1, x_2, x_3 \ge 0 \\ \lambda \in [0, 1] \end{cases} \tag{45}$$

where $w_1 = 0.22$and $w_1 = 0.44$.

Solving this model by Maple 2017, the solutions obtained are as follows:

$$\begin{cases} \lambda = 1,\ d_1^- =,\ d_2^- = 0, \\ x = (x_1 = 0\ x_2 = 11.774,\ \ x_3 = 3.681), \\ \mu_1(f_1) = 0.951,\ \mu_2(f_2) = 0.908, \\ f_1 = 73.914, f_2 = 56.805. \end{cases}$$

When the results obtained from nonlinear problems (42), (44) and their linear forms (43), (45) are compared, it is observed that the linearization procedures presented in this paper give accurate and efficient results.

Furthermore, all the results show that all the sums of the membership values generated by the suggested procedure are greater than that generated by the approaches presented in [18, 33].

According to Theorem 3, $x = (x_1 = 0.223,\ \ x_2 = 12.590,\ x_3 = 2.590)$ is an efficient solution to MONLPP (26) and then from Theorem 4, $x = (x_1 = 0.223,\ \ x_2 = 12.590,\ x_3 = 2.590)$ is an efficient solution for MONLPP with IT2 FNs (25).

Finally, the interactive solution procedure proposed in this study is applicable to obtain more and more profit with a needed time.

As seen in the results, the decision maker evaluates the results obtained, and if not accepts the results, intervenes in the solution process and tries to obtain the best solution in his(her) preferred direction. In the familiar FGP (40) and (41), it is habitually difficult to attain the aspired levels for DMs. Since interactive approaches present additional data to decision makers for producing better decisions, then the solutions of numerical examples are obtained by using the presented interactive FGP approach. Thus the proposed interactive solution procedure offers a practical way to attain appropriate aspiration levels reducing the computational complexity.

5 Conclusions

This paper presents a model of Multiobjective Nonlinear Programming Problems (MONLPP) with trapezoidal Interval type 2 Fuzzy Numbers (IT2 FNs). The most serious aspect of the modeled problem is that it is having two objective functions; one of them is to maximize the desired profit while the other is to minimize the integrated time. At first, MONLPP with IT2 FNs is converted into an equivalent crisp MONLPP by using an expected value function. Then the feasible region of MONLPP is transformed to linear inequalities using the limits of decision variable. Thus the nonlinear membership and constraint functions in the model are also converted into an equivalent linear function by using Taylor polynomial approximation. In this way, a linear fuzzy goal programming model for the MONLPP with IT2 FNs is constructed. This model is also solved to obtain the optimal solution by using the different approaches. Thus desired more profit at the minimum time is obtained by using the proposed interactive approach.

Consequently, applications of the proposed procedures are discussed with a numerical model, and the effectiveness of the solutions achieved by the interactive procedures is verified.

The proposed interactive fuzzy goal programming based on Taylor series may be assistant in solving decision making problems in the area of manufacturing, planning, and carrying, etc. In future, the presented approach can be applied to some other type of MONLPPs with different membership functions.

References

1. Aljarah I, Faris H, Mirjalili S, Al-Madi N (2016) Training radial basis function networks using biogeography-based optimizer. Neural Comput Appl. https://doi.org/10.1007/s00521-016-2559-2
2. Arfken GB, Weber HJ (2005) Mathematical methods for physicists international student edition, 5th edn. Academic press, New York
3. Arora SR, Gupta R (2009) Interactive fuzzy goal programming approach for bilevel programming problem. Eur J Oper Res 194(2):368–376
4. Castillo O, Melin P (2014) A review on interval type-2 fuzzy logic applications in intelligent control. Inf Sci 279:615–631
5. Castillo O, Amador -AL, Castro JR, Garcia- VM (2016) A comparative study of type-1 fuzzy logic systems, interval type-2 fuzzy logic systems and generalized type-2 fuzzy logic systems in control problems. Inf Sci 354:257–274
6. Cervantes L, Castillo O (2015) type-2 fuzzy logic aggregation of multiple fuzzy controllers for airplane flight control. Inf Sci 324:247–256
7. Charnes A, Cooper WW, Ferguson RO (1955) Optimal estimation of executive compensation by linear programming. Manag Sci 1(2):138–151
8. Charnes A, Cooper WW (1957) Management models and industrial applications of linear programming. Manage Sci 4(1):38–91
9. Dalman H, Köçken HG, Sivri M (2013) A solution proposal to indefinite quadratic interval transportation problem. New Trends Math Sci 1(2):07–12
10. de Jesús Rubio J (2016) Least square neural network model of the crude oil blending process. Neural Netw 78:88–96
11. de Jesús Rubio J, Elias I, Cruz DR, Pacheco J (2017) Uniform stable radial basis function neural network for the prediction in two mechatronic processes. Neurocomputing 227:122–130
12. de Jesús Rubio J (2017) USNFIS: Uniform Stable Neuro Fuzzy Inference System. Neurocomputing. https://doi.org/10.1016/j.neucom.2016.08.150
13. Dey S, Roy TK (2015) Intuitionistic fuzzy goal programming technique for solving non-linear multi-objective structural problem. J Fuzzy Set Valued Anal 3:179–193
14. Dutta A, Jana DK (2017) Expectations of the reductions for type-2 trapezoidal fuzzy variables and its application to a multi-objective solid transportation problem via goal programming technique. J Uncertain Anal Appl. https://doi.org/10.1186/s40467-017-0057-4
15. El-Wahed WFA, Lee SM (2006) Interactive fuzzy goal programming for multi-objective transportation problems. Omega 34(2):158–166
16. Ertuğrul İ, Tuş A (2007) Interactive fuzzy linear programming and an application sample at a textile firm. Fuzzy Optim Decis Making 6(1):29–49
17. Gupta JN, Ahmed NU (1988) A goal programming approach to job evaluation. Comput Ind Eng 14(2):147–152
18. Gupta M, Bhattacharjee D (2012) Two weighted fuzzy goal programming methods to solve multiobjective goal programming problem. J Appl Math. https://doi.org/10.1155/2012/796028
19. Hosseinzadeh E, Hassanpour H, Arefi M (2016) A weighted goal programming approach to estimate the linear regression model in full quasi type-2 fuzzy environment. J Intell Fuzzy Syst 30(3):1319–1330
20. Hu J, Zhang Y, Chen X, Liu Y (2013) July) Multi-criteria decision making method based on possibility degree of interval type-2 fuzzy number. Knowl-Based Syst 43:21–29
21. Jin F, Ni Z, Pei L, Chen H, Li Y (2016) Goal programming approach to derive intuitionistic multiplicative weights based on intuitionistic multiplicative preference relations. Int J Mach Learn Cybernet. https://doi.org/10.1007/s13042-016-0590-3
22. Kahraman C, Öztayşi B, Sarı İU, Turanoğlu E (2014) Fuzzy analytic hierarchy process with interval type-2 fuzzy sets. Knowl-Based Syst 59:48–57
23. Khalili-DK, Sadi-NS, Tavana M (2013) Solving multi-period project selection problems with fuzzy goal programming based on TOPSIS and a fuzzy preference relation. Inf Sci 252:42–61
24. Khan I, Aggarwal A, Mehra A, Chandra S (2017) Solving matrix games with Atanassov's I-fuzzy goals via indeterminacy resolution approach. J Inf Optimization Sci 38(2):259–287

25. Lachhwani K (2015) Modified FGP approach for multi-level multi objective linear fractional programmingproblems. Appl Math Comput 266:1038–1049
26. Li S, Hu C (2009) Satisfying optimization method based on goal programming for fuzzy multiple objectiveoptimization problem. Eur J Oper Res 197(2):675–684
27. Li H, Pan Y, Zhou Q (2015) Filter design for interval type-2 fuzzy systems with D stability constraints under a unified frame. Fuzzy Syst IEEE Trans 23(3):719–725
28. Lin CC, Liu YT, Chen AP (2016) Hedging an option portfolio with minimum transaction lots: a fuzzy goal programming problem. Appl Soft Comput 47:295–303
29. Liu Q, Yin J, Leung VC, Zhai JH, Cai Z, Lin J (2016) Applying a new localized generalization error model to design neural networks trained with extreme learning machine. Neural Comput Appl 27(1):59–66
30. Mehrbod M, Tu N, Miao L, Wenjing D (2012) Interactive fuzzy goal programming for a multi-objective closed-loop logistics network. Ann Oper Res 201(1):367–381
31. Mendel JM, John RI, Liu F (2006) Interval type-2 fuzzy logic systems made simple. Fuzzy Syst IEEE Trans 14(6):808–821
32. Mendel JM (2009) On answering the question “Where do I start in order to solve a new problem involving interval type-2 fuzzy sets?”. Inf Sci 179(19):3418–3431
33. Mohamed RH (1997) The relationship between goal programming and fuzzy programming. Fuzzy Sets Syst 89(2):215–222
34. Nan JX, Li DF (2013) Linear programming approach to matrix games with intuitionistic fuzzy goals. Int J Comput Intell Syst 6(1):186–197
35. Narasimhan R (1980) Goal programming in a fuzzy environment. Decis Sci 11(2):325–336
36. Own CM (2009) Switching between type-2 fuzzy sets and intuitionistic fuzzy sets: an application in medical diagnosis. Appl Intell 31(3):283–291
37. Pal BB, Moitra BN, Maulik U (2003) A goal programming procedure for fuzzy multiobjective linear fractional programming problem. Fuzzy Sets Syst 139(2):395–405
38. Qin J, Liu X, Pedrycz W (2015) An extended VIKOR method based on prospect theory for multiple attribute decision making under interval type-2 fuzzy environment. Knowl-Based Syst 86:116–130
39. Razmi J, Jafarian E, Amin SH (2016) An intuitionistic fuzzy goal programming approach for finding pareto-optimal solutions to multi-objective programming problems. Expert Syst Appl 65:181–193
40. Sanchez MA, Castillo O, Castro JR (2015) Information granule formation via the concept of uncertainty-based information with Interval type-2 fuzzy sets representation and Takagi–Sugeno–Kang consequents optimized with Cuckoo search. Appl Soft Comput 27:602–609
41. Singh SK, Yadav SP (2015) Intuitionistic fuzzy non linear programming problem: modeling and optimization in manufacturing systems. J Intell Fuzzy Syst 28(3):1421–1433
42. Tang J, Wang D (1996) Modelling and optimization for a type of fuzzy nonlinear programming problems in manufacturing systems. In Decision and Control, 1996. Proceedings of the 35th IEEE Conference on, Vol. 4, pp. 4401–4405, IEEE
43. Tiwari RN, Dharmar S, Rao JR (1987) Fuzzy goal programming—an additive model. Fuzzy Sets Syst 24(1):27–34
44. Toksarı MD, Bilim Y (2015) Interactive fuzzy goal programming based on jacobian matrix to solve decentralized bi-level multi-objective fractional programming problems. Int J Fuzzy Syst 17(4):499–508
45. Wan SP, Wang F, Xu GL, Dong JY, Tang J (2016) An intuitionistic fuzzy programming method for group decision making with interval-valued fuzzy preference relations. Fuzzy Optim Decis Making. https://doi.org/10.1007/s10700-016-9250-z
46. Werners B (1987) An interactive fuzzy programming system. Fuzzy Set Syst 23:131–147
47. Zadeh LA (1975) The concept of a linguistic variable and its application to approximate reasoning—I. Inf Sci 8(3):199–249
48. Zangiabadi M, Maleki HR (2013) Fuzzy goal programming technique to solve multiobjective transportation problems with some non-linear membership functions. Iran J Fuzzy Syst 10(1):61–74
49. Zimmermann HJ (1978) Fuzzy programming and linear programming with several objective functions. Fuzzy Sets Syst 1(1):45–55