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NEW INSIGHT INTO QUATERNIONS AND THEIR MATRICES

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Abstract. This paper aims to bring together quaternions and generalized complex numbers. Generalized quaternions with generalized complex number components are expressed and their algebraic structures are examined. Several matrix representations and computational results are introduced. An alternative approach for a generalized quaternion matrix with elliptic number entries has been developed as a crucial part.

1. Introduction

Hamilton introduced the Hamiltonian quaternions for representing vectors in the space, [1, 2]. The real quaternion is written as $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ are components and $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ are versors, [3]. The set of real quaternions, as an extension of complex numbers, is an associative but non-commutative Clifford algebra used in many fields of applied mathematics. The associative quaternions will be divided into two classes: in the first class, there are the non-commutative quaternions (Hamiltonian, hyperbolic, split, generalized quaternions [4–11] etc.), and in the second class, there are the commutative quaternions (generalized Segré quaternions [12, 13], dual quaternions, [14–18] etc.).

The algebra of generalized quaternions as a non-commutative system, denoted by $\mathcal{Q}_{\alpha,\beta}$, includes a variety of well-known four-dimensional algebras as special cases.

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The conditions of the versors for them are given by:

$$\begin{array}{ccc} \mathbf{i}^2 = -\alpha, & \mathbf{j}^2 = -\beta, & \mathbf{k}^2 = -\alpha\beta, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, & \mathbf{jk} = -\mathbf{kj} = \beta\mathbf{i}, & \mathbf{ki} = -\mathbf{ik} = \alpha\mathbf{j}, \end{array} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$. For $\alpha = \beta = 1$ Hamiltonian quaternions, $\alpha = 1, \beta = -1$ split quaternions, $\alpha = 1, \beta = 0$ semi-quaternions, $\alpha = -1, \beta = 0$ split semi-quaternions, and $\alpha = \beta = 0$ quasi-quaternions are obtained.

Additionally, the general bidimensional hypercomplex systems (namely generalized complex numbers ($\mathcal{GCN}$)) over the field of real numbers $\mathbb{R}$ are given by the ring ([19–24]):

$$\frac{\mathbb{R}[X]}{\langle h(X)\rangle} \cong \left\{ z = x_1 + x_2 I : \ I^2 = I\mathfrak{q} + \mathfrak{p}, \ \ \mathfrak{p}, \mathfrak{q}, x_1, x_2 \in \mathbb{R}, I \notin \mathbb{R} \right\},$$

where $h(X) = X^2 - \mathfrak{q}X - \mathfrak{p}$ is monic quadratic. By denoting this set with $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, it is well known that the sign of $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$ determines the properties of the general bidimensional systems. These systems are ring isomorphic with one of the following three types:

- for $\Delta > 0$ the *hyperbolic system*; the canonical system is the system of hyperbolic (double, split complex, perplex) numbers $\mathbb{H} \cong \mathbb{C}_{0,1}$ with $\mathfrak{p} = 1$, $\mathfrak{q} = 0$, [25–28],
- for $\Delta < 0$ the *elliptic system*; the canonical system is the system of complex (ordinary) numbers $\mathbb{C} \cong \mathbb{C}_{0,-1}$ with $\mathfrak{p} = -1$, $\mathfrak{q} = 0$, [28, 29],
- for $\Delta = 0$ the *parabolic system*; the canonical system is the system of dual numbers $\mathbb{D} \cong \mathbb{C}_{0,0}$ with $\mathfrak{p} = 0$, $\mathfrak{q} = 0$, [28, 30, 31].

Regarding the value $\mathcal{D}_z = z\overline{z} = (x_1 + x_2 I)(x_1 - x_2 I) = {x_1}^2 - \mathfrak{p}{x_2}^2 + \mathfrak{q}x_1x_2$, which is called the characteristic determinant, $z \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ can be classified into three types, [20]. Hence $z \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ is called timelike, spacelike or null where $\mathcal{D}_z < 0$, $\mathcal{D}_z > 0$ and $\mathcal{D}_z = 0$, respectively. Then all of the elements of the set $\mathbb{C}_{0,-1}$ are spacelike. For $\mathfrak{q} = 0$, $I^2 = \mathfrak{p} \in \mathbb{R}$, the generalized complex number system is denoted by $\mathbb{C}_{\mathfrak{p}}$ and called $\mathfrak{p}$-complex plane, [23].

In this paper, we aim to design generalized quaternions by taking the components as elements of $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$. Moreover, the algebraic structures and properties of these quaternions are investigated, and several types of matrix representations are introduced. Also, an alternative approach for the generalized quaternion matrix with elliptic number entries is considered as a further result.

2. Generalized Quaternions with *Gcn* Components

In this section, we present mathematical formulations of improved quaternions: *generalized quaternions with* $\mathcal{GCN}$ and examine special matrix correspondences.

Definition 1. *For $\alpha, \beta \in \mathbb{R}$, the set of generalized quaternions with $\mathcal{GCN}$ components are denoted by $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and the element of this set is defined as in the form:*

$$\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $a_0, a_1, a_2, a_3 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ *and* $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ *are generalized quaternion versors that satisfy the properties in equations* (1)*.*

Axiomatically, the generalized complex unit I commutes with the three quaternion versors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, that is $\mathbf{i}I = I\mathbf{i}$, $\mathbf{j}I = I\mathbf{j}$ and $\mathbf{k}I = I\mathbf{k}$. It is obvious that for $\mathfrak{q} = 0, \mathfrak{p} = -1, \alpha = 1$, the usual complex operator is distinct from quaternion versor $\mathbf{i}$. Moreover $\mathbf{i}$ distinct from the usual hyperbolic unit for $\mathfrak{q} = 0, \mathfrak{p} = 1, \alpha = -1$ and distinct from the usual dual unit for $\mathfrak{q} = 0, \mathfrak{p} = 0, \alpha = 0$. This conditions can also be extended for the other versors.

Throughout this section, $\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\widetilde{p} = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ are considered. Due to the generalized quaternions with $\mathcal{GCN}$ components are an extension of generalized quaternions, many properties of them are familiar. For any $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, $S_{\widetilde{q}} = a_0$ is the scalar part and $V_{\widetilde{q}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is the vector part. Equality of two improved quaternions is as follows: $\widetilde{p} = \widetilde{q} \Leftrightarrow S_{\widetilde{p}} = S_{\widetilde{q}}, V_{\widetilde{p}} = V_{\widetilde{q}}$. Addition (and hence subtraction) of $\widetilde{q}$ to another quaternion $\widetilde{p}$ acts in a componentwise way:

$$\begin{aligned} \widetilde{q} + \widetilde{p} &= (a_0 + b_0) + (a_1 + b_1)\,\mathbf{i} + (a_2 + b_2)\,\mathbf{j} + (a_3 + b_3)\,\mathbf{k} \\ &= S_{\widetilde{p}} + S_{\widetilde{q}} + V_{\widetilde{p}} + V_{\widetilde{q}}. \end{aligned} \tag{2}$$

The conjugate of $\widetilde{q}$ is the following quaternion:

$$\overline{\widetilde{q}} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k} = S_{\widetilde{q}} - V_{\widetilde{q}}. \tag{3}$$

The scalar multiplication of $\widetilde{q}$ with a scalar $c \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ gives another improved quaternion as:

$$c\widetilde{q} = ca_0 + ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k} = cS_{\widetilde{q}} + cV_{\widetilde{q}}. \tag{4}$$

Multiplication of the two quaternions is carried out as follows:

$$\begin{aligned} \widetilde{q}\,\widetilde{p} &= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3) \\ &\quad + (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ &\quad + (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \\ &\quad + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k}. \end{aligned} \tag{5}$$

Proposition 1. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ *is a* 4*-dimensional module over* $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ *with base* $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ *and is an* 8*-dimensional vector space over* $\mathbb{R}$ *with base* $\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\}$*.*

Definition 2. *For any* $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$*, the scalar and vector products on* $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ *are, respectively, defined by:*

$$\begin{aligned} \langle \widetilde{q}, \widetilde{p} \rangle_g &= S_{\widetilde{q}}S_{\widetilde{p}} + \langle V_{\widetilde{q}}, V_{\widetilde{p}} \rangle_g = a_0b_0 + \alpha a_1b_1 + \beta a_2b_2 + \alpha\beta a_3b_3 = S_{\widetilde{q}\overline{\widetilde{p}}}, \\ \widetilde{q} \times_g \widetilde{p} &= S_{\widetilde{q}}V_{\overline{\widetilde{p}}} + S_{\overline{\widetilde{p}}}V_{\widetilde{q}} - V_{\widetilde{q}} \times_g V_{\widetilde{p}} = V_{\widetilde{q}\overline{\widetilde{p}}}, \end{aligned}$$

where $\langle , \rangle_g$ *and* $\times_g$ *represent generalized scalar product and generalized vector products*[1] *for* $\alpha, \beta \in \mathbb{R}^+$*, respectively.*

[1]For a more general description of the generalized inner and cross product, see [7].

Definition 3. *The norm of $\widetilde{q}$ is defined as:*

$$N_{\widetilde{q}} = \widetilde{q}\,\overline{\widetilde{q}} = \overline{\widetilde{q}}\,\widetilde{q} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}. \tag{6}$$

Definition 4. *The inverse of $\widetilde{q}$ is calculated by:*

$$(\widetilde{q})^{-1} = \frac{\overline{\widetilde{q}}}{N_{\widetilde{q}}}$$

for non-null $N_{\widetilde{q}}$ that is $\mathcal{D}_{N_{\widetilde{q}}} \neq 0$.

Proposition 2. *For any $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $c_1, c_2 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, the conjugate and norm hold the following properties:*

i. $\overline{\overline{\widetilde{q}}} = \widetilde{q}$,
ii. $\overline{c_1\widetilde{p} + c_2\widetilde{q}} = c_1\overline{\widetilde{p}} + c_2\overline{\widetilde{q}}$,
iii. $\overline{\widetilde{q}\,\widetilde{p}} = \overline{\widetilde{p}}\,\overline{\widetilde{q}}$,
iv. $N_{c_1\widetilde{q}} = c_1^2 N_{\widetilde{q}}$,
v. $N_{\widetilde{q}\,\widetilde{p}} = N_{\widetilde{q}}\, N_{\widetilde{p}}$.

Proof. Taking into account equations (2), (3) and (4), items i and ii are obvious.

iii. Considering the conjugate of equation (5), we have:

$$\begin{aligned}\overline{\widetilde{q}\widetilde{p}} \;&= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3) \\ &\quad - (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ &\quad - (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \\ &\quad - (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k}.\end{aligned}$$

Using equations (1), it is easy to check that

$$\overline{\widetilde{p}}\,\overline{\widetilde{q}} = (b_0 - b_1\mathbf{i} - b_2\mathbf{j} - b_3\mathbf{k})(a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}) = \overline{\widetilde{q}\,\widetilde{p}}.$$

iv. Having item ii and equation (6), we get: $N_{c_1\widetilde{q}} = (c_1\widetilde{q})\,\overline{(c_1\widetilde{q})} = c_1^2 N_{\widetilde{q}}$.

v. Using item iii and equation (6), we obtain:

$$N_{\widetilde{q}\,\widetilde{p}} = (\widetilde{q}\,\widetilde{p})\,\overline{(\widetilde{q}\,\widetilde{p})} = \widetilde{q}\,\widetilde{p}\overline{\widetilde{p}}\,\overline{\widetilde{q}} = N_{\widetilde{q}} N_{\widetilde{p}}.$$

□

Remark 1. *As an another perspective to $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, the following can be calculated:*

$$\begin{aligned}\widetilde{q} \;&= a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ &= (x_{01} + x_{02}I) + (x_{11} + x_{12}I)\,\mathbf{i} + (x_{21} + x_{22}I)\,\mathbf{j} + (x_{31} + x_{32}I)\,\mathbf{k} \\ &= q_0 + q_1 I,\end{aligned} \tag{7}$$

where $a_i = x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, $q_{j-1} = x_{0j} + x_{1j}\mathbf{i} + x_{2j}\mathbf{j} + x_{3j}\mathbf{k} \in \mathcal{Q}_{\alpha,\beta}$ for $0 \leq i \leq 3$, $1 \leq j \leq 2$. For $\widetilde{q} = q_0 + q_1 I$ and $\widetilde{p} = p_0 + p_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, if $\widetilde{p} = \widetilde{q}$, then $p_0 = q_0, p_1 = q_1$. The addition is $\widetilde{p} + \widetilde{q} = (p_0 + q_0) + (p_1 + q_1)\,I$. The conjugate and anti conjugate are $\widetilde{q}^{\dagger_1} = q_0 + \mathfrak{q}q_1 - q_1 I$ and $\widetilde{q}^{\dagger_2} = q_1 - q_0 I$, respectively. Additionally, $c\widetilde{q} = cq_0 + cq_1 I$, $c \in \mathbb{R}$ and

$$\widetilde{q}\,\widetilde{p} = (q_0p_0 + \mathfrak{p}q_1p_1) + (q_0p_1 + q_1p_0 + \mathfrak{q}q_1p_1)\,I.$$

It is worthy to note that $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is a 2-dimensional module over $\mathcal{Q}_{\alpha,\beta}$ (skew-field) with base $\{1, I\}$. The moduli is

$$N_{\widetilde{q}}^{\dagger_1} = \widetilde{q}\,\widetilde{q}^{\dagger_1} \tag{8}$$

and the inverse is $(\widetilde{q})^{-1} = \frac{\widetilde{q}^{\dagger_1}}{N_{\widetilde{q}}^{\dagger_1}}$ for non-null $N_{\widetilde{q}}^{\dagger_1}$. The analogue of the scalar product on $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ can also defined by as follows:

$$\langle \widetilde{q}, \widetilde{p} \rangle_g \quad = S_{q_0\overline{p}_0} + \mathfrak{p} S_{q_1\overline{p}_1} + \left(S_{q_0\overline{p}_1} + S_{q_1\overline{p}_0} + \mathfrak{q} S_{q_1\overline{p}_1} \right) I.$$

Proposition 3. *The followings hold for $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $c_1, c_2 \in \mathbb{R}$:*

i. $\left(\widetilde{q}^{\dagger_1}\right)^{\dagger_1} = \widetilde{q}$,
ii. $\left(\widetilde{q}^{\dagger_2}\right)^{\dagger_2} = -\widetilde{q}$,
iii. $(c_1\widetilde{q} \pm c_2\widetilde{p})^{\dagger_1} = c_1\widetilde{q}^{\dagger_1} \pm c_2\widetilde{p}^{\dagger_1}$,
iv. $(c_1\widetilde{q} \pm c_2\widetilde{p})^{\dagger_2} = c_1\widetilde{q}^{\dagger_2} \pm c_2\widetilde{p}^{\dagger_2}$,
v. $\widetilde{q} + \widetilde{q}^{\dagger_1} = 2q_0 + \mathfrak{q} q_1$,
vi. $(\widetilde{q}\,\widetilde{p})^{\dagger_1} \neq \widetilde{p}^{\dagger_1}\,\widetilde{q}^{\dagger_1}$,
vii. $N_{c_1\widetilde{q}}^{\dagger_1} = {c_1}^2 N_{\widetilde{q}}^{\dagger_1}$,
viii. $N_{\widetilde{q}\widetilde{p}}^{\dagger_1} \neq N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1}$.

Proof. vi. Let us consider $\widetilde{q} = (1+\mathbf{i})I$ and $\widetilde{p} = \mathbf{j}+I$. As it is seen the followings:

$$\widetilde{q}\,\widetilde{p} = \mathfrak{p}(1+\mathbf{i}) + (\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} + \mathbf{k})I,$$

$$(\widetilde{q}\,\widetilde{p})^{\dagger_1} = \mathfrak{p}(1+\mathbf{i}) + \mathfrak{q}(\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} + \mathbf{k})I,$$

and

$$\begin{aligned} \widetilde{p}^{\dagger_1}\,\widetilde{q}^{\dagger_1} = &\ (\mathbf{j} + \mathfrak{q} - I)(\mathfrak{q}(1+\mathbf{i}) - (1+\mathbf{i})I) \\ = &\ (\mathfrak{p} + \mathfrak{q}^2) + (\mathfrak{p} + \mathfrak{q}^2)\mathbf{i} + \mathfrak{q}\mathbf{j} - \mathfrak{q}\mathbf{k} - (\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} - \mathbf{k})I. \end{aligned}$$

It follows that $(\widetilde{q}\,\widetilde{p})^{\dagger_1} \neq \widetilde{p}^{\dagger_1}\,\widetilde{q}^{\dagger_1}$.

viii. From equation (8), we have the following equations:

$$N_{\widetilde{q}\widetilde{p}}^{\dagger_1} = (\widetilde{q}\,\widetilde{p})(\widetilde{q}\,\widetilde{p})^{\dagger_1}$$

and

$$N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1} \quad = \left(\widetilde{q}\,\widetilde{q}^{\dagger_1}\right)\left(\widetilde{p}\,\widetilde{p}^{\dagger_1}\right).$$

On account of the generalized quaternions are non-commutative and item vi, we find $N_{\widetilde{q}\widetilde{p}}^{\dagger_1} \neq N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1}$. One can also see this inequality considering $\widetilde{q} = \mathbf{i}I$ and $\widetilde{p} = \mathbf{j}$ as $N_{\widetilde{q}\widetilde{p}}^{\dagger_1} = \mathfrak{p}\alpha\beta = -N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1}$.

The proof of the other items is a simple calculation considering Remark 1. □

2.1. **Matrix Correspondences.** In this subsection, we formulate 2×2, 4×4 and 8×8 matrix correspondences which provide an alternative formulation of multiplication.

Theorem 1. *Every generalized quaternion with $\mathcal{GCN}$ components can be represented by a 2×2 quaternionic matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta})$.*

Proof. For $\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, $\mathcal{L} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{R}, \widetilde{q} \mapsto \mathcal{A}_{\widetilde{q}}$ is linear map, where

$$\mathcal{R} := \left\{ \mathcal{A}_{\widetilde{q}} \in \mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta}) : \mathcal{A}_{\widetilde{q}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix} \right\} \tag{9}$$

is a subset of $\mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta})$. So there exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{R}$ via the map $\mathcal{L}$. Hence, 2×2 quaternionic matrix representation of $\widetilde{q}$ is $\mathcal{A}_{\widetilde{q}}$.

□

Corollary 1. *$\mathcal{L}$ can be determined as the following representation:*

$$\mathcal{L}(a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = a_0 I_2 + a_1\mathsf{I} + a_2\mathsf{J} + a_3\mathsf{K}, \tag{10}$$

where

$$\mathsf{I} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \mathsf{J} = \begin{bmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix}, \mathsf{K} = \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix}.$$

Thus

$$\begin{array}{ccc} \mathsf{I}^2 = -\alpha I_2, & \mathsf{J}^2 = -\beta I_2, & \mathsf{K}^2 = -\alpha\beta I_2, \\ \mathsf{IJ} = -\mathsf{JI} = \mathsf{K}, & \mathsf{JK} = -\mathsf{KJ} = -\beta\mathsf{I}, & \mathsf{KI} = -\mathsf{IK} = \alpha\mathsf{J}. \end{array}$$

Theorem 2. *For $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{R}$, then the following identities hold:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{A}_{\widetilde{q}} = \mathcal{A}_{\widetilde{p}}$,
ii. $\mathcal{A}_{\widetilde{q}+\widetilde{p}} = \mathcal{A}_{\widetilde{q}} + \mathcal{A}_{\widetilde{p}}$,
iii. $\mathcal{A}_{\lambda\widetilde{q}} = \lambda(\mathcal{A}_{\widetilde{q}})$,
iv. $\mathcal{A}_{\widetilde{q}\widetilde{p}} = \mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$.

Proof. The proof is obvious considering the matrix form given in equation (9). However let us discuss the proof of the item iv for better understanding:

iv. Considering equation (5), we can write:

$$\mathcal{A}_{\widetilde{q}\widetilde{p}} = \begin{bmatrix} \begin{array}{c} a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3 \\ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k} \end{array} & \begin{array}{c} (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ +\,(a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \end{array} \\ \\ \begin{array}{c} (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ +\,(a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \end{array} & \begin{array}{c} a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3 \\ +\,(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k} \end{array} \end{bmatrix}. \tag{11}$$

Computing $\mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$ as

$$\mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix} \begin{bmatrix} b_0 + b_3\mathbf{k} & b_1\mathbf{i} + b_2\mathbf{j} \\ b_1\mathbf{i} + b_2\mathbf{j} & b_0 + b_3\mathbf{k} \end{bmatrix}$$

gives equation (11) quickly. We thus get $\mathcal{A}_{\widetilde{q}\widetilde{p}} = \mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$.

□

Theorem 3. *Every generalized quaternion with $\mathcal{GCN}$ components can be represented by a 4×4 generalized complex matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_4(\mathbb{C}_{\mathfrak{q},\mathfrak{p}})$.*

Proof. For $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, denote $\mathcal{K}$ as a subset of $\mathbb{M}(\mathbb{C}_{\mathfrak{q},\mathfrak{p}})$ given by:

$$\mathcal{K} := \left\{ \mathcal{B}^l_{\widetilde{q}} \in \mathbb{M}_4(\mathbb{C}_{\mathfrak{q},\mathfrak{p}}) : \mathcal{B}^l_{\widetilde{q}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \right\} \tag{12}$$

and define linear the map $\mathcal{N} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{K}, \widetilde{q} \mapsto \mathcal{B}^l_{\widetilde{q}}$. There exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{K}$ via the map $\mathcal{N}$. $\mathcal{B}^l_{\widetilde{q}}$ is the 4×4 left generalized complex matrix representation of $\widetilde{q}$ according to the standard base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

4×4 right generalized complex matrix representation of $\widetilde{q}$ can be calculated similarly[2]. Throughout this paper $\mathcal{B}^l_{\widetilde{q}}$ will be considered. □

Corollary 2. *Considering the base* $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$*, the column matrix representation of* $\widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ *is given by* $\widetilde{p} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}^T$. *Using* $\mathcal{B}^l_{\widetilde{q}}$*, the multiplication of* $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ *can also be written by:* $\widetilde{q}\widetilde{p} = \mathcal{B}^l_{\widetilde{q}} \widetilde{p}$.

Theorem 4. *Let* $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. $\mathcal{B}^l_{\widetilde{q}}$ *can be determined as:*

$$\mathcal{B}^l_{\widetilde{q}} = a_0 I_4 + a_1 \mathbf{I} + a_2 \mathbf{J} + a_3 \mathbf{K},$$

where

$$\mathbf{I} = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Undoubtedly, $\mathbf{I}, \mathbf{J}, \mathbf{K}$ *satisfy the generalized quaternion versors conditions in equations* (1).

Using $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ as $\widetilde{q} = (a_0 + a_1\mathbf{i}) + (a_2 + a_3\mathbf{i})\,\mathbf{j}$ and considering a different conjugate related to this form, we can write the following theorem:

Theorem 5. *Let* $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. *Then, we have* $\sigma \mathcal{B}^l_{\widetilde{q}} \sigma = \mathcal{B}^l_{\widetilde{q}^*}$*, where* $\sigma = \mathrm{diag}(1, 1, -1, -1)$ *and* $\widetilde{q}^* = (a_0 + a_1\mathbf{i}) - (a_2 + a_3\mathbf{i})\mathbf{j} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$.

---

[2] 4×4 right generalized complex matrix representation of $\widetilde{q}$ is:

$$\mathcal{B}^r_{\widetilde{q}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Proof. An easy computation shows that

$$\begin{aligned}\sigma\mathcal{B}^{l}_{\widetilde{q}}\sigma &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a_0 & -\alpha a_1 & \beta a_2 & \alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ -a_2 & -\alpha a_3 & a_0 & -\alpha a_1 \\ -a_3 & a_2 & a_1 & a_0 \end{bmatrix}.\end{aligned}$$

Hence, one can see that the last matrix is $\mathcal{B}^{l}_{\widetilde{q}^*}$. □

Theorem 6. *Let $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{C}_{\mathbf{q},\mathbf{p}}$, the following properties are satisfied:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{B}^{l}_{\widetilde{q}} = \mathcal{B}^{l}_{\widetilde{p}}$,
ii. $\mathcal{B}^{l}_{\widetilde{q}+\widetilde{p}} = \mathcal{B}^{l}_{\widetilde{q}} + \mathcal{B}^{l}_{\widetilde{p}}$,
iii. $\mathcal{B}^{l}_{\lambda\widetilde{q}} = \lambda(\mathcal{B}^{l}_{\widetilde{q}})$,
iv. $\mathcal{B}^{l}_{\widetilde{q}\widetilde{p}} = \mathcal{B}^{l}_{\widetilde{q}}\mathcal{B}^{l}_{\widetilde{p}}$,
v. $\det(\mathcal{B}^{l}_{\widetilde{q}}) = N^2_{\widetilde{q}}$,
vi. $\operatorname{tr}(\mathcal{B}^{l}_{\widetilde{q}}) = 4S_{\widetilde{q}}$.

Proof. By considering the matrix form given in equation (12), the proof is clear. As well let us discuss the proof of the item iv for better understanding:

iv. Using equation (5), we obtain the following matrix for $\mathcal{B}^{l}_{\widetilde{q}\widetilde{p}}$:

$$\begin{bmatrix} \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} & \begin{matrix} -\alpha\,(a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2) \end{matrix} & \begin{matrix} -\beta\,(a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} & \begin{matrix} -\alpha\beta\,(a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} \\ \begin{matrix} a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2 \end{matrix} & \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} & \begin{matrix} -\beta\,(a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} & \begin{matrix} \beta\,(a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} \\ \begin{matrix} (a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} & \begin{matrix} \alpha\,(a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} & \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} & \begin{matrix} -\alpha\,(a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2) \end{matrix} \\ \begin{matrix} (a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} & \begin{matrix} -\,(a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} & \begin{matrix} a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2 \end{matrix} & \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} \end{bmatrix}. \tag{13}$$

Multiplying $\mathcal{B}^{l}_{\widetilde{q}}$ and $\mathcal{B}^{l}_{\widetilde{p}}$ as:

$$\mathcal{B}^{l}_{\widetilde{q}}\mathcal{B}^{l}_{\widetilde{p}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 & -\alpha b_1 & -\beta b_2 & -\alpha\beta b_3 \\ b_1 & b_0 & -\beta b_3 & \beta b_2 \\ b_2 & \alpha b_3 & b_0 & -\alpha b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{bmatrix}$$

gives equation (13) quickly. Hence we get $\mathcal{B}^{l}_{\widetilde{q}\widetilde{p}} = \mathcal{B}^{l}_{\widetilde{q}}\mathcal{B}^{l}_{\widetilde{p}}$.

□

Theorem 7. *Let $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\widetilde{q}^{-1}$ be the inverse of $\widetilde{q}$. Then,*

$$\mathcal{B}^{l}_{\widetilde{q}^{-1}} = \frac{1}{\sqrt{\det(\mathcal{B}^{l}_{\widetilde{q}})}}\mathcal{B}^{l}_{\overline{\widetilde{q}}}.$$

Proof. Taking into account Definition 4 and Theorem 6 items iii and v, the proof is obvious. □

Theorem 8. *Every $\mathcal{GCN}$ with generalized quaternion components can be represented by a 2×2 generalized quaternion matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_2(\mathcal{Q}_{\alpha,\beta})$.*

Proof. For $\widetilde{q}=q_0+q_1 I\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$, denote $\mathcal{T}$ as a subset of $\mathbb{M}_2(\mathcal{Q}_{\alpha,\beta})$ given by:

$$\mathcal{T}:=\left\{\mathcal{D}_{\widetilde{q}}\in\mathbb{M}_2(\mathcal{Q}_{\alpha,\beta}):\mathcal{D}_{\widetilde{q}}=\begin{bmatrix} q_0 & \mathfrak{p}q_1\\ q_1 & q_0+\mathfrak{q}q_1\end{bmatrix}\right\}, \tag{14}$$

and define the linear map $\mathcal{M}:\widetilde{\mathcal{Q}}_{\alpha,\beta}\to\mathcal{T}$, $\widetilde{q}\mapsto\mathcal{D}_{\widetilde{q}}$. It can be concluded that there exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{T}$ via the map $\mathcal{M}$. Hence, 2×2 generalized complex matrix representation of $\widetilde{q}$ with respect to the standard base $\{1,I\}$ is the matrix $\mathcal{D}_{\widetilde{q}}$. □

By using $\mathcal{D}_{\widetilde{q}}$ and $\widetilde{p}=\begin{bmatrix} p_0 & p_1\end{bmatrix}^T$, we have: $\widetilde{q}\widetilde{p}=\mathcal{D}_{\widetilde{q}}\widetilde{p}$. Moreover, $\mathcal{D}_{\widetilde{q}}$ is also in the form $\mathcal{D}_{\widetilde{q}}=q_0 I_2+q_1\mathsf{I}$, where $\mathsf{I}=\begin{bmatrix} 0 & \mathfrak{p}\\ 1 & \mathfrak{q}\end{bmatrix}$ is the representation of I. It should be noted that there are many ways to choose I, for instance: $\mathsf{I}=\begin{bmatrix} \mathfrak{q} & 1\\ \mathfrak{p} & 0\end{bmatrix}$ (see in [32]).

Theorem 9. *For any $\widetilde{q}=q_0+q_1 I$ and $\widetilde{p}=p_0+p_1 I\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda\in\mathbb{R}$, the following properties are satisfied:*

i. $\widetilde{q}=\widetilde{p}\Leftrightarrow\mathcal{D}_{\widetilde{q}}=\mathcal{D}_{\widetilde{p}}$,
ii. $\mathcal{D}_{\widetilde{q}+\widetilde{p}}=\mathcal{D}_{\widetilde{q}}+\mathcal{D}_{\widetilde{p}}$,
iii. $\mathcal{D}_{\lambda\widetilde{q}}=\lambda(\mathcal{D}_{\widetilde{q}})$,
iv. $\mathcal{D}_{\widetilde{q}\widetilde{p}}=\mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}}$,
v. $\mathsf{det}(\mathcal{D}_{\widetilde{q}})=q_0^2+\mathfrak{q}q_1q_0-\mathfrak{p}q_1^2$, *where the notation* det *represents the determinant of the quaternion matrix*[3].

Proof. The proof is obvious considering the matrix form given in equation (14).

iv. Using equation (1), we obtain:

$$\mathcal{D}_{\widetilde{q}\widetilde{p}}=\begin{bmatrix} q_0p_0+\mathfrak{p}q_1p_1 & \mathfrak{p}(q_0p_1+q_1p_0+\mathfrak{q}q_1p_1)\\ q_0p_1+q_1p_0+\mathfrak{q}q_1p_1 & q_0p_0+\mathfrak{p}q_1p_1+\mathfrak{q}(q_0p_1+q_1p_0+\mathfrak{q}q_1p_1)\end{bmatrix}. \tag{15}$$

Also, the computation of the following multiplication

$$\mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}}=\begin{bmatrix} q_0 & \mathfrak{p}q_1\\ q_1 & q_0+\mathfrak{q}q_1\end{bmatrix}\begin{bmatrix} p_0 & \mathfrak{p}p_1\\ p_1 & p_0+\mathfrak{q}p_1\end{bmatrix}$$

gives equation (15). Hence we have $\mathcal{D}_{\widetilde{q}\widetilde{p}}=\mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}}$.

[3]The determinant of an arbitrary 2×2 quaternion matrix is defined by $\mathsf{det}\left(\begin{bmatrix} \mathsf{a} & \mathsf{b}\\ \mathsf{c} & \mathsf{d}\end{bmatrix}\right)=\mathsf{da}-\mathsf{cb}$, [33].

□

Definition 5. *Let $\widetilde{q} = q_0 + q_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. The vector representation of $\widetilde{q}$ is defined as*

$$\overrightarrow{\widetilde{q}} = \begin{bmatrix} \overrightarrow{q_0}^T & \overrightarrow{q_1}^T \end{bmatrix}^T = \begin{bmatrix} \overrightarrow{q_0} \\ \overrightarrow{q_1} \end{bmatrix} \in \mathbb{M}_{8\times 1}(\mathbb{R}),$$

where $q_{j-1} = x_{0j} + x_{1j}\mathbf{i} + x_{2j}\mathbf{j} + x_{3j}\mathbf{k} \in \mathcal{Q}_{\alpha,\beta}$ and

$$\overrightarrow{q_{j-1}} = (x_{0j}, x_{1j}, x_{2j}, x_{3j})^{\ T} = [x_{0j}\ x_{1j}\ x_{2j}\ x_{3j}]^T$$

are vectors (matrices) for $1 \leq j \leq 2$.

Theorem 10. *Let $\widetilde{q} = q_0 + q_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. Then*

i. $\overrightarrow{\widetilde{q}^{\dagger_1}} = \mathcal{X}\overrightarrow{\widetilde{q}}$, *where* $\mathcal{X} = \begin{bmatrix} I_4 & \mathfrak{q} I_4 \\ 0 & -I_4 \end{bmatrix} \in \mathbb{M}_8(\mathbb{R})$.

ii. $\overrightarrow{\widetilde{q}^{\dagger_2}} = \mathcal{Y}\overrightarrow{\widetilde{q}}$, *where* $\mathcal{Y} = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix} \in \mathbb{M}_8(\mathbb{R})$.

Proof.

i. Computing $\overrightarrow{\widetilde{q}^{\dagger_1}}$ and $\mathcal{X}\overrightarrow{\widetilde{q}}$ gives the equality as: $\overrightarrow{\widetilde{q}^{\dagger_1}} = \begin{bmatrix} \overrightarrow{q_0} + \mathfrak{q}\overrightarrow{q_1} \\ -\overrightarrow{q_1} \end{bmatrix}$ and

$$\mathcal{X}\overrightarrow{\widetilde{q}} = \begin{bmatrix} I_4 & \mathfrak{q} I_4 \\ 0 & -I_4 \end{bmatrix}\begin{bmatrix} \overrightarrow{q_0} \\ \overrightarrow{q_1} \end{bmatrix} = \begin{bmatrix} \overrightarrow{q_0} + \mathfrak{q}\overrightarrow{q_1} \\ -\overrightarrow{q_1} \end{bmatrix}.$$

With the same manner the other item can be proved. □

By applying the map $\Gamma(x_{i1} + x_{i2}I) = \begin{bmatrix} x_{i1} & \mathfrak{p}x_{i2} \\ x_{i2} & x_{i1} + \mathfrak{q}x_{i2} \end{bmatrix}$ to $\mathcal{B}^l_{\widetilde{q}}$, where $a_i = x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, for $0 \leq i \leq 3$, the left real matrix representation $\mathcal{C}^l_{\widetilde{q}}$ of $\widetilde{q}$ (see in equation (7)) with respect to the base $\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\}$ can be easily found. So, $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_8(\mathbb{R})$.

Example 1. *Take $\widetilde{q} \in \widetilde{\mathcal{Q}}_{2,1}$ with $\mathcal{GCN}$ components for $\mathfrak{p} = -1$ and $\mathfrak{q} = 1$:*

$$\widetilde{q} = 1 + (-1 + I)\,\mathbf{i} + I\mathbf{j} + (1 + 2I)\,\mathbf{k}.$$

Then,

$$\mathcal{A}_{\widetilde{q}} = \begin{bmatrix} 1 + (1+2I)\mathbf{k} & (-1+I)\,\mathbf{i} + I\mathbf{j} \\ (-1+I)\,\mathbf{i} + I\mathbf{j} & 1 + (1+2I)\mathbf{k} \end{bmatrix},$$

$$\mathcal{B}^l_{\widetilde{q}} = \begin{bmatrix} 1 & -2(-1+I) & -I & -2(1+2I) \\ -1+I & 1 & -1-2I & I \\ I & 2(1+2I) & 1 & -2(-1+I) \\ 1+2I & -I & -1+I & 1 \end{bmatrix},$$

$$\mathcal{C}^l_{\widetilde{q}} = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 1 & -2 & 4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -4 & -6 \\ -1 & -1 & 1 & 0 & -1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 & -2 & -3 & 1 & 1 \\ 0 & -1 & 2 & -4 & 1 & 0 & 2 & 2 \\ 1 & 1 & 4 & 6 & 0 & 1 & -2 & 0 \\ 1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 \\ 2 & 3 & -1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{D}_{\widetilde{q}} = \begin{bmatrix} 1-\mathbf{i}+\mathbf{k} & -\mathbf{i}-\mathbf{j}-2\mathbf{k} \\ \mathbf{i}+\mathbf{j}+2\mathbf{k} & 1+\mathbf{j}+3\mathbf{k} \end{bmatrix},$$

$$\mathcal{B}^l_{\widetilde{q}^{-1}} = \frac{1}{\sqrt{-189+45I}} \begin{bmatrix} 1 & 2(-1+I) & I & 2(1+2I) \\ 1-I & 1 & 1+2I & -I \\ -I & -2(1+2I) & 1 & 2(-1+I) \\ -1-2I & I & 1-I & 1 \end{bmatrix}.$$

Also, the vector representation of $\widetilde{q}^{\dagger_1}$ *is computed by:*

$$\begin{aligned} \overrightarrow{\widetilde{q}^{\dagger_1}} = \mathcal{X}\overrightarrow{\widetilde{q}} &= \begin{bmatrix} I_4 & I_4 \\ 0 & -I_4 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}^T \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 & 3 & 0 & -1 & -1 & -2 \end{bmatrix}^T. \end{aligned}$$

3. Further Result: An Alternative Matrix Approach

The questions about numbers, hypercomplex numbers and quaternions included questions about their matrices. Inspired by matrix forms in the study [34], we give an answer for the question of the alternative representation of generalized quaternion matrix with elliptic number entries (see elliptic biquaternions in [35]). So this matrix is in the form:

$$\widetilde{Q} = A_0 I_2 + A_1\mathcal{I} + A_2\mathcal{J} + A_3\mathcal{K},$$

where $A_0, A_1, A_2, A_3 \in \mathbb{C}_\mathfrak{p}$ are elliptic numbers for $\mathfrak{p} < 0$. The base elements can be defined as follows:

Case 1: For $\alpha, \beta \in \mathbb{R}^+$

$$\mathcal{I} = \begin{bmatrix} \sqrt{\frac{\alpha}{|\mathfrak{p}|}}I & 0 \\ 0 & -\sqrt{\frac{\alpha}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{\beta} \\ -\sqrt{\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I \\ \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 2: For $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^-$

$$\mathcal{I} = \begin{bmatrix} \sqrt{\frac{\alpha}{|\mathfrak{p}|}}I & 0 \\ 0 & -\sqrt{\frac{\alpha}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{-\beta} \\ \sqrt{-\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I \\ -\sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 3: For $\alpha \in \mathbb{R}^-$, $\beta \in \mathbb{R}^+$

$$\mathcal{I} = \begin{bmatrix} 0 & \sqrt{-\alpha} \\ \sqrt{-\alpha} & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} -\sqrt{\frac{\beta}{|\mathfrak{p}|}} I & 0 \\ 0 & \sqrt{\frac{\beta}{|\mathfrak{p}|}} I \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I \\ -\sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 4: For $\alpha, \beta \in \mathbb{R}^-$

$$\mathcal{I} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha}{|\mathfrak{p}|}} I \\ -\sqrt{\frac{-\alpha}{|\mathfrak{p}|}} I & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{-\beta} \\ \sqrt{-\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \\ 0 & -\sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I \end{bmatrix}.$$

These elements satisfy the following conditions:

$$\begin{array}{ll} \mathcal{I}^2 = -\alpha I_2, & \mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K}, \\ \mathcal{J}^2 = -\beta I_2, & \mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J} = \beta\mathcal{I}, \\ \mathcal{K}^2 = -\alpha\beta I_2, & \mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K} = \alpha\mathcal{J}. \end{array}$$

Taking into account Case 1, $\widetilde{Q}$ is rewritten as

$$\widetilde{Q} = \begin{bmatrix} A_0 + \sqrt{\frac{\alpha}{|\mathfrak{p}|}} I A_1 & \sqrt{\beta} A_2 + \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I A_3 \\ -\sqrt{\beta} A_2 + \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I A_3 & A_0 - \sqrt{\frac{\alpha}{|\mathfrak{p}|}} I A_1 \end{bmatrix}.$$

One can see this matrix in Tian's paper [36] related to biquaternions (complexified quaternion) for $\alpha = \beta = 1$ and $\mathfrak{p} = -1$.

The conjugate (same as the adjoint), transpose, the elliptic conjugate, the total conjugate and determinant $\widetilde{Q}$ can be given as follows:

$$\begin{array}{ll} \overline{\widetilde{Q}} & = A_0 I_2 - A_1 \mathcal{I} - A_2 \mathcal{J} - A_3 \mathcal{K} = \operatorname{Adj} \widetilde{Q}, \\ \widetilde{Q}^T & = A_0 I_2 + A_1 \mathcal{I} - A_2 \mathcal{J} + A_3 \mathcal{K}, \\ \widetilde{Q}^{\mathbb{C}_\mathfrak{p}} & = A_0 I_2 - A_1 \mathcal{I} + A_2 \mathcal{J} - A_3 \mathcal{K} = \overline{\widetilde{Q}}^T, \\ \overline{\widetilde{Q}}^{\mathbb{C}_\mathfrak{p}} & = A_0 I_2 + A_1 \mathcal{I} - A_2 \mathcal{J} + A_3 \mathcal{K} = \overline{\left(\widetilde{Q}^{\mathbb{C}_\mathfrak{p}}\right)}, \end{array}$$

and

$$\begin{array}{ll} \det \widetilde{Q} & = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \\ & = A_0^2 + A_1^2 \det \mathcal{I} + A_2^2 \det \mathcal{J} + A_3^2 \det \mathcal{K}. \end{array}$$

For $\det \widetilde{Q} \neq 0$, the inverse of $\widetilde{Q}$ is defined by:

$$\widetilde{Q}^{-1} = \frac{1}{\det \widetilde{Q}} \overline{\widetilde{Q}} = \frac{1}{A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2} (A_0 I_2 - A_1 \mathcal{I} - A_2 \mathcal{J} - A_3 \mathcal{K}).$$

Similar calculations can be given for the other cases. Additionally, the relationships between the above operations and some properties of generalized quaternion matrices with elliptic number entries can be easily proved. We omit them for the sake of brevity. For $A_0, A_1, A_2, A_3 \in \mathbb{C}_{-1}$, we refer to [37] under the condition that $\alpha = \beta = 1$ and $\alpha = 1, \beta = -1$.

4. Concluding Remarks

Our paper is motivated by the question: What happens if the components of quaternions become $\mathcal{GCN}$? Based on this question, we develop the theory of generalized quaternions (non-commutative system) with $\mathcal{GCN}$ components $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$. Also, we investigate the algebraic structures and properties by considering them as a $\mathcal{GCN}$ and a quaternion. With specific values of α and β, we obtained different types of quaternions with $\mathcal{GCN}$ components in Section 2. Additionally, we establish matrix representations and give a numerical example. In Section 3, we also come up with a different way to deal with a generalized quaternion matrix with elliptic number entries.

The crucial part of this paper is that one can reduce the calculations to mentioned types of quaternions by considering hyperbolic, elliptic and parabolic number components for $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$ (see Table 1). As a natural consequence of this situation, taking into account special conditions, the definition of special quaternions mentioned in the papers [38–47] are generalized via Definition 1, the papers [48–53] are generalized from the viewpoint of definition, algebraic properties, relations and matrix representations of quaternions and finally, different matrix forms in the papers [35–37] are generalized in Section 3. All of these situations can be examined in Table 2. For instance, all of the obtained calculations agree with complex quaternions for $\alpha = \beta = 1, \mathfrak{q} = 0, \mathfrak{p} = -1$.

With this unified method, we believe that these results give rise to ease of calculation via mathematical concordance, and in future studies, we intend to investigate other commutative and non-commutative quaternions created with $\mathcal{GCN}$ components in this manner. Now, the necessary and sufficient condition for similarity, co-similarity and semi-similarity for elements of the generalized quaternions with $\mathcal{GCN}$ components for $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$ is an open problem for researchers.

Table 1. Basic classification regarding components

$\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$	**Type of components**	**References**
$\Delta < 0$	elliptic	biquaternion [35, 50] (for $\mathfrak{q} = 0$)
$\Delta = 0$	parabolic	[41, 51] (for $\mathfrak{q} = 0$)
$\Delta > 0$	hyperbolic	

TABLE 2. Classification considering components with regard to the value of $\mathfrak{p}, \mathfrak{q}, \alpha$ and β

Condition	α	β	Component	Quaternion	Ref.
$\mathfrak{q} = 0$, $\mathfrak{p} = -1$	1	1	complex	Hamiltonian	[14, 36, 44]
	1	-1	complex	split	[49, 53]
	1	0	complex	semi	[38, 39]
	-1	0	complex	split semi	
	0	0	complex	quasi	
$\mathfrak{q} = 0$, $\mathfrak{p} = 0$	1	1	dual	Hamiltonian	[45, 47, 54]
	1	-1	dual	split	[46]
	1	0	dual	semi	[42, 52]
	-1	0	dual	split semi	
	0	0	dual	quasi	
$\mathfrak{q} = 0$, $\mathfrak{p} = 1$	1	1	hyperbolic	Hamiltonian	[40]
	1	-1	hyperbolic	split	[43]
	1	0	hyperbolic	semi	
	-1	0	hyperbolic	split semi	[48]
	0	0	hyperbolic	quasi	

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