APPROXIMATION PROPERTIES OF
λ-BERNSTEIN-KANTOROVICH-STANCU OPERATORS

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ABSTRACT. The goal of this paper is to construct a new type of Bernstein operators depending on the shape parameter $\lambda \in [-1, 1]$. For these new type operators a uniform convergence result is presented. Furthermore, order of approximation in the sense of local approximation is investigated and Voronovskaja type theorem is proved. Lastly, some graphical results are given to show the rate of convergence of constructed operators to a given function f.

1. Introduction

In 1912, the Russian mathematician S. N. Bernstein gave one of the most known proof of the Weierstrass's theorem [17], in [5], by defining the following operators; $B_n \colon C[0,1] \to C[0,1]$,

$$B_n(f; x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0, 1].$$

In 1987, Lupaş [12] introduced the first q-analogue of Bernstein operators and investigated its approximating and shape-preserving properties. In 1996, Phillips [14] proposed another q-variant of the classical Bernstein operators, the so-called Phillips q-Bernstein operators. We have to remind that the q-variants of Bernstein polynomials provide one shape parameter for constructing free-form curves and surfaces, Phillips q-Bernstein operator was applied well in this area, as Khan and Lobiyal [11] indicated before.

Leonid V. Kantorovich, in 1930, introduced a generalization of Bernstein operators as $K_n \colon L_1[0,1] \to C[0,1]$

$$K_n(f; x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t. \tag{1.2}$$

K_n operators are obtained from the classical Bernstein operators (1.1), by replacing the approximating function value $f(\frac{k}{n})$ with integral of f in a neighborhood of $\frac{k}{n}$ [10].

In 1968, D. D. Stancu [16] constructed positive linear operators $S_n^{\alpha,\beta} \colon C[0,1] \to C[0,1]$

$$S_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right). \tag{1.3}$$

Here $f \in C[0,1]$, α and β real parameters such that $0 \le \alpha \le \beta$ and $n \in \mathbb{N}$. Note that for $\alpha = \beta = 0$ the operators turn to classical Bernstein operators [5]. Some recent papers mentioning different types of generalizations of Benstein operators and its applications are cited in [11] and [13].

In 2004, D. Barbosu [2] gave a Kantorovich-Stancu form as below:

$$K_n^{\alpha,\beta}(f;x) = (n+\beta+1) \sum_{k=0}^{n} b_{n,k}(x) \int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\mathrm{d}t. \tag{1.4}$$

Some interesting approximation properties of Stancu type operators were given by Barbosu in [3,4].

Constructing a new kind of Bézier curve or a new type of Bernstein operators by using different basis is currently of high interest to researchers. In 2010, Zhangxiang Ye and his colleagues [18] proposed their study not only constructing a new kind of basis functions of Bézier curve with single shape parameter, but also providing a practical algorithm of curve modeling. They discussed some important properties of the basis functions and the corresponding curves. Thus, they explained why a curve or a surface with shape parameters is a good choice.

In 2018, inspiring by a paper in [18], Cai et al. [6] introduced a new generalization of Bernstein operators as follows

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{1.5}$$

where $\lambda \in [-1,1]$ and $\tilde{b}_{n,k}$ for $k = 0,1,\ldots,$ are defined by

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\
\tilde{b}_{n,i}(\lambda;x) &= b_{n,i}(x) + \lambda\left(\frac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x)\right), \\
\tilde{b}_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x).
\end{aligned} \tag{1.6}$$

It is easy to verify that $\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) = 1$ and these operators become the classical Bernstein operators for $\lambda = 0$. They obtained some approximation properties of λ-Bernstein operators and established an asymptotic formula. Also, they gave some numerical examples to show the rate of convergence of $B_{n,\lambda}(f;x)$ to $f(x)$.

Very recently Acu et al. [1] presented a Kantorovich modification of λ-Bernstein operators. They considered the positive linear operators as

$$K_{n,\lambda}(f;x) = (n+1) \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) \int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t. \tag{1.7}$$

They proved a quantitative Voronovskaja type theorem by means of Ditzian-Totik modulus of smoothness. Also, a Grüss-Voronovskaja type theorem for λ-Kantorovich operators were given by them. They studied approximation properties of the Kantorovich variant of λ-Bernstein operators. Some articles can be given about related work [7, 8, 15]. According to this progress, we defined Kantorovich-Stancu type λ-Bernstein operators as below

$$S_{n,\lambda}^{\alpha,\beta}(f;x) = (n+\beta+1)\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)\int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)\mathrm{d}t, \tag{1.8}$$

where Bézier bases $\tilde{b}_{n,k}(\lambda;x)$ are defined in (1.6). Here, as we mentioned above α and β real parameters such that $0 \le \alpha \le \beta$ and $n \in \mathbb{N}$. Besides, it should be taken into consideration that

(i) if we chose $\alpha = \beta = 0$, the sequence of operators defined in (1.8) reduce to operators in (1.7),

(ii) if we take $\lambda = 0$ in (1.8), then the Kantorovich-Stancu type λ-Bernstein operators tend to Kantorovich-Stancu type operators in (1.4),

(iii) For $\alpha = \beta = \lambda = 0$ in (1.8), then the Kantorovich-Stancu type λ-Bernstein operators reduce to Kantorovich type operators in (1.2).

The present work is organized as follows: We demonstrate moments of (1.8) operators can be obtained using the concept of moment generating function. A uniform of these constructed operators (1.8) is mentioned. After, we investigate order of approximation in the sense of local approximation with using a classical approach, the second modulus of continuity, Peetre's K-functional and Lipschitz type function. Then, a Voronovskaja type result is presented for given operators. Some graphical examples are also given in this paper in order to show the rate of convergence of (1.8) operators.

2. Preliminary results

In this part, we present the moments and the central moments of Kantorovich-Stancu type λ-Bernstein operators to use the proofs of the main theorems.

LEMMA 2.1. *Let $e_i(t) := t^i$, $i = 0, 1, 2 \ldots$. Then the λ-Kantorovich-Stancu operators $S_{n,\lambda}^{\alpha,\beta}$ satisfy*

$$S_{n,\lambda}^{\alpha,\beta}(e_0; x) = 1,$$

$$S_{n,\lambda}^{\alpha,\beta}(e_1; x) = \frac{nx+\alpha}{n+\beta+1} + \frac{1}{2(n+\beta+1)} + \lambda\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{(n+\beta+1)(n-1)},$$

$$S_{n,\lambda}^{\alpha,\beta}(e_2; x) = \frac{n(n-1)x^2}{(n+\beta+1)^2} + \frac{2(1+\alpha)nx}{(n+\beta+1)^2} + \frac{\alpha^2+\alpha+\frac{1}{3}}{(n+\beta+1)^2}$$
$$+ \lambda\frac{2nx(1-2x)+2x^{n+1}(n+1)-2x}{(n+\beta+1)^2(n-1)}$$
$$+ \lambda\frac{2\alpha(1-2x+x^{n+1}-(1-x)^{n+1})}{(n+\beta+1)^2(n-1)},$$

$$S_{n,\lambda}^{\alpha,\beta}(e_3;x) = \frac{1}{4(n+\beta+1)^3}\left\{1+2nx(7-9x+9nx+4x^2-6nx^2+2n^2x^2)\right.$$
$$\left.+4\alpha(1+6nx-12nx^2+12n^2x^2+12nx\alpha+3\alpha/2+\alpha^2)\right\}$$
$$+\frac{\lambda(n+3)}{(n^2-1)(n+\beta+1)^3}(1-(1-x)^{n+1}-x^{n+1})$$
$$+\frac{\lambda}{2(n-1)(n+\beta+1)^3}\left\{-1+(1-x)^{n+1}-12x+6nx-30nx^2\right.$$
$$\left.+6n^2x^2+12nx^3-12n^2x^3+13x^{n+1}+12nx^{n+1}+6n^2x^{n+1}\right\}$$
$$+\frac{3\lambda\alpha}{2(n-1)(n+\beta+1)^3}\left\{-4x+4nx-8nx^2+4x^{n+1}+4nx^{n+1}\right.$$
$$\left.+2\alpha-6(1-x)^{n+1}\alpha-4x\alpha+2x^{n+1}\alpha\right\},$$

$$S_{n,\lambda}^{\alpha,\beta}(e_4;x) = \frac{1}{5(n+\beta+1)^4}\left\{5n^4x^4-30n^3x^4+40n^3x^3+55n^2x^4-120n^2x^3\right.$$
$$\left.+80nx^3-75nx^2+30nx+1\right\}+\frac{\alpha}{(n+\beta+1)^4}\left\{4n^3x^3-12n^2x^3\right.$$
$$+8nx^3+18n^2x^2-18nx^2+14nx+1+6n^2x^2\alpha-6nx^2\alpha$$
$$\left.+4nx\alpha^2+12nx\alpha+\alpha^3+2\alpha^2+2\alpha\right\}$$
$$+\frac{2\lambda(n+3)}{(n^2-1)(n+\beta+1)^4}(1-(1-x)^{n+1}-x^{n+1}+2\alpha-2\alpha(1-x)^{n+1}-2\alpha x^{n+1})$$
$$+\frac{\lambda}{(n+\beta+1)^4}\left\{x-nx^2+6n^2x^2-44nx^3-2n^2x^3-16nx^4\right.$$
$$\left.-8n^2x^4-x^{n+1}+29nx^{n+1}+4n^2x^{n+1}-24nx^3\alpha+24nx^{n+1}\alpha\right\}$$
$$+\frac{\lambda}{(n-1)(n+\beta+1)^4}\left\{-2+2(1-x)^{n+1}-9x+nx(5-49x+13nx)\right.$$
$$\left.+x^{n+1}(11+44n-13n^2))\right\}$$
$$+\frac{\lambda\alpha}{(n+\beta+1)^4}\left\{-2+2(1-x)^{n+1}-24x+12nx-60nx^2+12n^2x^2+26x^{n+1}\right.$$
$$+48nx^{n+1}-12n^2x^{n+1}-12x\alpha+12nx\alpha-24nx^2\alpha+12x^{n+1}\alpha+12nx^{n+1}\alpha$$
$$\left.+4\alpha^2-4(1-x)^{n+1}\alpha^2-8x\alpha^2+4x^{n+1}\alpha^2\right\}.$$

Lemma 2.2. *By considering* Lemma 2.1, *λ-Kantorovich-Stancu operators possess*

$$S_{n,\lambda}^{\alpha,\beta}(t-x;x) = \frac{\alpha-(\beta+1)x}{n+\beta+1}+\frac{1}{2(n+\beta+1)}+\lambda\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{(n+\beta+1)(n-1)} \tag{2.1}$$
$$= \varsigma(\alpha,\beta,n,\lambda)(x),$$

$$S_{n,\lambda}^{\alpha,\beta}((t-x)^2;x) = \frac{(\beta+1)^2-n}{(n+\beta+1)^2}x^2-\frac{(\beta+1)(2\alpha+1)-n}{(n+\beta+1)^2}x+\frac{\alpha^2+\alpha+\frac{1}{3}}{(n+\beta+1)^2}$$
$$+\frac{\lambda}{(n+\beta+1)^2(n-1)}\left\{2nx(1-2x)+2x^{n+1}(n+1)-2x\right. \tag{2.2}$$
$$\left.+2(\alpha-(n+\beta+1)x)(1-2x+x^{n+1}-(1-x)^{n+1})\right\}$$
$$= \sigma(\alpha,\beta,n,\lambda)(x).$$

Besides, the consecutive limits hold

$$\lim_{n\to\infty} nS_{n,\lambda}^{\alpha,\beta}(t-x;x) = \frac{2(\alpha - x(\beta+1))+1}{2}$$

and

$$\lim_{n\to\infty} nS_{n,\lambda}^{\alpha,\beta}\left((t-x)^2;x\right) = x(1-x).$$

3. Convergence results

In this main chapter, we search the rate of convergence by means of the modulus of continuity, Peetre's K-functional and elements of Lipschitz class. Here, $C[0,1]$ is the class of real valued functions defined on $[0,1]$ which is uniformly continuous with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

THEOREM 3.1. *For $f \in C[0,1]$, $\lambda \in [-1,1]$, Kantorovich-Stancu type λ-Bernstein operators $S_{n,\lambda}^{\alpha,\beta}$ converge uniformly to f on $[0,1]$.*

P r o o f. By the Bohmann-Korovkin theorem, it is sufficient to show that

$$\lim_{n\to\infty} \|S_{n,\lambda}^{\alpha,\beta}(t^i - x^i)\| = 0, \qquad i = 0,1,2.$$

It can be clearly obtained these three conditions using Lemma 2.1. □

Here, we will give some definitions to use in this section.

DEFINITION 1. For $\delta > 0$ and $f \in C[0,1]$, the modulus of continuity $\omega(f;\delta)$ of the function f is defined by

$$\omega(f;\delta) = \sup_{\substack{x,y\in[0,1] \\ |x-y|\le\delta}} |f(x) - f(y)|.$$

DEFINITION 2. The second modulus of continuity of $f \in C[0,1]$ is defined by

$$\omega_2(f;\sqrt{\delta}) := \sup_{0<t\le\sqrt{\delta}} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C[0,1]}.$$

DEFINITION 3 ([9]). Peetre's K-functional of the function $f \in C[0,1]$ is defined by

$$K_2(f;\delta) := \inf_{g\in W^2[0,1]} \{\|f - g\|_{C[0,1]} + \delta\|g''\|_{C[0,1]}\},$$

where

$$W^2[0,1] := \{g \in C[0,1] : g', g'' \in C[0,1]\}.$$

The following inequality

$$K_2(f;\delta) \le M\{\omega_2(f;\sqrt{\delta})\}$$

is valid for all $\delta > 0$. The positive constant M is independent of f and δ.

In the following theorem, the modulus of continuity is used to estimate the order of approximation to the function f.

THEOREM 3.2. *Let $f \in C[0,1]$, then*

$$\left| S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x) \right| \le 2\omega(f;\delta),$$

where $\delta := \delta(\alpha,\beta,n,\lambda) = \sqrt{\sigma(\alpha,\beta,n,\lambda)(x)}$.

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P r o o f. Keeping in mind the following property of modulus of continuity

$$|f(t) - f(x)| \leq \omega(f; \delta)\Big(\frac{(t-x)^2}{\delta^2} + 1\Big).$$

We obtain

$$|S_{n,\lambda}^{\alpha,\beta}(f; x) - f(x)| \leq S_{n,\lambda}^{\alpha,\beta}(|f(t) - f(x)|; x)$$

$$\leq \omega(f; \delta)\Big(1 + \frac{1}{\delta^2}S_{n,\lambda}^{\alpha,\beta}((t-x)^2; x)\Big).$$

Lastly, choosing $\delta = \sqrt{\sigma(\alpha, \beta, n, \lambda)(x)}$, so

$$|S_{n,\lambda}^{\alpha,\beta}(f; x) - f(x)| \leq 2\omega(f; \delta).$$

Hence, the proof is completed. □

THEOREM 3.3. *If $g \in C^1[0,1]$, then*

$$|S_{n,\lambda}^{\alpha,\beta}(g; x) - g(x)| \leq \varsigma(\alpha, \beta, n, \lambda)(x)|g'(x)| + 2\sqrt{\sigma(\alpha, \beta, n; \lambda)(x)}\omega(g', \sqrt{\sigma(\alpha, \beta, n; \lambda)(x)}).$$

P r o o f. Let $g \in C^1[0,1]$. For any $x, t \in [0,1]$, we have

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t (g'(y) - g'(x))\,\mathrm{d}y,$$

so, applying the operators at above equality, we get

$$S_{n,\lambda}^{\alpha,\beta}(g(t) - g(x); x) = g'(x)S_{n,\lambda}^{\alpha,\beta}(t - x; x) + S_{n,\lambda}^{\alpha,\beta}\Big(\int_x^t (g'(y) - g'(x))\mathrm{d}y; x\Big). \qquad (3.1)$$

Taking into account the following property of modulus of continuity

$$|g(y) - g(x)| \leq \omega(g; \delta)\Big(\frac{|y - x|}{\delta} + 1\Big), \qquad \delta > 0,$$

we can write

$$\Big|\int_x^t |g'(y) - g'(x)|\mathrm{d}y\Big| \leq \omega(g'; \delta)\Big[\frac{(t-x)^2}{\delta} + |t - x|\Big]. \qquad (3.2)$$

By using (3.2) in (3.1), we get

$$|S_{n,\lambda}^{\alpha,\beta}(g; x) - g(x)| \leq |g'(x)| \cdot |S_{n,\lambda}^{\alpha,\beta}(t - x; x)|$$

$$+ \omega(g'; \delta)\Big\{\frac{1}{\delta}S_{n,\lambda}^{\alpha,\beta}((t-x)^2; x) + S_{n,\lambda}^{\alpha,\beta}(|t - x|; x)\Big\}.$$

Using Cauchy-Schwarz inequality then we reach

$$|S_{n,\lambda}^{\alpha,\beta}(g; x) - g(x)| \leq |g'(x)||S_{n,\lambda}^{\alpha,\beta}(t - x; x)|$$

$$+ \omega(g', \delta)\Big\{\frac{1}{\delta}\sqrt{S_{n,\lambda}^{\alpha,\beta}((t-x)^2; x)} + 1\Big\}\sqrt{S_{n,\lambda}^{\alpha,\beta}((t-x)^2; x)}$$

$$\leq |g'(x)|\varsigma(\alpha, \beta, n, \lambda)(x) + \omega(g', \delta)\Big\{\frac{1}{\delta}\sqrt{\sigma(\alpha, \beta, n; \lambda)(x)} + 1\Big\}\sqrt{\sigma(\alpha, \beta, n; \lambda)(x)}.$$

Choosing $\delta = \sqrt{\sigma(\alpha, \beta, n, \lambda)(x)}$, we find the desired inequality. □

Now, we interest in the rate of convergence by the known method K-functional.

Theorem 3.4. *If* $f \in C[0,1]$, *then*

$$\left| S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x) \right| \le M\omega_2\left(f, \frac{1}{2}\sqrt{\sigma(\alpha,\beta,n,\lambda)(x) + \varsigma^2(\alpha,\beta,n,\lambda)(x)} \right) + \omega\left(f, \varsigma(\alpha,\beta,n,\lambda)(x) \right),$$

where M *is a positive constant.*

P r o o f. Denoting

$$\varepsilon_{n,\lambda}^{\alpha,\beta}(x) = \frac{nx+\alpha}{n+\beta+1} + \frac{1}{2(n+\beta+1)} + \lambda\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{(n+\beta+1)(n-1)}$$

and

$$\tilde{S}_{n,\lambda}^{\alpha,\beta}(f;x) = S_{n,\lambda}^{\alpha,\beta}(f;x) + f(x) - f(\varepsilon_{n,k}(x)). \tag{3.3}$$

It follows immediately

$$\tilde{S}_{n,\lambda}^{\alpha,\beta}(e_0;x) = S_{n,\lambda}^{\alpha,\beta}(e_0;x) = 1$$
$$\tilde{S}_{n,\lambda}^{\alpha,\beta}(e_1;x) = S_{n,\lambda}^{\alpha,\beta}(e_1;x) + x - \varepsilon_{n,\lambda}(x) = x.$$

Applying $\tilde{S}_{n,\lambda}^{\alpha,\beta}$ to Taylor's formula, we get

$$\tilde{S}_{n,\lambda}^{\alpha,\beta}(g;x) = g(x) + \tilde{S}_{n,\lambda}^{\alpha,\beta}\left(\int_x^t (t-y)g''(y)\mathrm{d}y; x \right).$$

Therefore,

$$\tilde{S}_{n,\lambda}^{\alpha,\beta}(g;x) = g(x) + S_{n,\lambda}^{\alpha,\beta}\left(\int_x^t (t-y)g''(y)\mathrm{d}y; x \right) - \int_x^{\varepsilon_{n,k}(x)} (\varepsilon_{n,k}(x) - y)\,g''(y)\mathrm{d}y.$$

This implies that

$$|\tilde{S}_{n,\lambda}^{\alpha,\beta}(g;x) - g(x)| \le \left| S_{n,\lambda}^{\alpha,\beta}\left(\int_x^t (t-y)g''(y)\mathrm{d}y; x \right) \right| + \left| \int_x^{\varepsilon_{n,\lambda}^{\alpha,\beta}(x)} \left(\varepsilon_{n,\lambda}^{\alpha,\beta}(x) - y \right) g''(y)\mathrm{d}y \right|$$

$$\le S_{n,\lambda}^{\alpha,\beta}((t-x)^2;x)\,\|g''\| + \left(\varepsilon_{n,\lambda}^{\alpha,\beta}(x) - x \right)^2 \|g''\|$$

$$\le \left[\sigma(\alpha,\beta,n,\lambda)(x) + \varsigma^2(\alpha,\beta,n,\lambda)(x) \right] \|g''\|.$$

In view of (3.3), we obtain

$$|\tilde{S}_{n,\lambda}^{\alpha,\beta}(f;x)| \le |S_{n,\lambda}^{\alpha,\beta}(f;x)| + |f(x)| + |f(\varepsilon_{n,\lambda}^{\alpha,\beta}(x))| \le 3\,\|f\|. \tag{3.4}$$

Now, for $f \in C[0,1]$ and $g \in W^2[0,1]$, using (3.3) and (3.4), we get

$$|S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x)| = \left| \tilde{S}_{n,\lambda}^{\alpha,\beta}(f;x) - f(x) + f(\varepsilon_{n,\lambda}^{\alpha,\beta}(x)) - f(x) \right|$$

$$\le \left| \tilde{S}_{n,\lambda}^{\alpha,\beta}(f-g;x) \right| + \left| \tilde{S}_{n,\lambda}^{\alpha,\beta}(g;x) - g(x) \right| + |g(x) - f(x)| + \left| f(\varepsilon_{n,\lambda}^{\alpha,\beta}(x)) - f(x) \right|$$

$$\le 4\,\|f-g\| + \left[\sigma(\alpha,\beta,n,\lambda)(x) + \varsigma^2(\alpha,\beta,n,\lambda)(x) \right] \|g''\|$$
$$+ \omega\left(f, \varsigma(\alpha,\beta,n,\lambda)(x) \right).$$

Taking the infimum on the right side over all $g \in W^2[0,1]$, we have

$$|S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x)| \leq 4K_2\Big(f, \frac{1}{4}\big(\sigma(\alpha,\beta,n,\lambda)(x) + \varsigma^2(\alpha,\beta,n,\lambda)(x)\big)\Big)$$
$$+ \omega\left(f, \varsigma(\alpha,\beta,n,\lambda)(x)\right).$$

Conclusively, when using the relation between K-functional and the second modulus of continuity as shown in Definition 3, we reach the desired result. □

We investigate the rate of convergence with the help of the functions of Lipschitz class. For this, we have to give below definition primarily.

DEFINITION 4. Let f be a real valued continuous function defined on $[0,\infty)$. Then the following statement holds for f in order γ $(0 < \gamma \leq 1)$ on $[0,\infty)$,

$$|f(x) - f(y)| \leq M|x - y|^\gamma$$

for all $x, y \in [0,\infty)$ and $M > 0$. These set of Lipschitz continuous functions of order γ with Lipschitz constant M is denoted by $Lip_M(\gamma)$.

THEOREM 3.5. Let $f \in Lip_M(\gamma)$, $x \in [0,1]$ and $\lambda \in [-1,1]$, then we have

$$\left|S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x)\right| \leq M\left[\sigma(\alpha,\beta,n,\lambda)(x)\right]^{\frac{\gamma}{2}}$$

where $\sigma(\alpha,\beta,n,\lambda)(x)$ is defined in (2.2).

P r o o f. For the positive linear operators $S_{n,\lambda}^{\alpha,\beta}(f;x)$ and $f \in Lip_M(\gamma)$, while $\sum\limits_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) = 1$, and keeping in mind well-known Hölder's inequality, it can be written directly

$$\left|S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x)\right| \leq S_{n,\lambda}^{\alpha,\beta}\big(|f(t) - f(x)|;x\big)$$

$$= (n+\beta+1)\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) \int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |f(t) - f(x)| \mathrm{d}t$$

$$\leq M(n+\beta+1)\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) \int\limits_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |t - x|^\gamma \mathrm{d}t$$

$$\leq M\left[\sigma(\alpha,\beta,n,\lambda)(x)\right]^{\frac{\gamma}{2}}.$$

Hence, the proof is completed. □

4. Voronovskaja type theorem

Now, we prove the Voronovskaja type theorem for the operators $S_{n,\lambda}^{\alpha,\beta}$.

THEOREM 4.1. Let f be a bounded function on $[0,1]$, f', f'' derivatives exist for any $x \in (0,1)$ and $\lambda \in [-1,1]$. Then, we have

$$\lim_{n\to\infty} 2n\left(S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x)\right) = (2(\alpha - x(\beta+1)) + 1)f'(x) + x(1-x)f''(x).$$

P r o o f. The Taylor's expansion

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + h(t,x)(t-x)^2,$$

where $h(.) \in C[0,1]$ and $\lim_{t \to x} h(t;x) = 0$. Taking into account the linearity of the operators $S_{n,\lambda}^{\alpha,\beta}$ and applying it to both sides of the above Taylor's expansion with a simple calculations, we obtain

$$n\left(S_{n,\lambda}^{\alpha,\beta}f(x) - f(x)\right) = nS_{n,\lambda}^{\alpha,\beta}(t-x;x)f'(x)$$

$$+ nS_{n,\lambda}^{\alpha,\beta}((t-x)^2;x)\frac{f''(x)}{2}$$

$$+ nS_{n,\lambda}^{\alpha,\beta}(h(t,x)(t-x)^2;x).$$

Therefore, by using Lemma 2.2, we get

$$\lim_{n \to \infty} n\left(S_{n,\lambda}^{\alpha,\beta}(f;x) - f(x)\right) = \frac{(2(\alpha - x(\beta+1)) + 1)}{2}f'(x)$$

$$+ \frac{x(1-x)}{2}f''(x) \tag{4.1}$$

$$+ \lim_{n \to \infty} nS_{n,\lambda}^{\alpha,\beta}(h(t,x)(t-x)^2;x).$$

Hence, by the Cauchy-Schwarz inequality, we have

$$n|S_{n,\lambda}^{\alpha,\beta}(h(t,x)(t-x)^2;x)| \leq \sqrt{S_{n,\lambda}^{\alpha,\beta}(h^2(t,x);x)}\sqrt{n^2 S_{n,\lambda}^{\alpha,\beta}((t-x)^4;x)}. \tag{4.2}$$

Since $\lim_{n \to \infty} n^2 S_{n,\lambda}^{\alpha,\beta}((t-x)^4;x)$ is bounded by Lemma 2.1 and $h^2(x,x) = 0$, by considering (4.2) in (4.1), we obtain

$$\lim_{n \to \infty} nS_{n,\lambda}^{\alpha,\beta}(h(t,x)(t-x)^2;x) = 0$$

which completes the proof. $\square$

5. Graphical results

Finally, in this section , we provide some graphical examples that show the convergence of Kantorovich-Stancu type λ-Bernstein operators. With these given graphicals examples, it is possible to understand better how our operators converge to given functions.

***Example* 1.** The first illustration demonstrates the convergence of the operators $S_{n,\lambda}^{\alpha,\beta}$ depending on n. Here, taken $\lambda = 0.4$, $\alpha = 0.2, \beta = 0.3$ values and approximating function of $f(x) = 1 - \sin(\frac{7}{2}\pi x)$, we see the operators respectively for $n = 20$, $n = 50$, $n = 80$ values in Figure 1.
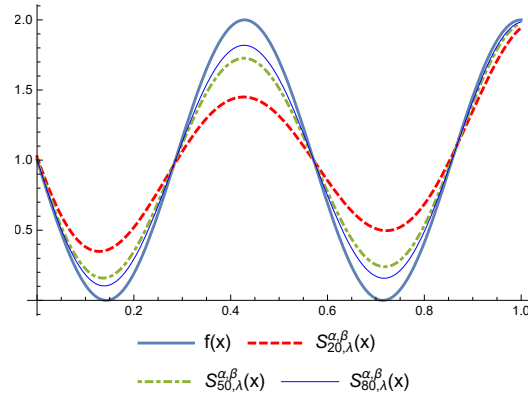
FIGURE 1. Convergence of Kantorovich-Stancu type λ-Bernstein operators for $n = 20$, $n = 50$ and $n = 80$

Example 2. Another example is illustrated in Figure 2 to show impact of shape parameters λ. For $\alpha = 0.2, \beta = 0.3$, $n = 6$ be fixed and $\lambda = -1$, $\lambda = 0$, $\lambda = 1$ approximant $S_{n,\lambda}^{\alpha,\beta}$ convergence to $f(x) = 1 - \sin(\frac{7}{2}\pi x)$.
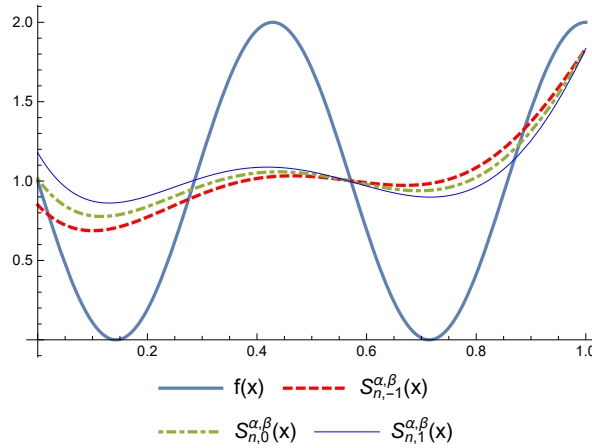

FIGURE 2. Convergence of Kantorovich-Stancu type λ-Bernstein operators for $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$

Now, we would like to take an attention to influence of different real parameters α (also can be shown different β) which is sense of Stancu.

Example 3. For $\lambda = 0.3$, $n = 20$, $\beta = 2$ and $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$ the convergence of Kantorovich-Stancu type λ-Bernstein operators to $f(x) = 1 - \sin(\frac{7}{2}\pi x)$ are illustrated in Figure 3.
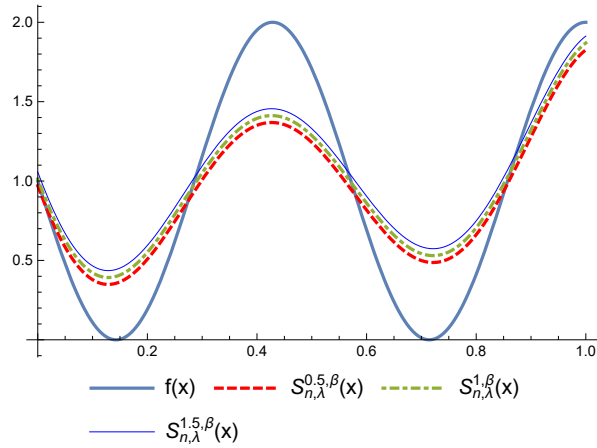
FIGURE 3. Convergence of Kantorovich-Stancu type λ-Bernstein operators for $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$

Example 4. The purpose of the last example as represented in Figure 4 is making a comparison between Kantorovich-Stancu type λ-Bernstein operators and classical Bernstein Kantorovich operators about approximating to $f(x) = 1 - \sin(\frac{7}{2}\pi x)$. Here, $\lambda = 1$, $n = 20$, $\alpha = 0.2$, $\beta = 0.3$.
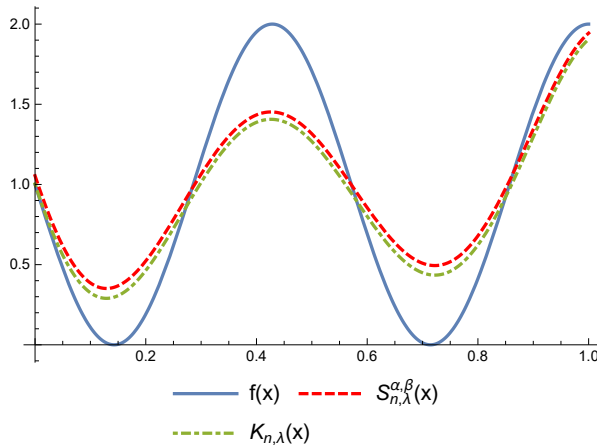

FIGURE 4. Comparison between two the convergence of operators

REFERENCES

[1] ACU, A. M.—MANAV, N.—SOFONEA, D. F.: *Approximation properties of λ-Kantorovich operators*, J. Inequal. Appl. (2018), Art. 202.

[2] BARBOSU, D.: *Kantorovich-Stancu type operators*, J. Inequal. Pure Appl. Math. **5**(3) (2004), 1–6.

[3] BARBOSU, D.: *The Kantorovich form of Schurer Stancu operators*, Demonstr. Math. **37**(2) (2004), 383–391.

[4] BARBOSU, D.: *A survey regarding the approximation properties of the Schurer Stancu operators*, Carpathian J. Math. **20**(1) (2004), 1–5.

[5] BERNSTEIN, S. N.: *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Communications de la Société Mathematique de Kharkov **13**(1–2) (1913).

[6] CAI, Q. B.—LIAN, B. Y.—ZHOU, G.: *Approximation properties of λ-Bernstein operators*, J. Inequal. Appl. (2018), Art. 61.

[7] CAI, Q. B.: *The Bézier variant of Kantorovich type λ-Bernstein operators*, J. Inequal. Appl. (2018), Art. 90.

[8] CAI, Q. B.—ZHOU, G. J.: *Blending type approximation by GBS operators of bivariate tensor product of λ-Bernstein Kantorovich type*, J. Inequal. Appl. (2018), Art. 268.

[9] DITZIAN, Z.—TOTIK, V.: *Moduli of Smoothness*, Springer-Verlag, New York, 1987.

[10] KANTOROVICH, L. V.: *Sur certain développements suivant les polynômes de la forme de S. Bernstein, I. II.*, Dokl. Akad. Nauk. SSSR (1930), 563–568, 595–600.

[11] KHAN, K.—LOBIYAL, D. K.: *Bézier curves based on Lupaş (p,q)-analogue of Bernstein function in CAGD*, J. Comput. Appl. Math. **317** (2017), 458–477.

[12] LUPAŞ, A.: *A q-analogue of the Bernstein operator*, Seminar on Numerical and Statistical Calculus University of Cluj-Napoca **9** (1987), 85–92.

[13] MURSALEEN, M.—ANSARI, K. J.—KHAN, A.: *On (p,q)-analogue of Bernstein Operators*, Appl. Math. Comput. **266** (2015), 874–882; [Erratum: **278**(2016), 70–71].

[14] PHILLIPS, G. M.: *Bernstein polynomials based on the q-integers*, J. Numer. Math. **4** (1997), 511–518.

[15] SRIVASTAVA, H. M.—ÖZGER, F.—MOHIUDDINE, S. A.: *Construction of Stancu-Type Bernstein Operators Based on Bézier Bases with Shape Parameter λ*, Symmetry **11** (2019), Art. 316.

[16] STANCU, D. D.: *Approximation of function by a new class of polynomial operators*, Rev. Roum. Math. Pures Appl. **13**(8) (1968), 1173–1194.

[17] WEIERSTRASS, K.: *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsber. Akad. Wiss. Berlin. (1885), 633–639, 789–805.

[18] YE, Z.—LONG, X.—ZENG, X. M.: *Adjustment algorithms for Bézier curve and surface.* International Conference on Computer Science and Education, 2010, pp. 1712–1716.

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