






## A Study on Dual-Generalized Complex and Hyperbolic-Generalized Complex Numbers

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### Highlights

- This paper focuses on the theories of dual-generalized and hyperbolic-generalized complex numbers.
- The algebraic structures of dual-generalized and hyperbolic-generalized complex numbers are given.
- Dual-generalized complex and hyperbolic-generalized complex valued functions are defined.
- The matrix representations of dual-generalized and hyperbolic-generalized complex numbers are stated.
- An efficient classification includes complex-generalized complex numbers is examined.

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### Abstract

This work is intended to introduce the theories of dual-generalized complex and hyperbolic-generalized complex numbers. The algebraic properties of these numbers are taken into consideration. Besides, dual-generalized complex and hyperbolic-generalized complex valued functions are defined and different matrix representations of these numbers are examined. Moreover, a remarkable classification are given for special cases and the set of complex-generalized complex numbers are mentioned.

## 1. INTRODUCTION

The set of complex numbers is denoted by  $\mathbb{C}$  and defined as

$$\mathbb{C} = \{z = (x_1, x_2) = x_1 + x_2i \mid x_1, x_2 \in \mathbb{R}, i^2 = -1\}.$$

As it is well known that the set of complex numbers is a field, an associative and commutative algebra with unity. The complex numbers are applied to many aspects of real life, especially in applied mathematics, physics, quantum mechanics and electronics. The detailed information related to complex numbers can be found in [1].

Besides, different definitions of the complex unit “ $i$ ” conduce to new special number systems. Hence complex numbers are the building blocks of the other two dimensional number systems, such as hyperbolic numbers (double, split complex, perplex) [2,3], dual numbers [4, 5] and multicomponent number systems [6].

As a generalization of complex numbers, the set of generalized complex numbers is denoted by  $\mathbb{C}_p$  and defined as:

$$\mathbb{C}_p = \left\{ z = (x_1, x_2) = x_1 + x_2 J \mid x_1, x_2 \in \mathbb{R}, J^2 = p, -\infty < p < \infty \right\}.$$

The generalized complex numbers analogous with the set of complex numbers  $\mathbb{C}$  for  $p = -1$ , hyperbolic numbers  $\mathbb{H}$  for  $p = 1$ , dual numbers  $\mathbb{D}$  for  $p = 0$ . Also,  $\mathbb{C}_p$  is a vector space over real numbers  $\mathbb{R}$  and  $\{1, J\}$  is basis  $\mathbb{C}_p$ . The further information about generalized complex numbers can be found in [7].

The idea of investigating the number systems by writing the coefficients as elements of the sets  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{D}$  is a remarkable area for the researchers. Therefore, hyperbolic numbers with complex coefficients (complex-hyperbolic numbers or hyperbolic-complex numbers) are introduced by [8] and examined in [9, 10].  $n$ -dimensional hyperbolic-complex numbers are examined in [11]. Moreover, bicomplex numbers are investigated in [12-14]. Dual numbers with complex coefficients (complex-dual numbers or dual-complex numbers) are presented in [15, 16]. In [17], the concept of dual-complex numbers and their holomorphic functions are investigated. Moreover, the algebraic properties, functions and matrix representations of dual-hyperbolic numbers and complex-hyperbolic numbers are presented in [18]. Also, as an extension of dual numbers, hyperdual numbers are studied in [19-24]. Bihyperbolic numbers are examined in [25-28]. There exist quite studies in the literature related to types of numbers which consist of different combination of the coefficients, [29, 30].

The main point of this paper is to develop the theory of dual-generalized complex and hyperbolic-generalized complex numbers by using generalized complex number theory. So, the paper is organized as 3 main sections and conclusions section. Section 2 is devoted to the construction of the concept of dual-generalized complex numbers. In this section, we firstly introduce algebraic structures of dual-generalized complex numbers such as the conjugations and the modules. Then, dual-generalized complex valued functions are concerned. Besides, the real and generalized matrix representations of dual-generalized complex numbers and the real and dual matrix representations of generalized complex-dual numbers are stated. In Section 3, the concept of hyperbolic-generalized complex numbers are given. The algebraic structures, hyperbolic-generalized complex valued functions, and matrix representations of these numbers are concerned. In conclusions, a notable classification that includes complex-generalized complex numbers are addressed by considering special cases.

## 2. DUAL-GENERALIZED COMPLEX NUMBERS

In this original section, we construct the theory of dual-generalized complex numbers.

**Definition 1.** A dual-generalized complex number  $w$  is an ordered pair of generalized complex numbers  $(z_1, z_2)$  and denoted in the form

$$w = (z_1, z_2) = z_1 + z_2 \varepsilon, \quad (1)$$

where  $z_1 = x_1 + x_2 J \in \mathbb{C}_p$ ,  $z_2 = y_1 + y_2 J \in \mathbb{C}_p$ ,  $J^2 = p$ ,  $-\infty < p < \infty$  and  $\varepsilon^2 = 0$ ,  $\varepsilon \neq 0$ . This special number is a dual number with generalized complex coefficients. The set of these numbers is denoted by  $\mathbb{DC}_p$  and defined as

$$\mathbb{DC}_p = \left\{ w = z_1 + z_2 \varepsilon \mid z_1, z_2 \in \mathbb{C}_p, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}.$$

If  $x_1 \neq 0, x_2 \neq 0$  and  $y_1 = y_2 = 0$ , then  $x_1 + x_2 J$  is called proper generalized-complex number. If all values are non-zero,  $w$  is called a proper dual-generalized complex number.

**Definition 2.** Let  $w_1, w_2 \in \mathbb{DC}_p$  and  $\lambda \in \mathbb{R}$ . Then, the addition, subtraction, multiplication, scalar multiplication and equality of dual-generalized complex numbers are defined respectively as in the following Table:

**Table 1.** Operations for dual-generalized complex numbers

Addition	$\oplus: \mathbb{DC}_p \times \mathbb{DC}_p \rightarrow \mathbb{DC}_p$ $(w_1, w_2) \rightarrow w_1 \oplus w_2$	$w_1 \oplus w_2 = (z_{11} + z_{12}\varepsilon) \oplus (z_{21} + z_{22}\varepsilon)$ $= (z_{11} + z_{21}) + (z_{12} + z_{22})\varepsilon$
Substraction	$\ominus: \mathbb{DC}_p \times \mathbb{DC}_p \rightarrow \mathbb{DC}_p$ $(w_1, w_2) \rightarrow w_1 \ominus w_2$	$w_1 \ominus w_2 = (z_{11} + z_{12}\varepsilon) \ominus (z_{21} + z_{22}\varepsilon)$ $= (z_{11} - z_{21}) + (z_{12} - z_{22})\varepsilon$
Multiplication	$\Delta: \mathbb{DC}_p \times \mathbb{DC}_p \rightarrow \mathbb{DC}_p$ $(w_1, w_2) \rightarrow w_1 \Delta w_2$	$w_1 \Delta w_2 = (z_{11} + z_{12}\varepsilon) \Delta (z_{21} + z_{22}\varepsilon)$ $= (z_{11}z_{21}) + (z_{11}z_{22} + z_{12}z_{21})\varepsilon$
Scalar Multiplication	$\odot: \mathbb{R} \times \mathbb{DC}_p \rightarrow \mathbb{DC}_p$ $(\lambda, w_1) \rightarrow \lambda \odot w_1$	$\lambda \odot w_1 = \lambda \odot (z_{11} + z_{12}\varepsilon)$ $= (\lambda z_{11}) + (\lambda z_{12})\varepsilon$
Equality	$w_1 = w_2 \Leftrightarrow z_{11} + z_{12}\varepsilon = z_{21} + z_{22}\varepsilon \Leftrightarrow z_{11} = z_{21} \wedge z_{12} = z_{22}$ .	

**Proposition 1.** The algebraic properties of  $\mathbb{DC}_p$  are given as follows:

- i)  $(\mathbb{DC}_p, \oplus, \Delta)$  is a commutative ring with unity under addition and multiplication as defined in Definition 2. The additive and multiplicative identities of this ring are  $(0, 0) = 0 + 0\varepsilon$  and  $(1, 0) = 1 + 0\varepsilon$ , respectively. But,  $\mathbb{DC}_p$  is not a field, because of the fact that the elements of the form  $w = 0 + z_2\varepsilon$  are not invertible. All elements of this form are called zero divisors.
- ii)  $\{\mathbb{DC}_p, \oplus, \mathbb{R}, +, \cdot, \odot\}$  is a vector space over  $\mathbb{R}$  with addition and scalar multiplication defined in Definition 2.
- iii) The set  $\{1, J, \varepsilon, J\varepsilon\}$  (or  $\{1, \varepsilon, J, \varepsilon J\}$ ) is a basis for  $\mathbb{DC}_p$ , i.e.  $\mathbb{DC}_p = span\{1, J, \varepsilon, J\varepsilon\}$ . Then,  $dim\mathbb{DC}_p = 4$ .
- iv) The elements of the basis  $\{1, J, \varepsilon, J\varepsilon\}$  satisfy the properties of the commutative multiplication scheme given in Table 2.

**Table 2.** Multiplication scheme of dual-generalized complex numbers

$\Delta$	1	J	$\varepsilon$	$J\varepsilon$
1	1	J	$\varepsilon$	$J\varepsilon$
J	J	p	$J\varepsilon$	$p\varepsilon$
$\varepsilon$	$\varepsilon$	$J\varepsilon$	0	0
$J\varepsilon$	$J\varepsilon$	$p\varepsilon$	0	0

- v) A dual-generalized complex number can be given in terms of its basis elements  $\{1, J, \varepsilon, J\varepsilon\}$  (or  $\{1, \varepsilon, J, \varepsilon J\}$ ) by using Table 2 as  $w = x_1 + x_2J + y_1\varepsilon + y_2J\varepsilon$ , where J indicates the pure generalized complex unit,  $\varepsilon$  is the pure dual unit and  $J\varepsilon$  is the dual-generalized complex unit. Moreover,  $x_1$  is called the real part,  $x_2$  is called the generalized complex part,  $y_1$  is called the dual part and  $y_2$  is called the dual-generalized complex part of  $w$ .

vi)  $\{\mathbb{D}\mathbb{C}_p, \oplus, \mathbb{R}, +, \cdot, \odot, \Delta\}$  is an algebra with operations defined in Definition 2.

**Remark 1.**  $\forall w \in \mathbb{D}\mathbb{C}_p$ , it can be written that

$$w = z_1 + z_2\mathcal{E} = (x_1 + x_2J) + (y_1 + y_2J)\mathcal{E} = (x_1 + y_1\mathcal{E}) + (x_2 + y_2\mathcal{E})J = d_1 + d_2J,$$

where  $d_1, d_2 \in \mathbb{D}$ . Hence the set of generalized complex-dual numbers is denoted by  $\mathbb{C}_p\mathbb{D}$  and defined as:

$$\mathbb{C}_p\mathbb{D} = \{w = d_1 + d_2J \mid d_1, d_2 \in \mathbb{D}, J^2 = \mathfrak{p}, -\infty < \mathfrak{p} < \infty\}.$$

So, there is no difference between dual-generalized complex numbers and generalized complex-dual numbers.

Conjugation of a number plays a significant role for its algebraic and geometric properties. For dual-generalized complex numbers, there are five different conjugations and four modulus in  $\mathbb{D}\mathbb{C}_p$  are as follows:

**Definition 3.** For any  $w = z_1 + z_2\mathcal{E} \in \mathbb{D}\mathbb{C}_p$ , the conjugations and modules of  $w$  are defined as in Table 3. (Note that, for  $z = x_1 + x_2J \in \mathbb{C}_p$ ,  $x_1$  and  $x_2$  are called real and imaginary part of  $z$  and denoted by  $\text{Re}(z) = x_1$  and  $\text{Im}(z) = x_2$ , respectively.  $\bar{z} = x_1 - x_2J$  indicates conjugate and  $|z|^2$  indicates the module of  $z \in \mathbb{C}_p$ , [7].)

**Table 3.** Conjugations and modules of dual-generalized complex numbers

Generalized complex conjugation	$w^{\dagger_1} = \bar{z}_1 + \bar{z}_2\mathcal{E}$
Generalized complex module	$ w _{\dagger_1}^2 = w_{\Delta}w^{\dagger_1} =  z_1 ^2 + 2\text{Re}(z_1\bar{z}_2)\mathcal{E} \in \mathbb{D}$
Dual conjugation	$w^{\dagger_2} = z_1 - z_2\mathcal{E}$
Dual module	$ w _{\dagger_2}^2 = w_{\Delta}w^{\dagger_2} = z_1^2 \in \mathbb{C}_p$
Coupled conjugation	$w^{\dagger_3} = \bar{z}_1 - \bar{z}_2\mathcal{E}$
Coupled module	$ w _{\dagger_3}^2 = w_{\Delta}w^{\dagger_3} =  z_1 ^2 - 2\text{Im}(z_1\bar{z}_2)J\mathcal{E} \in \mathbb{D}\mathbb{C}_p$
Dual-generalized complex conjugation ( $w$ is not a zero divisor)	$w^{\dagger_4} = \bar{z}_1 \begin{pmatrix} 1 - \frac{z_2}{z_1}\mathcal{E} \\ z_1 \end{pmatrix}$
Dual-generalized complex module	$ w _{\dagger_4}^2 = w_{\Delta}w^{\dagger_4} =  z_1 ^2 \in \mathbb{R}$
Anti-dual conjugation	$w^{\dagger_5} = z_2 - z_1\mathcal{E}$

**Proposition 2.** Let  $w_1, w_2$  and  $w = z_1 + z_2\mathcal{E} \in \mathbb{DC}_p$ . Then, the following assertions hold:

- i)  $(w^{\dagger k})^{\dagger k} = w, k = 1, 2, 3, 4, (z_1 \neq 0 \text{ for } w^{\dagger 4}),$
- ii)  $(w^{\dagger 5})^{\dagger 5} = -w,$
- iii)  $(w_1 \oplus w_2)^{\dagger k} = w_1^{\dagger k} \oplus w_2^{\dagger k}, k = 1, 2, 3, 5,$
- iv)  $(w_1 \ominus w_2)^{\dagger k} = w_1^{\dagger k} \ominus w_2^{\dagger k}, k = 1, 2, 3, 5,$
- v)  $(w_1 \Delta w_2)^{\dagger k} = w_1^{\dagger k} \Delta w_2^{\dagger k}, (z_1 \neq 0 \text{ for } w^{\dagger 4}),$
- vi)  $\left(\frac{1}{w}\right)^{\dagger k} = \frac{1}{w^{\dagger k}}, k = 2, 4, (z_1 \neq 0 \text{ for } w^{\dagger 4}),$
- vii)  $w \oplus w^{\dagger 1} = 2 \operatorname{Re}(z_1) + 2 \operatorname{Re}(z_2)\mathcal{E} \in \mathbb{D},$
- viii)  $w \oplus w^{\dagger 2} = 2z_1 \in \mathbb{C}_p,$
- ix)  $w \oplus w^{\dagger 3} = 2 \operatorname{Re}(z_1) + 2 \operatorname{Im}(z_2)J\mathcal{E} \in \mathbb{DC}_p,$
- x)  $z_1 \cdot w^{\dagger 4} = \bar{z}_1 \cdot w^{\dagger 2} \in \mathbb{DC}_p, \text{ where } \cdot : \mathbb{C}_p \times \mathbb{DC}_p \rightarrow \mathbb{DC}_p,$
- xi)  $z_1 = w \ominus (w^{\dagger 5} \Delta \mathcal{E}), z_2 = w^{\dagger 5} \oplus (w \Delta \mathcal{E}).$

**Definition 4.** Under the condition that  $w$  is not a zero divisor, the inverse of any dual-generalized complex number  $w$  is defined as

$$\frac{1}{w} = \frac{w^{\dagger 2}}{|w|_{\dagger 2}^2} = \frac{w^{\dagger 4}}{|w|_{\dagger 4}^2}.$$

## 2.1. Dual-Generalized Complex Valued Functions

Here, we examine the dual-generalized complex valued function. In order to give this examination, we firstly give the following definition:

**Definition 5.** The following statements can be given for the set  $\mathbb{DC}_p$  :

- i)  $\Omega$  is called a dual-generalized complex subset of  $\mathbb{DC}_p$  if there exists a subset  $S \subset \mathbb{C}_p$  such that  $\Omega \approx S \times \mathbb{C}_p$ . Here  $S$  is called generator of  $\Omega$ .
- ii)  $\Omega$  is an open dual-generalized complex subset of  $\mathbb{DC}_p$  if the generator of  $\Omega$  is an open subset of  $\mathbb{C}_p$ .
- iii)  $\Omega$  is a closed dual-generalized complex subset of  $\mathbb{DC}_p$  if its complement is an open subset of  $\mathbb{DC}_p$ .

**Definition 6.** A dual-generalized complex valued function  $f$  is a mapping such that:

$$f : \Omega \subset \mathbb{DC}_p \rightarrow \mathbb{DC}_p$$

$$z_1 + z_2\mathcal{E} \rightarrow f(z_1 + z_2\mathcal{E}),$$

where  $\Omega$  is an open dual-generalized complex subset of  $\mathbb{DC}_p$ .

**Definition 7.** The dual-generalized complex valued function  $f$  is called continuous at  $w_0 = z_{01} + z_{02}\varepsilon$  if  $\lim_{w \rightarrow w_0} f(w) = f(w_0)$ . Moreover,  $f$  is called continuous in open subset  $\Omega \subset \mathbb{DC}_p$  if it is continuous at every point of  $\Omega$ .

**Definition 8.** The dual-generalized complex valued function  $f$  is differentiable at  $w_0 = z_{01} + z_{02}\varepsilon$  if the following limit exists:

$$\frac{df}{dw}(w_0) = \lim_{w \rightarrow w_0} \frac{f(w) - f(w_0)}{w - w_0}.$$

$\frac{df}{dw}(w_0)$  is called the derivative of  $f$  at the point  $w_0$ . Besides,  $f$  is differentiable in an open subset  $\Omega \subset \mathbb{DC}_p$  if it is differentiable at every point of  $\Omega$ .

**Theorem 1.** Let  $f$  be an dual-generalized complex infinitely differentiable function in open subset  $\Omega \subset \mathbb{DC}_p$ . For every  $w = z_1 + z_2\varepsilon \in \Omega$ , it can be written as:

$$f(z_1 + z_2\varepsilon) = f(z_1) + z_2 f'(z_1)\varepsilon.$$

**Proof.** The Taylor series of  $f(z_1 + z_2\varepsilon)$  at  $w=0$  can be written as:

$$f(w) = f(0) + f'(0)(z_1 + z_2\varepsilon) + f''(0) \frac{z_1^2 + 2z_1z_2\varepsilon}{2!} + \dots + f^{(k)}(0) \frac{z_1^k + (kz_1^{k-1}z_2)\varepsilon}{k!} + \dots$$

If we rearrange the above equation, we get

$$f(w) = \left( f(0) + f'(0)z_1 + f''(0) \frac{z_1^2}{2!} + \dots + f^{(k)}(0) \frac{z_1^k}{k!} + \dots \right) + z_2 \left( f'(0) + f''(0)z_1 + \dots + f^{(k)}(0) \frac{z_1^{k-1}}{(k-1)!} + \dots \right) \varepsilon.$$

Then, we have  $f(w) = f(z_1) + z_2 f'(z_1)\varepsilon$ .

**Example 1.** Some basic examples for the dual-generalized complex valued functions are given as follows:

- i)  $e^{z_1 + z_2\varepsilon} = e^{z_1} + z_2 e^{z_1} \varepsilon,$
- ii)  $\sqrt{z_1 + z_2\varepsilon} = \sqrt{z_1} + z_2 \frac{1}{2\sqrt{z_1}} \varepsilon,$
- iii)  $\ln(z_1 + z_2\varepsilon) = \ln z_1 + \frac{z_2}{z_1} \varepsilon,$
- iv)  $(z_1 + z_2\varepsilon)^n = z_1^n + n z_2 z_1^{n-1} \varepsilon.$

## 2. 2. Matrix Representations of Dual-Generalized Complex Numbers

Here, we first define the following transformation

$$f : \mathbb{DC}_p \rightarrow \mathbb{DC}_p$$

$$w_1 \rightarrow f_w(w_1) = w_{\Delta} w_1,$$

for every  $w = z_1 + z_2\varepsilon = x_1 + x_2J + y_1\varepsilon + y_2J\varepsilon \in \mathbb{D}\mathbb{C}_p$ . A trivial computation shows that  $f$  is a linear transformation. The real and generalized complex matrix representations of  $f$  (of  $w$  as well) with respect to the basis  $\{1, J, \varepsilon, J\varepsilon\}$  and  $\{1, \varepsilon\}$  (basis for  $\mathbb{D}$ ) are respectively given as follows:

$$A_{\mathbb{R}} = \begin{bmatrix} x_1 & px_2 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ y_1 & py_2 & x_1 & px_2 \\ y_2 & y_1 & x_2 & x_1 \end{bmatrix}, A_{\mathbb{C}_p} = \begin{bmatrix} x_1 + x_2J & 0 \\ y_1 + y_2J & x_1 + x_2J \end{bmatrix} = \begin{bmatrix} z_1 & 0 \\ z_2 & z_1 \end{bmatrix}.$$

Moreover,  $\det(A_{\mathbb{C}_p}) = z_1^2 = |w|_{\mathbb{C}_p}^2$ .

There is no loss of generality in assuming that  $w = z_1 + z_2\varepsilon = d_1 + d_2J \in \mathbb{C}_p\mathbb{D}$ . Similarly, we can write the matrices of  $w$  with respect to the basis  $\{1, \varepsilon, J, \varepsilon J\}$  and  $\{1, J\}$ , respectively:

$$B_{\mathbb{R}} = \begin{bmatrix} x_1 & 0 & px_2 & 0 \\ y_1 & x_1 & py_2 & px_2 \\ x_2 & 0 & x_1 & 0 \\ y_2 & x_2 & y_1 & x_1 \end{bmatrix}, B_{\mathbb{D}} = \begin{bmatrix} x_1 + y_1\varepsilon & p(x_2 + y_2\varepsilon) \\ x_2 + y_2\varepsilon & x_1 + y_1\varepsilon \end{bmatrix} = \begin{bmatrix} d_1 & pd_2 \\ d_2 & d_1 \end{bmatrix}.$$

Besides,  $\det(B_{\mathbb{D}}) = d_1^2 - pd_2^2 = |w|_{\mathbb{D}}^2$ .

**Example 2.** The matrix representations of the dual-generalized complex number  $w = -2 + 5J + \varepsilon - 4J\varepsilon$  can be calculated with respect to basis  $\{1, J, \varepsilon, J\varepsilon\}$  and  $\{1, \varepsilon\}$ , respectively as follows:

$$A_{\mathbb{R}} = \begin{bmatrix} -2 & 5p & 0 & 0 \\ 5 & -2 & 0 & 0 \\ 1 & -4p & -2 & 5p \\ -4 & 1 & 5 & -2 \end{bmatrix}, A_{\mathbb{C}_p} = \begin{bmatrix} -2 + 5J & 0 \\ 1 - 4J & -2 + 5J \end{bmatrix},$$

and with respect to basis  $\{1, \varepsilon, J, \varepsilon J\}$  and  $\{1, J\}$ , respectively as follows:

$$B_{\mathbb{R}} = \begin{bmatrix} -2 & 0 & 5p & 0 \\ 1 & -2 & -4p & 5p \\ 5 & 0 & -2 & 0 \\ -4 & 5 & 1 & -2 \end{bmatrix}, B_{\mathbb{D}} = \begin{bmatrix} -2 + \varepsilon & p(5 - 4\varepsilon) \\ 5 - 4\varepsilon & -2 + \varepsilon \end{bmatrix}.$$

Also,  $\det(A_{\mathbb{C}_p}) = (4 + 25p) - 20J$  and  $\det(B_{\mathbb{D}}) = (4 - 25p) + (-4 + 40p)\varepsilon$ .

### 3. HYPERBOLIC-GENERALIZED COMPLEX NUMBERS

In this original section, we present the concept of hyperbolic-generalized complex numbers.

**Definition 9.** A hyperbolic-generalized complex number  $\tilde{w}$  is an ordered pair of generalized complex numbers  $(z_1, z_2)$  and denoted in the form

$$\tilde{w} = (z_1, z_2) = z_1 + z_2j, \tag{2}$$

where  $z_1 = x_1 + x_2J \in \mathbb{C}_p$ ,  $z_2 = y_1 + y_2J \in \mathbb{C}_p$ ,  $J^2 = p$ ,  $-\infty < p < \infty$  and  $j^2 = 1, j \neq \pm 1$ . This special number is a hyperbolic number with generalized complex coefficients. The set of these numbers is denoted by  $\mathbb{HC}_p$  and defined as

$$\mathbb{HC}_p = \{ \tilde{w} = z_1 + z_2j \mid z_1, z_2 \in \mathbb{C}_p, j^2 = 1, j \neq \pm 1 \}.$$

If all values are non-zero,  $\tilde{w}$  is called a proper hyperbolic-generalized complex number.

**Definition 10.** Let  $\tilde{w}_1, \tilde{w}_2 \in \mathbb{HC}_p$  and  $\lambda \in \mathbb{R}$ . Then, the addition, subtraction, multiplication, scalar multiplication and equality of hyperbolic-generalized complex numbers are defined respectively as in the following table:

**Table 4.** Operations for hyperbolic-generalized complex numbers

Addition	$\oplus : \mathbb{HC}_p \times \mathbb{HC}_p \rightarrow \mathbb{HC}_p$ $(\tilde{w}_1, \tilde{w}_2) \rightarrow \tilde{w}_1 \oplus \tilde{w}_2$	$\tilde{w}_1 \oplus \tilde{w}_2 = (z_{11} + z_{12}j) \oplus (z_{21} + z_{22}j)$ $= (z_{11} + z_{21}) + (z_{12} + z_{22})j$
Substraction	$\ominus : \mathbb{HC}_p \times \mathbb{HC}_p \rightarrow \mathbb{HC}_p$ $(\tilde{w}_1, \tilde{w}_2) \rightarrow \tilde{w}_1 \ominus \tilde{w}_2$	$\tilde{w}_1 \ominus \tilde{w}_2 = (z_{11} + z_{12}j) \ominus (z_{21} + z_{22}j)$ $= (z_{11} - z_{21}) + (z_{12} - z_{22})j$
Multiplication	$\Delta : \mathbb{HC}_p \times \mathbb{HC}_p \rightarrow \mathbb{HC}_p$ $(\tilde{w}_1, \tilde{w}_2) \rightarrow \tilde{w}_1 \Delta \tilde{w}_2$	$\tilde{w}_1 \Delta \tilde{w}_2 = (z_{11} + z_{12}j) \Delta (z_{21} + z_{22}j)$ $= (z_{11}z_{21} + z_{12}z_{22}) + (z_{11}z_{22} + z_{12}z_{21})j$
Scalar Multiplication	$\odot : \mathbb{R} \times \mathbb{HC}_p \rightarrow \mathbb{HC}_p$ $(\lambda, \tilde{w}_1) \rightarrow \lambda \odot \tilde{w}_1$	$\lambda \odot \tilde{w}_1 = \lambda \odot (z_{11} + z_{12}j)$ $= (\lambda z_{11}) + (\lambda z_{12})j$
Equality	$\tilde{w}_1 = \tilde{w}_2 \Leftrightarrow z_{11} + z_{12}j = z_{21} + z_{22}j \Leftrightarrow z_{11} = z_{21} \wedge z_{12} = z_{22}$	

**Proposition 3.** The algebraic properties of the set of hyperbolic-generalized complex numbers  $\mathbb{HC}_p$  are given as follows:

- i)  $(\mathbb{HC}_p, \oplus, \Delta)$  is a commutative ring with unity under addition and multiplication as defined in Definition 10. The additive and multiplicative identities of this ring are  $(0,0) = 0 + 0j$  and  $(1,0) = 1 + 0j$ , respectively. But,  $\mathbb{HC}_p$  is not a field, owing to the fact that the elements of the form  $\tilde{w} = z_1 + z_2j$  are not invertible when  $z_1^2 - z_2^2 = 0$ . All elements of this form are called zero divisors.
- ii)  $\{ \mathbb{HC}_p, \oplus, \mathbb{R}, +, \cdot, \odot \}$  is a vector space over  $\mathbb{R}$  with addition and scalar multiplication defined in Definition 10.



- iii) The set  $\{1, J, j, Jj\}$  (or  $\{1, j, J, jJ\}$ ) is a basis for  $\mathbb{HC}_p$ , i.e.  $\mathbb{HC}_p = sp\{1, J, j, Jj\}$ . Then  $\dim \mathbb{HC}_p = 4$ .
- iv) The elements of the basis  $\{1, J, j, Jj\}$  satisfy the properties of the following commutative multiplication scheme given in Table 5.

**Table 5.** Multiplication scheme of hyperbolic-generalized complex numbers

$\Delta$	1	$J$	$j$	$Jj$
1	1	$J$	$j$	$Jj$
$J$	$J$	$\mathfrak{p}$	$Jj$	$\mathfrak{p}j$
$j$	$j$	$Jj$	1	$J$
$Jj$	$Jj$	$\mathfrak{p}j$	$J$	$\mathfrak{p}$

- v) A hyperbolic-generalized complex number can be given in terms of its basis elements  $\{1, J, j, Jj\}$  (or  $\{1, j, J, jJ\}$ ) by using Table 5 as  $\tilde{w} = x_1 + x_2J + y_1j + y_2Jj$ , where  $J$  indicates the pure generalized complex unit,  $j$  is the pure hyperbolic unit and  $Jj$  is the hyperbolic-generalized complex unit. Besides,  $x_1$  is called the real part,  $x_2$  is called the generalized complex part,  $y_1$  is called the hyperbolic part and  $y_2$  is called the hyperbolic-generalized complex part of  $\tilde{w}$ .
- vi)  $\{\mathbb{HC}_p, \oplus, \mathbb{R}, +, \cdot, \odot, \Delta\}$  is an algebra with operations defined in Definition 10.

**Remark 2.**  $\forall \tilde{w} \in \mathbb{HC}_p$ , the following equation can be written:

$$\tilde{w} = z_1 + z_2j = (x_1 + x_2J) + (y_1 + Jy_2)j = (x_1 + y_1j) + (x_2 + y_2j)J = h_1 + h_2J,$$

where  $h_1, h_2 \in \mathbb{H}$ . Then the set of generalized complex-hyperbolic numbers is denoted by  $\mathbb{C}_p\mathbb{H}$  and defined as:

$$\mathbb{C}_p\mathbb{H} = \left\{ \tilde{w} = h_1 + h_2J \mid h_1, h_2 \in \mathbb{H}, J^2 = \mathfrak{p}, -\infty < \mathfrak{p} < \infty \right\}.$$

Hence, there exists no difference between hyperbolic-generalized complex numbers and generalized complex-hyperbolic numbers.

On the other hand, every hyperbolic number can be written in terms of the basis  $\{1, j\}$ . Since the hyperbolic numbers have idempotent basis (see in [2]), we can express the hyperbolic-generalized complex number in terms of the elements of the set  $\left\{ j_+ = \frac{1+j}{2}, j_- = \frac{1-j}{2} \right\}$  as follows:

$$\begin{aligned} \tilde{w} = z_1 + z_2j &= (x_1 + x_2J) + (y_1 + y_2J)j \\ &= [(x_1 + y_1) + (x_2 + y_2)J] \left[ \frac{1+j}{2} \right] + [(x_1 - y_1) + (x_2 - y_2)J] \left[ \frac{1-j}{2} \right] \\ &= z_+j_+ + z_-j_-, \end{aligned}$$

where  $z_+ = (x_1 + y_1) + (x_2 + y_2)J$  and  $z_- = (x_1 - y_1) + (x_2 - y_2)J$ .

**Definition 11.** The set  $\left\{ j_+ = \frac{1+j}{2}, j_- = \frac{1-j}{2} \right\}$  is called an idempotent basis for hyperbolic-generalized complex numbers similar to hyperbolic numbers. Because of the fact that  $j_+^2 = j_+, j_-^2 = j_-$  and they are mutually annihilating because  $j_+ j_- = 0$ .

**Definition 12.**  $\forall \tilde{w} = z_1 + z_2 j \in \mathbb{H}\mathbb{C}_p$ , the conjugations and modules of  $\tilde{w}$  are defined as in Table 6.

**Table 6.** Conjugations and modules of hyperbolic-generalized complex numbers

Generalized complex conjugation	$\tilde{w}^{\dagger_1} = \bar{z}_1 + \bar{z}_2 j$
Generalized complex module	$ \tilde{w} _{\dagger_1}^2 = \tilde{w}_\Delta \tilde{w}^{\dagger_1} =  z_1 ^2 +  z_2 ^2 + 2\text{Re}(z_1 \bar{z}_2) j \in \mathbb{H}$
Hyperbolic conjugation	$\tilde{w}^{\dagger_2} = z_1 - z_2 j$
Hyperbolic module	$ \tilde{w} _{\dagger_2}^2 = \tilde{w}_\Delta \tilde{w}^{\dagger_2} = z_1^2 - z_2^2 \in \mathbb{C}_p$
Coupled conjugation	$\tilde{w}^{\dagger_3} = \bar{z}_1 - \bar{z}_2 j$
Coupled module	$ \tilde{w} _{\dagger_3}^2 = \tilde{w}_\Delta \tilde{w}^{\dagger_3} =  z_1 ^2 -  z_2 ^2 - 2\text{Im}(z_1 \bar{z}_2) J j \in \mathbb{H}\mathbb{C}_p$
Hyperbolic-generalized complex conjugation	$\tilde{w}^{\dagger_4} = \bar{z}_1 \left( 1 - \frac{z_2}{z_1} j \right)$
Hyperbolic-generalized complex module	$ \tilde{w} _{\dagger_4}^2 = \tilde{w}_\Delta \tilde{w}^{\dagger_4} =  z_1 ^2 - \frac{\bar{z}_1 z_2^2}{z_1}$
Anti-hyperbolic conjugation	$\tilde{w}^{\dagger_5} = z_2 - z_1 j$
Anti-hyperbolic module	$ \tilde{w} _{\dagger_5}^2 = \tilde{w}_\Delta \tilde{w}^{\dagger_5} = (z_2^2 - z_1^2) j \in \mathbb{H}\mathbb{C}_p$

**Proposition 4.** Let  $\tilde{w}_1, \tilde{w}_2$  and  $\tilde{w} = z_1 + jz_2 \in \mathbb{H}\mathbb{C}_p$ . Then, the following assertions hold:

- i)  $(\tilde{w}^{\dagger_k})^{\dagger_k} = \tilde{w}, k = 1, 2, 3, 4,$
- ii)  $(\tilde{w}^{\dagger_5})^{\dagger_5} = -\tilde{w},$
- iii)  $(\tilde{w}_1 \oplus \tilde{w}_2)^{\dagger_k} = \tilde{w}_1^{\dagger_k} \oplus \tilde{w}_2^{\dagger_k}, k = 1, 2, 3, 4, 5,$
- iv)  $(\tilde{w}_1 \ominus \tilde{w}_2)^{\dagger_k} = \tilde{w}_1^{\dagger_k} \ominus \tilde{w}_2^{\dagger_k}, k = 1, 2, 3, 4, 5,$
- v)  $(\tilde{w}_1 \Delta \tilde{w}_2)^{\dagger_k} = \tilde{w}_1^{\dagger_k} \Delta \tilde{w}_2^{\dagger_k}, k = 1, 2, 3,$
- vi)  $\left( \frac{1}{\tilde{w}} \right)^{\dagger_k} = \frac{1}{\tilde{w}^{\dagger_k}}, (\tilde{w} \text{ is not zero divisor}), k = 1, 2,$
- vii)  $\tilde{w} \oplus \tilde{w}^{\dagger_1} = 2\text{Re}(z_1) + j2\text{Re}(z_2) \in \mathbb{H},$
- viii)  $\tilde{w} \oplus \tilde{w}^{\dagger_2} = 2z_1 \in \mathbb{C}_p,$
- ix)  $\tilde{w} \oplus \tilde{w}^{\dagger_3} = 2\text{Re}(z_1) + 2j\text{Im}(z_2) \in \mathbb{H}\mathbb{C}_p,$
- x)  $z_1 \cdot \tilde{w}^{\dagger_4} = \bar{z}_1 \cdot \tilde{w}^{\dagger_2} \in \mathbb{H}\mathbb{C}_p, \text{ where } \cdot : \mathbb{C}_p \times \mathbb{H}\mathbb{C}_p \rightarrow \mathbb{H}\mathbb{C}_p,$
- xi)  $2z_1 = \tilde{w} \ominus (\tilde{w}^{\dagger_5} \Delta j), 2z_2 = \tilde{w}^{\dagger_5} \oplus (\tilde{w} \Delta j).$

**Definition 13.** Under the condition that  $\tilde{w}$  is not a zero divisor, the inverse of  $\tilde{w} \in \mathbb{HC}_p$  is defined as

$$\frac{1}{\tilde{w}} = \frac{\tilde{w}^{\dagger_1}}{|\tilde{w}|_{\dagger_1}^2} = \frac{\tilde{w}^{\dagger_2}}{|\tilde{w}|_{\dagger_2}^2}.$$

### 3. 1. Hyperbolic-Generalized Complex Valued Functions

Similar to Section 2.1, here we first define hyperbolic-generalized complex open subset  $\Omega \subset \mathbb{HC}_p$  and the mapping as follows:

**Definition 14.** The following statements can be given for the set  $\mathbb{HC}_p$ :

$$f : \Omega \subset \mathbb{HC}_p \rightarrow \mathbb{HC}_p \\ (z_1 + z_2 j) \rightarrow f(z_1 + z_2 j).$$

Then, it can be written by using idempotent basis as follows:

$$f(\tilde{w}) = f(z_+ j_+ + z_- j_-) = f(z_+) j_+ + f(z_-) j_-.$$

**Definition 15.** The hyperbolic-generalized complex valued function  $f$  is called continuous at  $\tilde{w}_0 = z_{01} + z_{02} j$  if  $\lim_{\tilde{w} \rightarrow \tilde{w}_0} f(\tilde{w}) = f(\tilde{w}_0)$ . Moreover,  $f$  is called continuous in open subset  $\Omega \subset \mathbb{HC}_p$  if it is continuous at every point of  $\Omega$ .

**Definition 16.** The hyperbolic-generalized complex valued function  $f$  is differentiable at  $\tilde{w}_0 = z_{01} + z_{02} j$  if the following limit exists:

$$\frac{df}{d\tilde{w}}(\tilde{w}_0) = \lim_{\tilde{w} \rightarrow \tilde{w}_0} \frac{f(\tilde{w}) - f(\tilde{w}_0)}{\tilde{w} - \tilde{w}_0}.$$

$\frac{df}{d\tilde{w}}(\tilde{w}_0)$  is called the derivative of  $f$  at the point  $\tilde{w}_0$ . Besides,  $f$  is differentiable in an open subset  $\Omega \subset \mathbb{HC}_p$  if it is differentiable at every point of  $\Omega$ .

**Example 3.** For  $\tilde{w} = z_1 + z_2 j \in \mathbb{HC}_p$ , some hyperbolic-generalized complex valued functions are given as follows:

- i)  $\|\tilde{w}\|^2 = \frac{1}{2}[\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2] + \frac{1}{2}[\|z_1 + z_2\|^2 - \|z_1 - z_2\|^2] j,$
- ii)  $e^{\tilde{z}_1 + \tilde{z}_2} j = \frac{1}{2}(e^{\tilde{z}_1 + \tilde{z}_2} + e^{\tilde{z}_1 - \tilde{z}_2}) + \frac{1}{2}(e^{\tilde{z}_1 + \tilde{z}_2} - e^{\tilde{z}_1 - \tilde{z}_2}) j,$
- iii)  $|z_1 + z_2 j| = \frac{1}{2}(|z_1 + z_2| + |z_1 - z_2|) + \frac{1}{2}(|z_1 + z_2| - |z_1 - z_2|) j,$
- iv)  $\ln(z_1 + z_2 j) = \frac{1}{2} \ln(z_1^2 - z_2^2) + \frac{1}{2} \ln\left(\frac{z_1 + z_2}{z_1 - z_2}\right) j, (z_1^2 - z_2^2) > 0, (z_1 - z_2) \neq 0, \left(\frac{z_1 + z_2}{z_1 - z_2}\right) > 0,$
- v)  $(z_1 + z_2 j)^n = (z_1 + z_2)^n \left(\frac{1+j}{2}\right) + (z_1 - z_2)^n \left(\frac{1-j}{2}\right).$

### 3. 2. Matrix Representations of Hyperbolic-Generalized Complex Numbers

Here, we first define the transformation  $f: \mathbb{H}\mathbb{C}_p \rightarrow \mathbb{H}\mathbb{C}_p$ ,  $f_{\tilde{w}}(\tilde{w}) = \tilde{w}\Delta\tilde{w}_1$ , for every  $\tilde{w} = z_1 + z_2j = x_1 + x_2J + y_1j + y_2Jj$ . It is obvious that  $f$  is linear. The real and generalized complex matrix representations of  $f$  (of  $\tilde{w}$  as well) with respect to the basis  $\{1, J, j, Jj\}$  and  $\{1, j\}$  (basis for  $\mathbb{H}$ ) can be given as below, respectively:

$$\tilde{A}_{\mathbb{R}} = \begin{bmatrix} x_1 & px_2 & y_1 & py_2 \\ x_2 & x_1 & y_2 & y_1 \\ y_1 & py_2 & x_1 & px_2 \\ y_2 & y_1 & x_2 & x_1 \end{bmatrix}, \tilde{A}_{\mathbb{C}_p} = \begin{bmatrix} x_1 + Jx_2 & y_1 + Jy_2 \\ y_1 + Jy_2 & x_1 + Jx_2 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_1 \end{bmatrix},$$

$$\text{where } \det(\tilde{A}_{\mathbb{C}_p}) = z_1^2 - z_2^2 = |\tilde{w}|_{\mathbb{C}_p}^2.$$

By writing  $\tilde{w} = z_1 + z_2j = h_1 + h_2J \in \mathbb{C}_p\mathbb{H}$ , the matrices of  $\tilde{w}$  with respect to the basis  $\{1, j, J, jJ\}$  and  $\{1, J\}$  are given respectively as follows:

$$\tilde{B}_{\mathbb{R}} = \begin{bmatrix} x_1 & y_1 & px_2 & py_2 \\ y_1 & x_1 & py_2 & px_2 \\ x_2 & y_2 & x_1 & y_1 \\ y_2 & x_2 & y_1 & x_1 \end{bmatrix}, \tilde{B}_{\mathbb{H}} = \begin{bmatrix} x_1 + jy_1 & p(x_2 + jy_2) \\ x_2 + jy_2 & x_1 + jy_1 \end{bmatrix} = \begin{bmatrix} h_1 & ph_2 \\ h_2 & h_1 \end{bmatrix},$$

$$\text{where } \det(\tilde{B}_{\mathbb{H}}) = h_1^2 - ph_2^2 = |\tilde{w}|_{\mathbb{H}}^2.$$

**Example 4.** The matrix representations of  $\tilde{w} = -2 + 5J + j - 4Jj$  with respect to the basis  $\{1, J, j, Jj\}$  and  $\{1, j\}$  can be given as below, respectively:

$$\tilde{A}_{\mathbb{R}} = \begin{bmatrix} -2 & 5p & 1 & -4p \\ 5 & -2 & -4 & 1 \\ 1 & -4p & -2 & 5p \\ -4 & 1 & 5 & -2 \end{bmatrix}, \tilde{A}_{\mathbb{C}_p} = \begin{bmatrix} -2 + 5J & 1 - 4J \\ 1 - 4J & -2 + 5J \end{bmatrix},$$

and with respect to the basis  $\{1, j, J, jJ\}$  and  $\{1, J\}$  respectively:

$$\tilde{B}_{\mathbb{R}} = \begin{bmatrix} -2 & 1 & 5p & -4p \\ 1 & -2 & -4p & 5p \\ 5 & -4 & -2 & 1 \\ -4 & 5 & 1 & -2 \end{bmatrix}, \tilde{B}_{\mathbb{H}} = \begin{bmatrix} -2 + j & p(5 - 4j) \\ 5 - 4j & -2 + j \end{bmatrix}.$$

Moreover,  $\det(\tilde{A}_{\mathbb{C}_p}) = (3 + 9p) - 12J \in \mathbb{C}_p$  and  $\det(\tilde{B}_{\mathbb{H}}) = (5 - 41p) + (-4 + 40p)j \in \mathbb{H}$ .

### 4. CONCLUSIONS

In this study, inspired from the theory of the number systems constructed by writing the coefficients as elements of the sets  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{D}$ , we present the theories of dual-generalized complex and hyperbolic-generalized complex numbers for  $\mathfrak{p} \in \mathbb{R}$ . By using similar thought, the expressions given in this paper can be given for the set of complex-generalized complex numbers:

$$\mathbb{CC}_p = \left\{ \tilde{w} = z_1 + z_2 i \mid z_1 = x_1 + x_2 J, z_2 = y_1 + y_2 J \in \mathbb{C}_p, J^2 = \mathfrak{p}, i^2 = -1 \right\}.$$

If the sets  $\mathbb{DC}_p$ ,  $\mathbb{HC}_p$  and  $\mathbb{CC}_p$  are examined for special values of  $\mathfrak{p} \in \{-1, 0, 1\}$ , then the classification in Table 7 can be given.

In the future studies, it is planned to adapt the theory undertaken in this paper to hybrid numbers created with any combination of the complex, hyperbolic and dual numbers, [31].

**Table 7.** The examination of the sets  $\mathbb{DC}_p$ ,  $\mathbb{HC}_p$  and  $\mathbb{CC}_p$  for  $\mathfrak{p} \in \{-1, 0, 1\}$

$i^2 = -1, \mathbf{i}^2 = -1$			$j^2 = 1, \mathbf{j}^2 = 1,$ $(\mathbf{j} \neq \pm 1, j \neq \pm 1)$	$\varepsilon^2 = 0, \boldsymbol{\varepsilon}^2 = 0,$ $(\varepsilon \neq 0, \boldsymbol{\varepsilon} \neq 0, \boldsymbol{\varepsilon} \boldsymbol{\varepsilon} \neq 0)$
$\mathbb{DC}_p$	DUAL-GENERALIZED COMPLEX NUMBERS	$w = x_1 + x_2 J + y_1 \boldsymbol{\varepsilon} + y_2 J \boldsymbol{\varepsilon}$	Dual-complex numbers (see in [15, 16, 17])	$w = x_1 + x_2 \mathbf{i} + y_1 \boldsymbol{\varepsilon} + y_2 \mathbf{i} \boldsymbol{\varepsilon}$ $J = \mathbf{i}, \mathfrak{p} = -1$
			Hyperdual numbers (see in [19, 20, 21, 22, 23, 24])	$w = x_1 + x_2 \boldsymbol{\varepsilon} + y_1 \boldsymbol{\varepsilon} + y_2 \boldsymbol{\varepsilon} \boldsymbol{\varepsilon}$ $J = \boldsymbol{\varepsilon}, \mathfrak{p} = 0$
			Dual-hyperbolic numbers (see in [18])	$w = x_1 + x_2 \mathbf{j} + y_1 \boldsymbol{\varepsilon} + y_2 \mathbf{j} \boldsymbol{\varepsilon}$ $J = \mathbf{j}, \mathfrak{p} = 1$
$\mathbb{HC}_p$	HYPERBOLIC-GENERALIZED COMPLEX NUMBERS	$\tilde{w} = x_1 + x_2 J + y_1 j + y_2 J j$	Hyperbolic-complex numbers (see in [8, 9, 10])	$\tilde{w} = x_1 + x_2 \mathbf{i} + y_1 j + y_2 \mathbf{i} j$ $J = \mathbf{i}, \mathfrak{p} = -1$
			Hyperbolic-dual numbers (see in [18])	$\tilde{w} = x_1 + x_2 \boldsymbol{\varepsilon} + y_1 j + y_2 \boldsymbol{\varepsilon} j$ $J = \boldsymbol{\varepsilon}, \mathfrak{p} = 0$
			Bihyperbolic numbers (see in [25, 26, 27, 28])	$\tilde{w} = x_1 + x_2 \mathbf{j} + y_1 j + y_2 \mathbf{j} j$ $J = \mathbf{j}, \mathfrak{p} = 1$
$\mathbb{CC}_p$	COMPLEX-GENERALIZED COMPLEX NUMBERS	$\tilde{\tilde{w}} = x_1 + x_2 J + y_1 i + y_2 J i$	Bicomplex numbers (see in [12, 13, 14])	$\tilde{\tilde{w}} = x_1 + x_2 \mathbf{i} + y_1 i + y_2 \mathbf{i} i$ $J = \mathbf{i}, \mathfrak{p} = -1$
			Complex-dual numbers (see in [15, 16, 17])	$\tilde{\tilde{w}} = x_1 + x_2 \boldsymbol{\varepsilon} + y_1 i + y_2 \boldsymbol{\varepsilon} i$ $J = \boldsymbol{\varepsilon}, \mathfrak{p} = 0$
			Complex-hyperbolic numbers (see in [8, 9, 10])	$\tilde{\tilde{w}} = x_1 + x_2 \mathbf{j} + y_1 i + y_2 \mathbf{j} i$ $J = \mathbf{j}, \mathfrak{p} = 1$

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## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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