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Traveling wave structures of some fourth-order nonlinear partial differential equations

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ABSTRACT

This study presents a large family of the traveling wave solutions to the two fourth-order nonlinear partial differential equations utilizing the Riccati-Bernoulli sub-ODE method. In this method, utilizing a traveling wave transformation with the aid of the Riccati-Bernoulli equation, the fourth-order equation can be transformed into a set of algebraic equations. Solving the set of algebraic equations, we acquire the novel exact solutions of the integrable fourth-order equations presented in this research paper. The physical interpretation of the nonlinear models are also detailed through the exact solutions, which demonstrate the effectiveness of the presented method.The Bäcklund transformation can produce an infinite sequence of solutions of the given two fourth-order nonlinear partial differential equations. Finally, 3D graphs of some derived solutions in this paper are depicted through suitable parameter values.

1. Introduction

In addition to the fact that obtaining solutions of ordinary differential equations (ODE) is important in many engineering fields [1–2], mathematical models of nonlinear partial differential equations (NLPDEs) are common in the last decades to model scientific problems which arise in engineering, biology and several other fields, even real world problems. Besides, there are many studies in the literature to find soliton solutions of various kinds ODEs and NLPDEs [3–7]. The examination of different types of wave solutions for NLPDEs has a significant role in analyzing nonlinear physical phenomena [8–13]. In recent years, a few effective methods have been presented to produce traveling wave solutions of NLPDEs, such as the multiple Exp-function method [14], the new extended direct algebraic approach [15–17], the sine-Gordon expansion method [18–19], the Bäcklund transformation method [20], the sub-equation method [21–22], the modified simple equation method [23–24], the modified Kudryashov method [25–26], the Riccati Bernoulli sub-ODE method [27–33], the extended trial function scheme [34].

This research paper is concerned with the following integrable fourth-order equations, which is presented as,

$$\psi_{tt} + \psi_{txxx} + \alpha(\psi_x\psi_t)_x = 0 \tag{1.1}$$

and

$$\psi_{tt} + \psi_{txxx} + \alpha(\psi_x\psi_t)_x + \beta\psi_{xx} = 0. \tag{1.2}$$

Herein, Eqs. (1.1)-(1.2) are known fourth-order NLPDEs [35]. The parameters α and β are nonzero real numbers. For $\beta = 0$, Eq. (1.1) is derived from Eq. (1.2). However, the two equations have independently distinct features. The simplified Hirota's algorithm is applied to examine multiple soliton solutions of both equations and the complex form of the simplified Hirota's algorithm is developed to get multiple complex soliton solutions of both equations in [35].

The integrable fourth-order equations models both right- and left-going waves in a like manner to the Boussinesq equation. The Boussinesq equation originates to represent the long waves transmitting on the surface of shallow water [36–39].The Boussinesq-like models are utilized generally for simulating surface water waves in shallow seas and harbors, dune, inundation, ocean basin-scale tsunami propagation, wave over-topping, and near-shore wave processing in ocean and coastal engineering [40–43]. The Boussinesq equation and its related models have a very crucial role in interpreting divergent physical processes in the field of fluid dynamics, plasma physics, and ocean engineering. Thus, the

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analysis of analytical solutions of the models in the mentioned fields has importance. Thus, it is wanted to show here that physical behaviors of the integrable fourth-order equations can be explored by their analytical solutions. In this research paper, inspiring by the ongoing research, the soliton solutions of the two different integrable fourth-order equations using the Riccati-Bernoulli Sub-ODE technique are examined. Riccati-Bernoulli Sub-ODE technique is one of the robust techniques to examine the exact solutions of NLPDEs. For example, Yang et al. examined exact traveling wave solutions of the generalized Ostrovsky equation, the Eckhaus equation, the generalized Zakharov-Kuznetsov-Burgers equationand the nonlinear fractional Klein-Gordon equation utilizing the Riccati-Bernoulli Sub-ODE method [26].

The rest of this study is formed as follows: We describe the Riccati-Bernoulli Sub-ODE technique in Section 2. The Riccati-Bernoulli Sub-ODE technique is applied to get soliton solutions of two fourth-order equations and soliton solutions for these equations are acquired in Section 3. Moreover, the 3D graphs for some solutions are depicted with the aid of the MAPLE software, which is an extremely useful mathematical software tool for research and education that analyzes and visualizes mathematical models and solves mathematical problems. The conclusion of this paper is given in Section 4.

2. Riccati Bernoulli sub-ode method

In this section, we summarize the main steps of the Riccati Bernoulli sub-ODE method.

Any NLPDE in two independent variables x and t is written in following form:

$$Q(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{tt}, \psi_{xt}, \dots) = 0. \tag{2.1}$$

Herein, $Q$is a polynomial which consisting $\psi(x,t)$ and its partial derivatives.

Step 1: In order to obtain the solitary wave solution of Eqs. (1.1)-(1.2), we employ the traveling wave transformation,

$$\psi(x,t) = \psi(\zeta), \zeta = k(x \pm \upsilon t) \tag{2.2}$$

in which $\psi(x,t) = \psi(\zeta)$is an unknown function to be found, k is defined as the width of the traveling wave and υ is identified as the velocity of the soliton. Then, Eq. (2.1) is turned into the following ODE:

$$Q\left(\psi, \psi', \psi'', \dots\right) = 0, \tag{2.3}$$

in which $\psi' = \frac{d\psi}{d\zeta}, \psi'' = \frac{d^2\psi}{d\zeta^2}$ and so on.

Step 2: Assume that Eq. (2.3) is the solution of the Riccati-Bernoulli equation of the form:

$$\psi' = a_1\psi + a_2\psi^{2-m} + a_3\psi^m \tag{2.4}$$

in which a_1, a_2, a_3 and m are constants. Utilizing from Eq. (2.4), we acquire

$$\psi'' = \psi^{-1-2m}\left(a_2\psi^2 + a_3\psi^{2m} + a_1\psi^{1+m}\right) \times \left(-a_2(-2+m)\psi^2 + a_3 m\psi^{2m} + a_1\psi^{1+m}\right) \tag{2.5}$$

and

$$\psi''' = \psi^{-2(1+m)}\left(a_1\psi + a_2\psi^{2-m} + a_3\psi^m\right)\left(a_2^2(-2+m)\right)(-3+2m)\psi^4 + a_3^2 m(-1+2m)\psi^{4m} + a_1a_2(-3+m)(-2+m)\psi^{3+m} + \left(a_1^2 + 2a_2a_3\right)\psi^{2+2m} + a_1a_3m(1+m)\psi^{1+3m} \tag{2.6}$$

and so on.

Remark 1. Eq. (2.4) is a Riccati equation when $a_2a_3 \neq 0$ and $m = 0$. Additionally, Eq. (2.4) is a Bernoulli equation when $a_2 \neq 0, a_3 = 0$ and $m \neq 1$.

The solutions of the Eq. (2.4) are as follows:

Case 1: If $m = 1$, Eq. (2.4) has the following solution

$$\psi(\zeta) = Ke^{(a_1+a_2+a_3)\zeta}. \tag{2.7}$$

Case 2: If $m \neq 1, a_1 = 0$ and $a_3 = 0$, Eq. (2.4) has the following solution

$$\psi(\zeta) = (a_2(m-1)(\zeta+K))^{\frac{1}{m-1}}. \tag{2.8}$$

Case 3: If $m \neq 1, a_1 \neq 0$ and $a_3 = 0$, Eq. (2.4) has the following solution

$$\psi(\zeta) = \left(Ke^{a_1(m-1)\zeta} - \frac{a_2}{a_1}\right)^{\frac{1}{m-1}}. \tag{2.9}$$

Case 4: If $m \neq 1, a_2 \neq 0$ and ${a_1}^2 - 4a_2a_3 < 0$, Eq. (2.4) has the following solutions

$$\psi(\zeta) = \left(-\frac{a_1}{2a_2} + \frac{\sqrt{4a_2a_3 - {a_1}^2}}{2a_2}\tan\left[\frac{(1-m)\sqrt{4a_2a_3 - {a_1}^2}}{2}(\zeta+K)\right]\right)^{\frac{1}{1-m}} \tag{2.10}$$

and

$$\psi(\zeta) = \left(-\frac{a_1}{2a_2} - \frac{\sqrt{4a_2a_3 - {a_1}^2}}{2a_2}\cot\left[\frac{(1-m)\sqrt{4a_2a_3 - {a_1}^2}}{2}(\zeta+K)\right]\right)^{\frac{1}{1-m}} \tag{2.11}$$

Case 5: If $m \neq 1, a_2 \neq 0$ and ${a_1}^2 - 4a_2a_3 < 0$, Eq. (2.4) has the following solutions

$$\psi(\zeta) = \left(-\frac{a_1}{2a_2} - \frac{\sqrt{{a_1}^2 - 4a_2a_3}}{2a_2}\tanh\left[\frac{(1-m)\sqrt{{a_1}^2 - 4a_2a_3}}{2}(\zeta+K)\right]\right)^{\frac{1}{1-m}} \tag{2.12}$$

and

$$\psi(\zeta) = \left(-\frac{a_1}{2a_2} - \frac{\sqrt{{a_1}^2 - 4a_2a_3}}{2a_2}\coth\left[\frac{(1-m)\sqrt{{a_1}^2 - 4a_2a_3}}{2}(\zeta+K)\right]\right)^{\frac{1}{1-m}}. \tag{2.13}$$

Case 6: If $m \neq 1, a_2 \neq 0$ and ${a_1}^2 - 4a_2a_3 = 0$, Eq. (2.4) has the following solution

$$\psi(\zeta) = \left(\frac{1}{a_2(m-1)(\zeta+K)} - \frac{a_2}{a_1}\right)^{\frac{1}{1-m}} \tag{2.14}$$

in which K is a constant.

Step 3: Finally, if ψ and its derivatives are substituted into Eq. (2.3), we can get a set of algebraic equation consisting the powers of ψ. Equating the coefficients of each power of ψ to zero gives a system of algebraic equations for a_1, a_2, a_3, k and υ. When the parameters are substituted into Eqs. (2.7)-(2.14), the traveling wave solutions of Eq. (2.1) are acquired.

2.1. Bäcklund transformation

In this subsection, we will give a Bäcklund transformation of the Riccati Bernoulli sub ODE method.

If $\psi_n(\zeta)$ and $\psi_{n-1}(\zeta)$ are solutions of Eq. (2.4), then we acquire

$$\frac{d\psi_n(\zeta)}{d\zeta} = \frac{d\psi_n(\zeta)}{d\psi_{n-1}(\zeta)}\frac{d\psi_{n-1}(\zeta)}{d\zeta} = \frac{d\psi_n(\zeta)}{d\psi_{n-1}(\zeta)}\left(a_1\psi + a_2\psi^{2-m} + a_3\psi^m\right). \tag{2.15}$$

From Eq. (2.15), we can rewrite

$$\frac{d\psi_n(\zeta)}{a_1\psi_n + a_2\psi_n^{2-m} + a_3\psi_n^m} = \frac{d\psi_{n-1}(\zeta)}{a_1\psi_{n-1} + a_2\psi_{n-1}^{2-m} + a_3\psi_{n-1}^m}. \tag{2.16}$$

When Eq. (2.16) is integrated with respect to ζ, we acquire

$$\psi_n(\zeta)=\left(\frac{-a_3C_1+a_2C_2(\psi_{n-1}(\zeta))^{1-m}}{a_1C_1+a_2C_2+a_2C_1(\psi_{n-1}(\zeta))^{1-m}}\right)^{\frac{1}{1-m}} \tag{2.17}$$

in which C_1 and C_2 are arbitrary constants. Eq. (2.17) is defined as Bäcklund transformation. Utilizing Bäcklund transformation, we can obtain infinite sequences of solutions for Eq. (2.1).

3. Applications

3.1. Applications of Riccati-Bernoulli sub-ode method to Eq. (1.1)

In this subsection, to get the wave solution of Eq. (1.1), we exploit the following traveling wave transformation

$$\psi(x,t)=\psi(\zeta),\zeta=k(x+\nu t). \tag{3.1}$$

Eq. (1.1) is reduced to the following ODE:

$$k^4\nu\psi^{(\text{<ct>ı<ot>v})}+k^2\nu^2\psi''+2\alpha k^3\nu\psi\psi''=0. \tag{3.2}$$

Assume that the solution of Eq. (3.2) is the solution of Eq. (2.4). If ψ and its derivatives are substituted into Eq. (3.2), and we set $m=0$, we get the following equation,

$$\begin{aligned}&(4a_2{}^3 3\alpha k^3\nu+24k^4\nu a_2{}^4)\psi^5+(10a_1a_2{}^2\alpha k^3\nu+60k^4\nu a_1a_2{}^3)\psi^4+\\&(2a_2{}^2k^2\nu^2+k^4\nu(12a_3a_2{}^3+30a_1{}^2a_2{}^2+6a_2{}^2(a_1{}^2+2a_3a_2)+2a_2(3a_1{}^2a_2\\&+4(a_1{}^2+2a_3a_2)a_2))+2((a_1{}^2+2a_3a_2)a_2+3a_1{}^2a_2+2a_2{}^2a_3)\alpha k^3\nu)\psi^3+\\&(3a_1a_2k^2\nu^2+k^4\nu(2a_2((a_1{}^2+2a_3a_2)a_1+6a_1a_3a_2)+a_1(3a_1{}^2a_2+4(a_1{}^2+a_1a_3a_2)a_2)\\&+24a_3a_2{}^2a_1+6a_1a_2(a_1{}^2+2a_3a_2))+2(4a_1a_3a_2+(a_1{}^2+2a_2a_3)a_1)\alpha k^3\nu)\psi^2+\\&((a_1{}^2+2a_2a_3)k^2\nu^2+k^4\nu(6a_2a_3(a_1{}^2+2a_3a_2)+6a_1{}^2a_2a_3+2a_2(a_1{}^2a_3+2a_2a_3{}^2)\\&+a_1((a_1{}^2+2a_3a_2)a_1+6a_1a_3a_2))+2(a_1{}^2a_3+(a_1{}^2+2a_3a_2)a_3)\alpha k^3\nu)\psi+\\&a_1a_3k^2\nu^2+k^4\nu(a_1(a_1{}^2a_3+2a_2a_3{}^2)+6a_3{}^2a_2a_1)+2a_1a_3{}^2\alpha k^3\nu=0.\end{aligned} \tag{3.3}$$

If we collect all the coefficients of $\psi^i(i=0,1,2,3,4,5)$ and equate each to zero in Eq. (3.3), the following system is obtained:

ψ^0 **coefficient:**

$$a_1a_3k^2\nu^2+k^4\nu(a_1(a_1{}^2a_3+2a_2a_3{}^2)+6a_3{}^2a_2a_1)+2a_1a_3{}^2\alpha k^3\nu=0 \tag{3.4}$$

ψ^1 **coefficient:**

$$\begin{aligned}&((a_1{}^2+2a_3a_2)k^2\nu^2+k^4\nu(6a_3a_2(a_1{}^2+2a_3a_2)+6a_1{}^2a_2a_3\\&+2a_2(a_1{}^2a_3+2a_2a_3{}^2)+a_1((a_1{}^2+2a_3a_2)a_1+6a_1a_3a_2))\\&+2(a_1{}^2a_3+(a_1{}^2+2a_3a_2)a_3)\alpha k^3\nu)=0\end{aligned}$$

ψ^2 **coefficient:**

$$\begin{aligned}&(3a_1a_2k^2\nu^2+k^4\nu(2a_2((a_1{}^2+2a_3a_2)a_1+6a_1a_3a_2)+a_1(3a_1{}^2a_2\\&+4(a_1{}^2+a_1a_3a_2)a_2)+24a_3a_2{}^2a_1\\&+6a_1a_2(a_1{}^2+2a_3a_2))+2(4a_1a_3a_2+(a_1{}^2+2a_3a_2)a_1)\alpha k^3\nu)=0\end{aligned} \tag{3.6}$$

ψ^3 **coefficient:**

$$\begin{aligned}&(2a_2^2k^2\nu^2+k^4\nu(12a_3a_2^3+30a_1^2a_2^2+6a_2^2(a_1^2+2a_3a_2)+2a_2(3a_1^2a_2\\&+4(a_1^2+2a_3a_2)a_2))+2((a_1^2+2a_3a_2)a_2+3a_1^2a_2+2a_2^2a_3)\alpha k^3\nu)\\&=0\end{aligned} \tag{3.7}$$

ψ^4 **coefficient:**

$$(10a_1a_2^2\alpha k^3\nu+60k^4\nu a_1a_2^3)=0 \tag{3.8}$$

ψ^5 **coefficient:**

$$(4a_2^3 3\alpha k^3\nu+24k^4\nu a_2^4)=0 \tag{3.9}$$

When solving the system of algebraic equations in Eqs. (3.4)-(3.9), the following cases are obtained.

Case 1:

$$a_2=-\frac{\alpha}{6k},a_1=\mp\frac{\sqrt{3}\sqrt{a_3\alpha k}}{3k},\nu=-a_3\alpha k \tag{3.10}$$

If the parameters in Eq. (3.10) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):

$$\psi_{1,1}(x,t)=\frac{\sqrt{3}\sqrt{a_3\alpha k}}{\alpha}-\frac{3\sqrt{-\frac{\alpha a_3}{k}}k\tan\left(\frac{\sqrt{-\frac{\alpha a_3}{k}}(k(x-a_3\alpha kt)+K)}{2}\right)}{\alpha}, \tag{3.11}$$

$$\psi_{1,2}(x,t)=\frac{\sqrt{3}\sqrt{a_3\alpha k}}{\alpha}+\frac{3\sqrt{-\frac{\alpha a_3}{k}}k\cot\left(\frac{\sqrt{-\frac{\alpha a_3}{k}}(k(x-a_3\alpha kt)+K)}{2}\right)}{\alpha}. \tag{3.12}$$

The graph of the $\psi_{1,1}(x,t)$ and $\psi_{1,2}(x,t)$ are illustrated in Fig. 13.

$$\psi_{1,3}(x,t)=\frac{\sqrt{3}\sqrt{a_3\alpha k}}{\alpha}+\frac{3\sqrt{\frac{\alpha a_3}{k}}k\tanh\left(\frac{\sqrt{\frac{\alpha a_3}{k}}(k(x-a_3\alpha kt)+K)}{2}\right)}{\alpha}, \tag{3.13}$$

$$\psi_{1,4}(x,t)=\frac{\sqrt{3}\sqrt{a_3\alpha k}}{\alpha}+\frac{3\sqrt{\frac{\alpha a_3}{k}}k\coth\left(\frac{\sqrt{\frac{\alpha a_3}{k}}(k(x-a_3\alpha kt)+K)}{2}\right)}{\alpha}. \tag{3.14}$$

in which $a_3\alpha k>0$ for valid solitons in solutions Eq. (3.13) and (3.14).

These solutions are depicted in Figs. 1 and 2.

Case 2:

$$a_2=\frac{\nu}{6a_3k^2},a_1=\mp\frac{\sqrt{-3\nu}}{3k},\alpha=-\frac{\nu}{a_3k} \tag{3.15}$$

If the parameters in Eq. (3.15) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):

$$\psi_{2,1}(x,t)=\frac{\sqrt{-3\nu}ka_3}{\nu}+\frac{3\sqrt{\frac{\nu}{k^2}}k^2a_3\tan\left(\frac{\sqrt{\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu}, \tag{3.16}$$

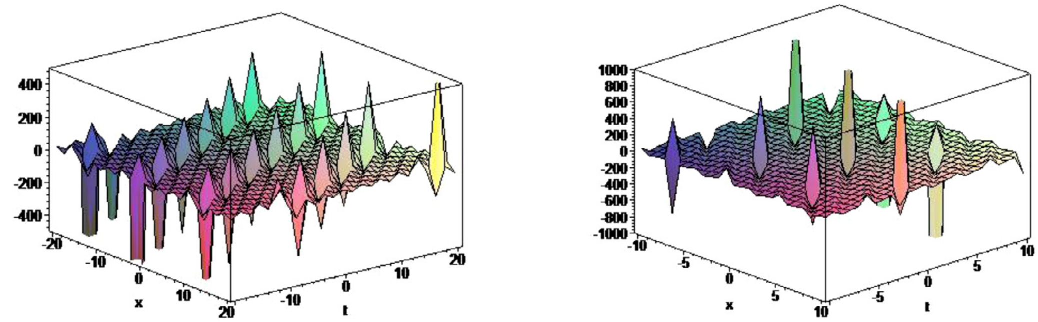
The graph of the $\psi_{2,1}(x,t)$ is depicted in Fig. 14.

$$\psi_{2,2}(x,t)=\frac{\sqrt{-3\nu}ka_3}{\nu}-\frac{3\sqrt{\frac{\nu}{k^2}}k^2a_3\cot\left(\frac{\sqrt{\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu}. \tag{3.17}$$

$$\psi_{2,3}(x,t)=\frac{\sqrt{-3\nu}ka_3}{\nu}-\frac{3\sqrt{-\frac{\nu}{k^2}}k^2a_3\tanh\left(\frac{\sqrt{-\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu}, \tag{3.18}$$

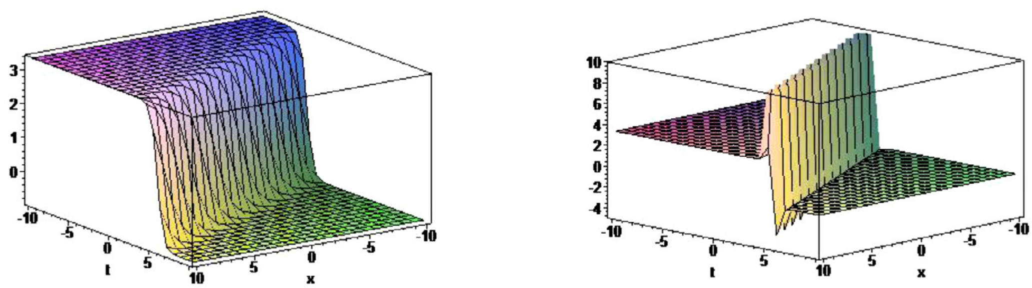
$$\psi_{2,4}(x,t)=\frac{\sqrt{-3\nu}ka_3}{\nu}-\frac{3\sqrt{-\frac{\nu}{k^2}}k^2a_3\coth\left(\frac{\sqrt{-\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu} \tag{3.19}$$

in which $\nu<0$ for valid solitons in solutions Eqs. (3.18) and (3.19).



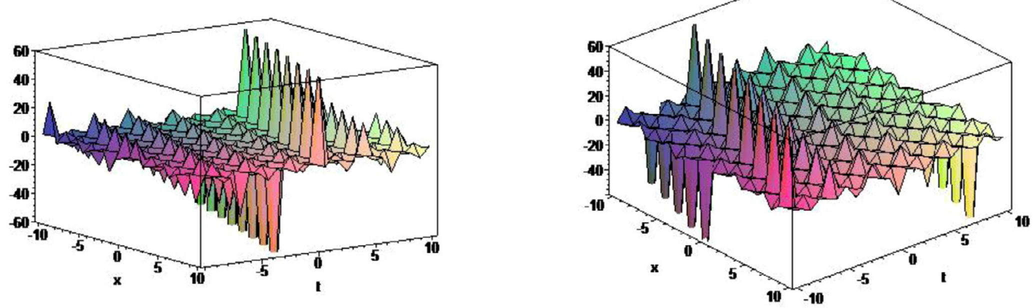
(a) (b)

Fig. 1. Graphs of the real part of solutions $\psi_{1,1}(x,t)$ and $\psi_{1,2}(x,t)$to Eq. (1.1), respectively, for the values $k=3, \alpha=-0.5, a_3=3$ and $K=4$.



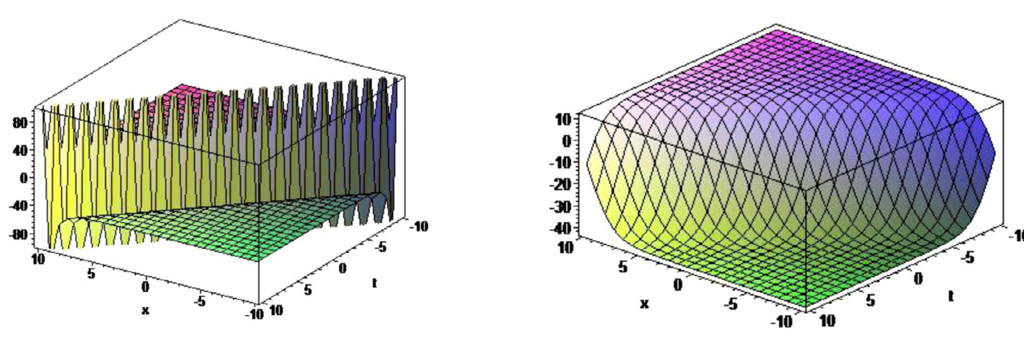
(a) (b)

Fig. 2. Graphs of the solutions $\psi_{1,3}(x,t)$ and $\psi_{1,4}(x,t)$of Eq. (1.1), respectively, for the values $k=3, \alpha=-0.5, a_3=3$ and $K=4$.



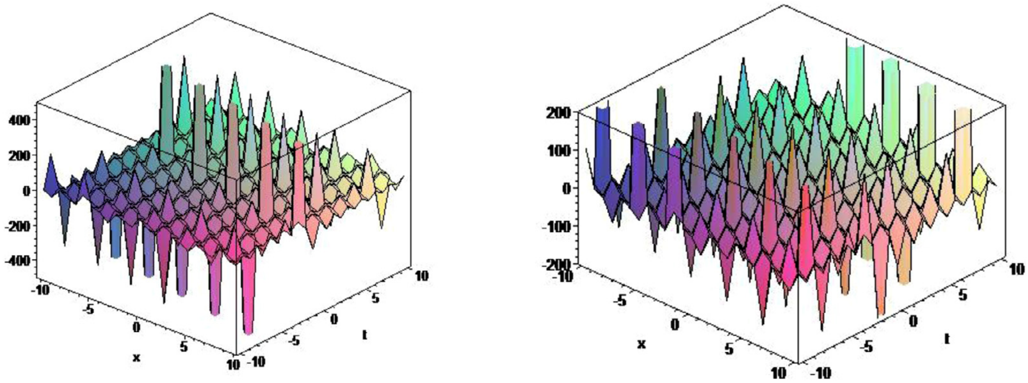
(a) (b)

Fig. 3. Graphs of the real part of solutions $\psi_{2,1}(x,t)$ and $\psi_{2,2}(x,t)$to Eq. (1.1), respectively, for the values $k=1, \nu=3, a_3=1$ and $K=1$.



(a) (b)

Fig. 4. Graphs of the solutions $\psi_{2,3}(x,t)$ and $\psi_{2,4}(x,t)$of Eq. (1.1), respectively, for the values $k=3, \nu=1, a_3=3$and $K=1$.



(a) (b)

Fig. 5. Graphs of the real part of solutions $\psi_{3,1}(x,t)$ and $\psi_{3,2}(x,t)$to Eq. (1.1), respectively, for the values $\alpha = 0.5, \nu = 5, a_3 = 3andK = 1$.

Fig. 3 illustrates the real parts of the solutions $\psi_{2,1}(x,t)$ and $\psi_{2,2}(x,t)$. Fig. 4 illustrates the solutions $\psi_{2,3}(x,t)$ and $\psi_{2,4}(x,t)$.

Case 3:

$$a_2 = \frac{\alpha^2 a_3}{6\nu}, a_1 = \mp\frac{a_3\alpha}{\sqrt{-3\nu}}, k = -\frac{\nu}{\alpha a_3} \tag{3.20}$$

If the parameters in Eq. (3.20) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):

$$\psi_{3,1}(x,t) = -\frac{3\nu}{\sqrt{-3\nu}\alpha} + \frac{3\sqrt{\frac{a_3^2\alpha^2}{\nu}}\,\nu\tan\left(\frac{\sqrt{\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{\nu(x+\nu t)}{\alpha a_3}\right)+K}{2}\right)}{\alpha^2 a_3} \tag{3.21}$$

$$\psi_{3,2}(x,t) = -\frac{3\nu}{\sqrt{-3\nu}\alpha} - \frac{3\sqrt{\frac{a_3^2\alpha^2}{\nu}}\,\nu\cot\left(\frac{\sqrt{\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{\nu(x+\nu t)}{\alpha a_3}\right)+K}{2}\right)}{\alpha^2 a_3} \tag{3.22}$$

$$\psi_{3,3}(x,t) = -\frac{3\nu}{\sqrt{-3\nu}\alpha} - \frac{3\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\,\nu\tanh\left(\frac{\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{\nu(x+\nu t)}{\alpha a_3}\right)+K}{2}\right)}{\alpha^2 a_3} \tag{3.23}$$

$$\psi_{3,4}(x,t) = -\frac{3\nu}{\sqrt{-3\nu}\alpha} - \frac{3\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\,\nu\coth\left(\frac{\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{\nu(x+\nu t)}{\alpha a_3}\right)+K}{2}\right)}{\alpha^2 a_3} \tag{3.24}$$

in which $\nu < 0$ for valid solutions.

Fig. 5 illustrates the real parts of the solutions $\psi_{3,1}(x,t)$ and $\psi_{3,2}(x,t)$. Fig. 6 illustrates the solutions $\psi_{3,3}(x,t)$ and $\psi_{3,4}(x,t)$.

Case 4:

$$a_2 = \frac{\alpha^2 a_3}{9\nu}, a_1 = 0, k = -\frac{3\nu}{2\alpha a_3} \tag{3.25}$$

If the parameters in Eq. (3.25) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):

$$\psi_{4,1}(x,t) = \frac{3\sqrt{\frac{a_3^2\alpha^2}{\nu}}\,\nu\tan\left(\frac{\sqrt{\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{3\nu(x+\nu t)}{2\alpha a_3}+K\right)}{3}\right)}{\alpha^2 a_3} \tag{3.26}$$

$$\psi_{4,2}(x,t) = -\frac{3\sqrt{\frac{a_3^2\alpha^2}{\nu}}\,\nu\cot\left(\frac{\sqrt{\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{3\nu(x+\nu t)}{2\alpha a_3}+K\right)}{3}\right)}{\alpha^2 a_3} \tag{3.27}$$

in which $\nu > 0$ valid solitons in solutions. These solutions are depicted in Fig. 7.

$$\psi_{4,3}(x,t) = -\frac{3\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\,\nu\tanh\left(\frac{\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{3\nu(x+\nu t)}{2\alpha a_3}+K\right)}{3}\right)}{\alpha^2 a_3} \tag{3.28}$$

$$\psi_{4,4}(x,t) = -\frac{3\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\,\nu\coth\left(\frac{\sqrt{-\frac{a_3^2\alpha^2}{\nu}}\left(-\frac{3\nu(x+\nu t)}{2\alpha a_3}+K\right)}{3}\right)}{\alpha^2 a_3} \tag{3.29}$$

in which $\nu < 0$ valid solitons in solutions. These solutions are depicted in Fig. 8.

Case 5:

$$a_2 = \frac{\nu}{4a_3k^2}, a_1 = 0, \alpha = -\frac{3\nu}{2a_3k} \tag{3.30}$$

If the parameters in Eq. (3.30) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):

$$\psi_{5,1}(x,t) = \frac{2\sqrt{\frac{\nu}{k^2}}a_3k^2\tan\left(\frac{\sqrt{\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu}, \tag{3.31}$$

$$\psi_{5,2}(x,t) = -\frac{2\sqrt{\frac{\nu}{k^2}}a_3k^2\cot\left(\frac{\sqrt{\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu}. \tag{3.32}$$

in which $\nu > 0$ valid solitons in solutions. These solutions are depicted in Fig. 9.

$$\psi_{5,3}(x,t) = -\frac{2\sqrt{-\frac{\nu}{k^2}}a_3k^2\tanh\left(\frac{\sqrt{-\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu}, \tag{3.33}$$

$$\psi_{5,4}(x,t) = -\frac{2\sqrt{-\frac{\nu}{k^2}}a_3k^2\coth\left(\frac{\sqrt{-\frac{\nu}{k^2}}(k(x+\nu t)+K)}{2}\right)}{\nu} \tag{3.34}$$

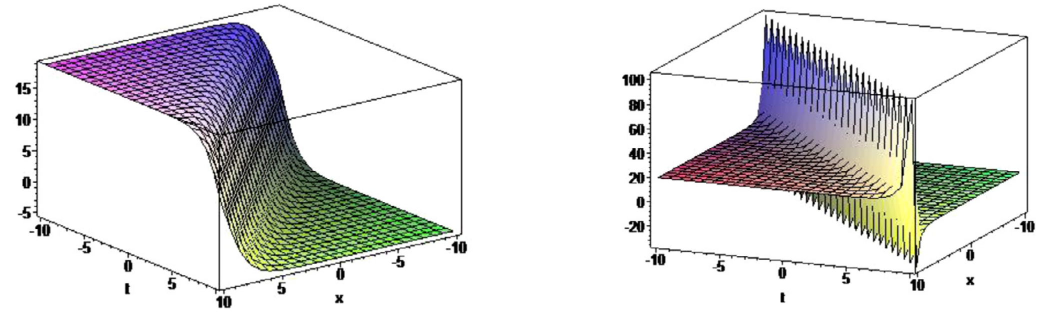
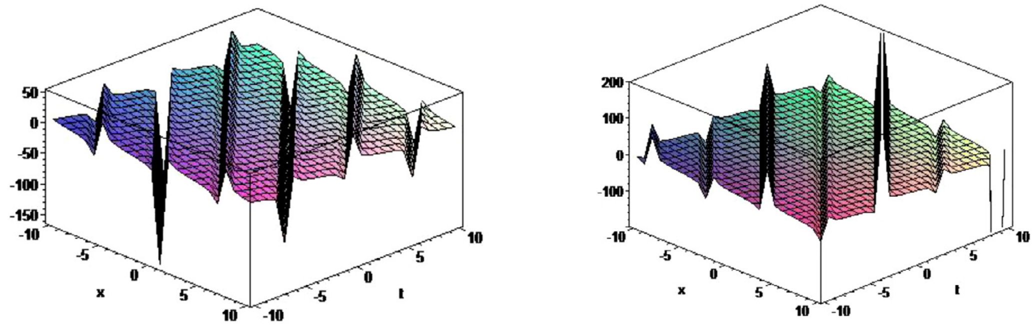
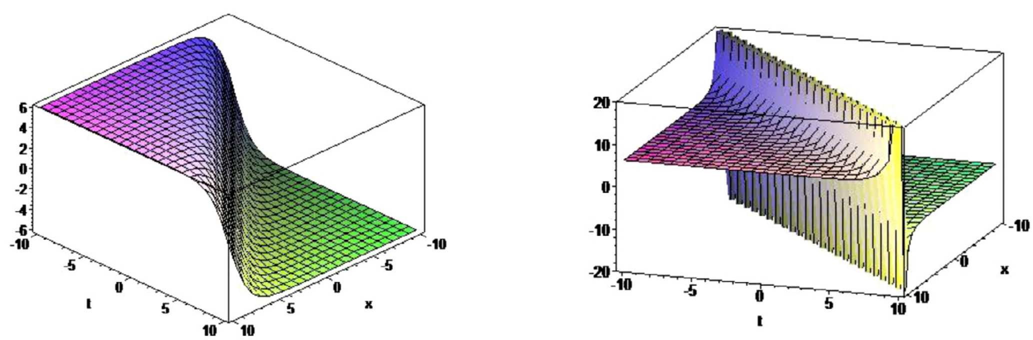


Fig. 6. Graphs of the solutions $\psi_{3,3}(x,t)$ and $\psi_{3,4}(x,t)$of Eq.(1.1), respectively, for the values $\alpha = 0.25, \nu = -1, a_3 = 1$ and $K = 1$.



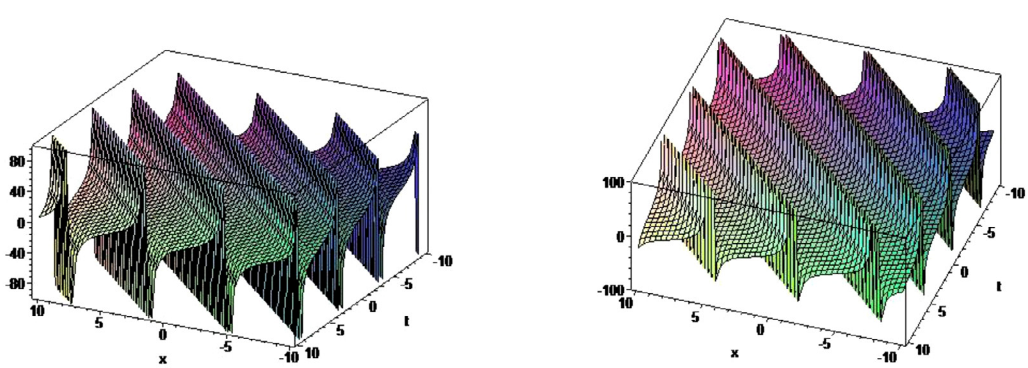
(a) (b)

Fig. 7. Graphs of the solutions $\psi_{4,1}(x,t)$ and $\psi_{4,2}(x,t)$of Eq. (1.1), respectively, for the values $\alpha = 0.5, \nu = 1, a_3 = 1$ and $K = 1$.



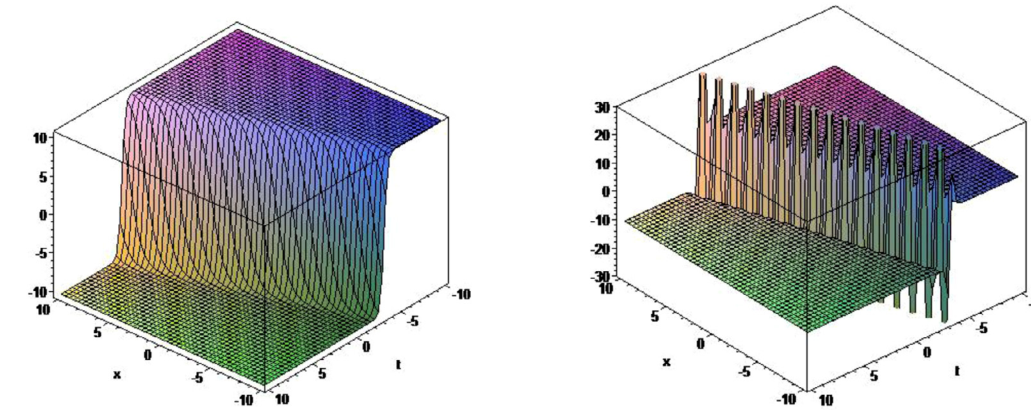
(a) (b)

Fig. 8. Graphs of the solutions $\psi_{4,3}(x,t)$ and $\psi_{4,4}(x,t)$of Eq. (1.1), respectively, for the values $\alpha = 0.5, \nu = -1, a_3 = 1$ and $K = 1$.



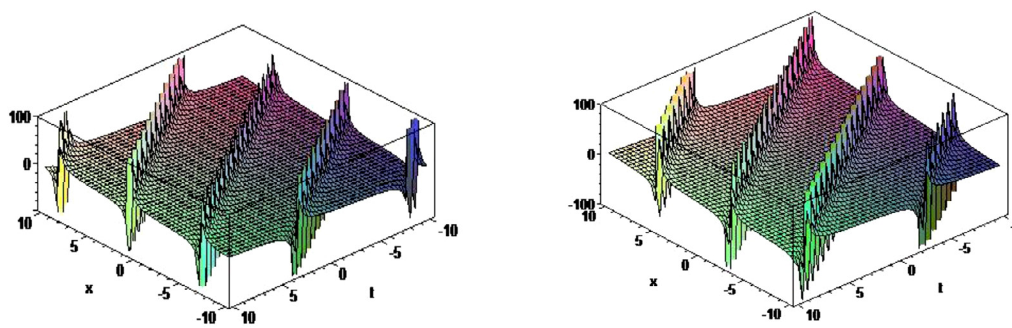
(a) (b)

Fig. 9. Graphs of the solutions $\psi_{5,1}(x,t)$ and $\psi_{5,2}(x,t)$of Eq. (1.1), respectively, for the values $k = -2, \nu = 1, a_3 = 3, K = 4$.



(a) (b)

Fig. 10. Graphs of the solutions $\psi_{5,3}(x,t)$ and $\psi_{5,4}(x,t)$of Eq. (1.1), respectively, for the values $k=3, \nu=-3, a_3=3$ and $K=-2$.



(a) (b)

Fig. 11. Graphs of the solutions $\psi_{6,1}(x,t)$ and $\psi_{6,2}(x,t)$of Eq. (1.1), respectively, for the values $\alpha=-0.5, k=1, a_3=2$and $K=4$.

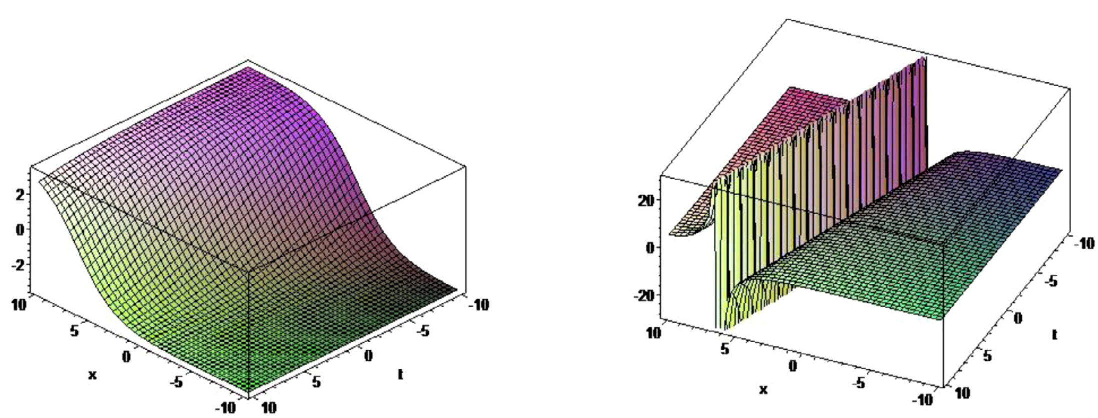


Fig. 12. Graphs of the solutions $\psi_{6,3}(x,t)$ and $\psi_{6,4}(x,t)$of Eq. (1.1), respectively, for the values $\alpha=0.5, k=-1, a_3=-1$ and $K=3$.

in which $\nu<0$ valid solitons in solutions. These solutions are depicted in Fig. 10.

Case 6:

$$a_2=\frac{\alpha}{6k}, a_1=0, \nu=-\frac{2a_3\alpha k}{3} \tag{3.35}$$

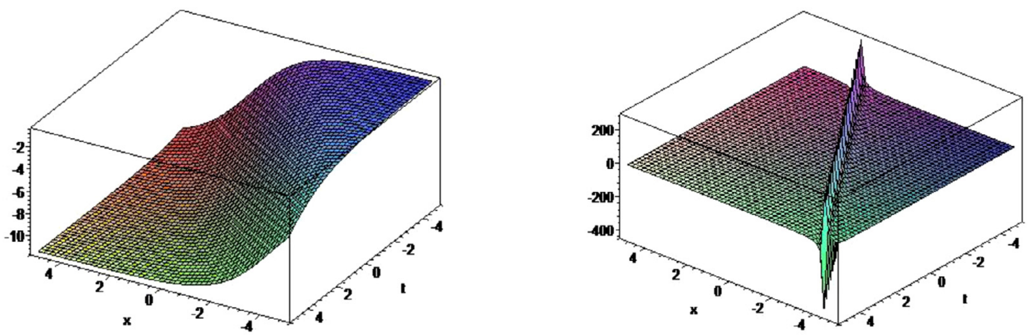
If the parameters in Eq. (3.35) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):

$$\psi_{6,1}(x,t)=-\frac{\sqrt{-\frac{6a_3\alpha}{k}}\,k\tan\left(\frac{\sqrt{-\frac{6a_3\alpha}{k}}\left(k\left(x-\frac{2a_3\alpha kt}{3}\right)+K\right)}{6}\right)}{\alpha}, \tag{3.36}$$

$$\psi_{6,2}(x,t)=\frac{\sqrt{-\frac{6a_3\alpha}{k}}\,k\cot\left(\frac{\sqrt{-\frac{6a_3\alpha}{k}}\left(k\left(x-\frac{2a_3\alpha kt}{3}\right)+K\right)}{6}\right)}{\alpha}. \tag{3.37}$$

in which $a_3\alpha k<0$ valid solitons in solutions. These solutions are depicted in Fig. 11.

$$\psi_{6,3}(x,t)=-\frac{\sqrt{6}\sqrt{\frac{a_3\alpha}{k}}\,k\tanh\left(\frac{\sqrt{6}\sqrt{\frac{a_3\alpha}{k}}\left(k\left(x-\frac{2a_3\alpha kt}{3}\right)+K\right)}{6}\right)}{\alpha}, \tag{3.38}$$



(a) (b)

Fig.13. Graphs of the solutions $\psi_{1,1}(x,t)$ and $\psi_{1,2}(x,t)$of Eq. (1.2), respectively, for the values $k=1, \alpha=-0.5, a_3=0.5, a_1=1, \beta=-1$and $K=1$.

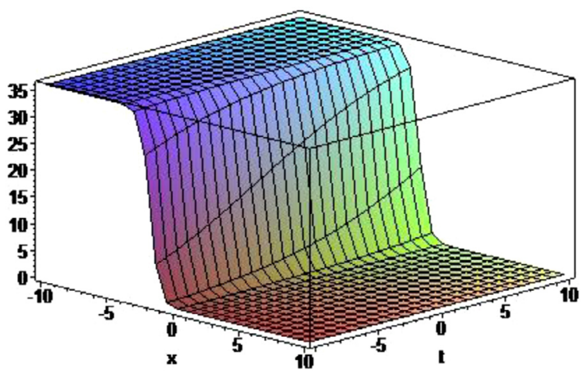


Fig. 14. Graph of the solution $\psi_{2,1}(x,t)$ of Eq. (1.2) for the values $\alpha=-0.5, \beta=0.5, a_1=-1, k=3$and $K=3$.

$$\psi_{6,4}(x,t) = -\frac{\sqrt{6}\sqrt{\frac{a_3\alpha}{k}}k\coth\left(\frac{\sqrt{6}\sqrt{\frac{a_3\alpha}{k}}\left(k\left(x-\frac{2a_3\alpha kt}{3}\right)+K\right)}{6}\right)}{\alpha} \quad (3.39)$$

in which $a_3\alpha k > 0$ valid solitons in solutions. These solutions are depicted in Fig. 12.

3.2. Applications of Riccati-Bernoulli sub-ode method to Eq. (1.2)

In this subsection, to obtain the wave solutions of Eq. (1.2), we utilize the following traveling wave transformation

$$\psi(x,t) = \psi(\zeta), \zeta = k(x+\upsilon t) \quad (3.40)$$

Eq. (1.2) is reduced to:

$$k^4\upsilon\psi^{(<ct>\iota<ot>\upsilon)} + k^2(\upsilon^2+\beta)\psi'' + 2\alpha k^3\upsilon\psi'\psi'' = 0 \quad (3.41)$$

Assume that the solution of Eq. (3.41) is the solution of Eq. (2.4). Substituting ψ and its derivatives into Eq. (3.41), and setting $m=0$, then we collect all the coefficients of $\psi^i (i=0,1,2,3,4,5)$ together and setting each to zero, the following system is obtained:

ψ^0 **coefficient:**

$$a_1a_3k^2(\upsilon^2+\beta) + k^4\upsilon(a_1(a_1^2a_3+2a_2a_3^2)+6a_3^2a_2a_1) +2a_1a_3^2\alpha k^3\upsilon = 0 \quad (3.42)$$

ψ^1 **coefficient:**

$$((a_1^2+2a_3a_2)k^2(\upsilon^2+\beta) + k^4\upsilon(6a_3a_2(a_1^2+2a_3a_2)+6a_1^2a_3a_2 +2a_2(a_1^2a_3+2a_2c^2)+a_1((a_1^2+2a_3a_2)a_1+6a_1a_3a_2)) +2(a_1^2a_3+(a_1^2+2a_3a_2)a_3)\alpha k^3\upsilon) = 0$$

ψ^2 **coefficient:**

$$(3a_1a_2k^2(\upsilon^2+\beta) + k^4\upsilon(2a_2((a_1^2+2a_3a_2)a_1+6a_1a_3a_2)+b(3a_1^2a_2 +4(a_1^2+a_1a_3a_2)a_2)+24a_3a_2^2a_1+6a_1a_2(a_1^2+2a_3a_2)) +2(4a_1a_3a_2+(a_1^2+2a_3a_2)a_1)\alpha k^3\upsilon) = 0 \quad (3.44)$$

ψ^3 **coefficient:**

$$(2a_2^2k^2(\upsilon^2+\beta) + k^4\upsilon(12a_3a_2^3+30a_1^2a_2^2+6a_2^2(a_1^2+2a_3a_2) +2a_2(3a_1^2a_2+4(a_1^2+2a_3a_2)a_2)) +2((a_1^2+2a_3a_2)a_2+3a_1^2a_2+2a_2^2a_3)\alpha k^3\upsilon) = 0 \quad (3.45)$$

ψ^4 **coefficient:**

$$(10a_1a_2^2\alpha k^3\upsilon + 60k^4\upsilon a_1a_2^3) = 0 \quad (3.46)$$

ψ^5 **coefficient:**

$$(4a_2^3\alpha k^3\upsilon + 24k^4\upsilon a_2^4) = 0 \quad (3.47)$$

When we solve the system of algebraic equations in Eqs. (3.42)-(3.47), then the following cases are acquired.

Case 1:

$$a_2 = -\frac{\alpha}{6k}, \upsilon = -\frac{a_1^2k^2}{2} - \frac{\alpha a_3k}{3} \mp \frac{\sqrt{9a_1^4k^4+12a_1^2a_3k^3\alpha+4a_3^2k^2\alpha^2-36\beta}}{6} \quad (3.48)$$

If the parameters in Eq. (3.48) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.2):

$$\psi_{1,1}(x,t) = \frac{3a_1k}{\alpha} - \frac{\sqrt{-\frac{6\alpha a_3}{k}-9a_1^2}k\tan\left(\frac{\sqrt{-\frac{6\alpha a_3}{k}-9a_1^2}\left(k\left(x+\left(-\frac{a_1^2k^2}{2}-\frac{\alpha a_3k}{3}+\frac{\sqrt{9a_1^4k^4+12a_1^2a_3k^3\alpha+4a_3^2k^2\alpha^2-36\beta}}{6}\right)t\right)+K\right)}{6}\right)}{\alpha}, \quad (3.49)$$

$$\psi_{1,2}(x,t) = \frac{3a_1k}{\alpha} + \frac{\sqrt{-\frac{6\alpha a_3}{k}-9a_1^2}k\cot\left(\frac{\sqrt{-\frac{6\alpha a_3}{k}-9a_1^2}\left(k\left(x+\left(-\frac{a_1^2k^2}{2}-\frac{\alpha a_3k}{3}+\frac{\sqrt{9a_1^4k^4+12a_1^2a_3k^3\alpha+4a_3^2k^2\alpha^2-36\beta}}{6}\right)t\right)+K\right)}{6}\right)}{\alpha} \quad (3.50)$$

in which $\frac{6\alpha a_3}{k}+9a_1^2 < 0$ for valid solutions.

$$\psi_{1,3}(x,t) = \frac{3a_1k}{\alpha} + \frac{\sqrt{\frac{6\alpha a_3}{k}+9a_1^2}k\tanh\left(\frac{\sqrt{\frac{6\alpha a_3}{k}+9a_1^2}\left(k\left(x+\left(-\frac{a_1^2k^2}{2}-\frac{\alpha a_3k}{3}+\frac{\sqrt{9a_1^4k^4+12a_1^2a_3k^3\alpha+4a_3^2k^2\alpha^2-36\beta}}{6}\right)t\right)+K\right)}{6}\right)}{\alpha}, \quad (3.51)$$

$$\psi_{1,4}(x,t) = \frac{3a_1k}{\alpha} + \frac{\sqrt{\frac{6\alpha a_3}{k} + 9a_1^2}\,k\coth\left(\frac{\sqrt{\frac{6\alpha a_3}{k}+9a_1^2}\left(k\left(x+\left(-\frac{a_1^2k^2}{2}-\frac{\alpha a_3k}{3}+\frac{\sqrt{9a_1^4k^4+12a_1^2a_3k^3\alpha+4a_3^2k^2\alpha^2-36\beta}}{6}\right)t\right)+K\right)}{6}\right)}{\alpha}, \quad (3.52)$$

in which $\frac{6\alpha a_3}{k} + 9a_1^2 > 0$ for valid solutions.

Case 2:

$$a_3 = 0, a_2 = -\frac{\alpha}{6k}, \nu = -\frac{k^2a_1^2}{2} \mp \frac{\sqrt{k^4a_1^4 - 4\beta}}{2} \quad (3.53)$$

If the parameters in Eq. (3.53) are substituted into (2.10)-(2.13), we acquire the following exact traveling wave solutions of Eq. (1.2):

$$\psi_{2,1}(x,t) = \frac{1}{Ke^{\left(-a_1k\left(x+\left(-\frac{k^2a_1^2}{2}+\frac{\sqrt{k^4a_1^4-4\beta}}{2}\right)t\right)\right)} + \frac{\alpha}{6ka_1}} \quad (3.54)$$

in which $k^4{a_1}^2 - 4\beta > 0$ for valid solutions.

4. Conclusion

In this research paper, the Riccati-Bernoulli sub-ODE technique is successfully utilized to determine the exact solutions of the integrable fourth-order equations. These equations are arising as a mathematical model in ocean engineering. To show the efficiency of the technique, two fourth-order equations are solved. As a result, we acquired some new exact solutions for these nonlinear models. Moreover, all the exact solutions in this paper satisfy the studied equations. The novel solitary solutions such as, the kink-type and the periodic wave solutions are acquired from the exact solutions. Moreover, the soliton survey of these solutions are demonstrate graphically. The solutions also show the productiveness of the presented method in investigating the behavior of waves in the studied models. Eventually, the proposed technique provides a very useful and effective mathematical technique for solving NLPDEs in mathematical physics and natural sciences.

5. Results and discussion

In this section, we present the obtained results. Implementing the Riccati Bernoulli sub-ODE scheme, new exact solitary wave solutions of the presented two fourth-order equations originated from the Boussinesq-type equation. This type of equation arises in many physical phenomena, like one-dimensional nonlinear lattice waves, electromagnetic waves in nonlinear dielectrics, oscillations in a nonlinear string, and ion sound waves in plasma. To our knowledge, these new solutions have not been reported earlier, and they are essential to clarify some physical phenomena.

In further studies, the classical NLPDEs models can be developed with fractional operators like in [44–52]. Riccati Bernoulli sub-ODE method can be applied to NLPDEs with different kinds of fractional operators. Furthermore, the comparison with distinct fractional operators can be investigated and differences between produced solutions can be presented with graphical visualization.

Declaration of competing interest

Authors declare that there is no conflict of interest whatsoever.

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