OSCILLATION PROPERTIES OF SOLUTIONS OF FRACTIONAL DIFFERENCE EQUATIONS

by

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In this article, studied the properties of the oscillation of fractional difference equations, and we obtain some results. The results we obtained are an expansion and further development of highly known results. Then we showed them with examples.

Key words: *fractional difference equation, oscillatory solutions, oscillation theory*

Introduction and preliminaries

In the investigations of qualitative properties for differential equations, research on time scales of the dynamic equations, oscillation of differential (or difference) equations and fractional differential equations have been a very important issue in the science and engineering. We refer to [1-25] and the references therein.

We first investigated following fractional difference equations:

$$\Delta\left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]+\sum_{i=1}^{n}q_i(t)\left[\sum_{s=t_0}^{t+\alpha-1}\frac{1}{\left(t-s-1\right)^{\alpha}}x(s)\right]^{\eta_i}=0 \tag{1}$$

We can rewrite eq. (1):

$$\Delta\left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]+\sum_{i=1}^{n}q_i(t)\,G^{\eta_i}(t)=0 \tag{2}$$

where

$$t\in \mathrm{N}_{t_0+1-\alpha},\quad G(t)=\sum_{s=t_0}^{\alpha+t-1}\frac{1}{(t-s-1)^{\alpha}}x(s),\quad \delta_1,\ \delta_2$$

and η_i are the division of two odd positive integers. The $\psi(t)$, $a(t)$, and $q_i(t)$ are positive coefficient sequences, and Δ^{α} demonstrate that the Riemann-Liouville fractional difference operator of order α where $0<\alpha\leq 1$. Therefore, in our results we use the following conditions:

C1. $$\sum_{s=t_0}^{\infty}\frac{1}{\psi^{1/\delta_1}(s)}=\infty \quad\text{and}\quad \sum_{s=t_0}^{\infty}\frac{1}{a^{1/\delta_2}(s)}=\infty \tag{3}$$

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C2. $$\sum_{s=t_0}^{\infty}\frac{1}{\psi^{1/\delta_1}(s)}<\infty \quad \text{and} \quad \sum_{s=t_0}^{\infty}\frac{1}{a^{1/\delta_2}(s)}<\infty \tag{4}$$

By a solution of eq. (2), we mean a real-valued sequence $x(t)$ satisfying eq. (2) for $t\in\mathbb{N}_{t_0}$. A solution $x(t)$ of eq. (2) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called non-oscillatory. Equation (2) is called oscillatory if all its solutions are oscillaory.

Definition 1. [26]. We define v^{th} fractional sum *f* as:

$$\Delta^{-v}f(t)=\left[\Gamma(v)\right]^{-1}\sum_{s=a}^{t-v}(t-s-1)^{v-1}f(s),\ \ v>0 \tag{5}$$

where we define *f* for $s\equiv a\,\text{mod}(1)$, $\Delta^{-v}f$ for $t\equiv(a+v)\,\text{mod}(1)$ and $t^{(v)}=\Gamma(1+t)/\Gamma(1+t-v)$. The fractional sum $\Delta^{-v}f$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+v}$, where $\mathbb{N}_t=\{t,\ t+1,\ t+2,...\}$.

Definition 2. [26] Let $m-1<\mu<m$ and $v>0$, where m denotes a positive integer, $m=\lceil\mu\rceil$. Set $v=m-\mu$. Then we define that μ^{th} fractional difference:

$$\Delta^{\mu}f(t)=\Delta^{m-v}f(t)=\Delta^{m}\Delta^{-v}f(t) \tag{6}$$

Oscillation properties of equation (2)

In this section, we work the oscillation properties of equation (2).

Lemma 1. [22]. Suppose that $x(t)$ be a solution of eq. (2) and let:

$$G(t)=\sum_{s=t_0}^{\alpha+t-1}(t-s-1)^{(-\alpha)}x(s) \tag{7}$$

then

$$\Delta\left[G(t)\right]=\Gamma(1-\alpha)\Delta^{\alpha}x(t) \tag{8}$$

Theorem 1. Assume C1 holds and furthermore, for all suficiently large *t*:

$$\sum_{s=t_3}^{\infty}\left\{\frac{1}{\psi(s)}\sum_{\tau=s}^{\infty}\left[\sum_{i=1}^{n}\frac{1}{a(\tau)}\sum_{\zeta=\tau}^{\infty}q_i(\zeta)\right]^{1/\delta_2}\right\}^{1/\delta_1}=\infty \tag{9}$$

and

$$\sum_{i=1}^{n}\sum_{s=t_3}^{\infty}q_i(s)\left[\sum_{\tau=t_2}^{s-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\tau)}\right]^{\eta_i}=\infty \tag{10}$$

Then every solution of eq. (2) is either oscillatory or $\lim_{t\to\infty}G(t)=0$.

Proof. Assume that the contrary that $x(t)$ is non-oscillatory solution of eq. (2). Then without loss of generality, we may assume that there is a solution $x(t)$ of eq. (2) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large, so that $G(t)>0$ on $[t_1,\infty)$. And all of $q_i(t)$'s are not identically zero on $[t_1,\infty)$ for $i=1,2,...,n$. From eq. (2), we have:

$$\Delta\left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]=-\sum_{i=1}^{n}q_i(t)G^{\eta_i}(t)<0 \tag{11}$$

In that case

$$a(t)\left(\Delta\left\{\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}$$

is an eventually non-increasing sequence on $[t_1,\infty)$. So, we understand that $\Delta\{\psi(t)[\Delta^{\alpha}x(t)]^{\delta_1}\}$ and $\Delta^{\alpha}x(t)$ are ultimately of one sign. For $t_2>t_1$ is big enough, $\Delta\{\psi(t)[\Delta^{\alpha}x(t)]^{\gamma_1}\}$ and $\Delta^{\alpha}x(t)$ have a fixed sign on $[t_2,\infty)$. We then consider the following conditions:

- Case 1. $\Delta^{\alpha}x(t)<0$ and $\Delta\{\psi(t)[\Delta^{\alpha}x(t)]^{\delta_1}\}<0$;
- Case 2. $\Delta\{\psi(t)[\Delta^{\alpha}x(t)]^{\delta_1}\}<0$ and $0<\Delta^{\alpha}x(t)$;
- Case 3. $\Delta\{\psi(t)[\Delta^{\alpha}x(t)]^{\delta_1}\}>0$ and $0>\Delta^{\alpha}x(t)$;
- Case 4. $\Delta\{\psi(t)[\Delta^{\alpha}x(t)]^{\delta_1}\}>0$ and $0<\Delta^{\alpha}x(t)$.

For the Case 1, we have:

$$\frac{G(t)}{\Gamma(1-\alpha)}=\frac{G(t_2)}{\Gamma(1-\alpha)}+\sum_{s=t_2}^{t-1}\frac{\left\{\psi(s)\left[\Delta^{\alpha}x(s)\right]^{\delta_1}\right\}^{1/\delta_1}}{\psi^{1/\delta_1}(s)}\le\frac{G(t_2)}{\Gamma(1-\alpha)}+$$
$$+\left\{\psi(t_2)\left[\Delta^{\alpha}x(t_2)\right]^{\delta_1}\right\}^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)}$$

Then, by C1, we obtain $\lim_{t\to\infty}G(t)=-\infty$ which contradicts with $0<G(t)$.

For the Case 2, we have from eq. (9):

$$\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}=\psi(t_2)\left[\Delta^{\alpha}x(t_2)\right]^{\delta_1}+\sum_{s=t_2}^{t-1}\frac{\left[a(s)\left(\Delta\left\{\psi(s)\left[\Delta^{\alpha}x(s)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]^{1/\delta_2}}{a^{1/\delta_2}(s)}\le$$
$$\le\psi(t_2)\left[\Delta^{\alpha}x(t_2)\right]^{\delta_1}+\left[a(t_2)\left(\Delta\left\{\psi(t_2)\left[\Delta^{\alpha}x(t_2)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]^{1/\delta_2}\sum_{s=t_2}^{t-1}\frac{1}{a^{1/\delta_2}(s)}$$

Then, by C1, we obtain $\lim_{t\to\infty}\psi(t)[\Delta^{\alpha}x(t)]^{\gamma_1}=-\infty$ which contradicts with $0<\Delta^{\alpha}x(t)$.

For the Case 3, we have $\lim_{t\to\infty}G(t)=k_1\ge 0$ and $\lim_{t\to\infty}\psi(t)[\Delta^{\alpha}x(t)]^{\delta_1}=k_2\le 0$. If we suppose that $k_1>0$, then $G(t)>k_1$ for $t\le t_3\le t_2$. Therefore, if we sum both sides of eq. (2) from t to ∞, we obtain:

$$a(t)\left(\Delta\left\{\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}<-\sum_{s=t}^{\infty}\sum_{i=1}^{n}q_i(s)G^{\eta_i}(s)\le-\sum_{i=1}^{n}k_1^{\eta_i}\sum_{s=t}^{\infty}q_i(s)$$

that is:

$$\Delta\left\{\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}\right\}\le-\left[\sum_{i=1}^{n}\frac{k_1^{\eta_i}}{a(t)}\sum_{s=t}^{\infty}q_i(s)\right]^{1/\delta_2}\tag{12}$$

If we sum both sides of the eq. (12) from t to ∞, we have:

$$\psi(t)\left[\Delta^{\alpha}x(t)\right]^{\delta_1}-k_2\le-\sum_{s=t}^{\infty}\left[\sum_{i=1}^{n}\frac{k_1^{\eta_i}}{a(s)}\sum_{\tau=s}^{\infty}q_i(\tau)\right]^{1/\delta_2}$$

which means for $k_2 \le 0$:

$$\Delta^{\alpha} x(t) \le -\left(\left[\psi(t)^{-1}\right]\sum_{s=t}^{\infty}\left\{\sum_{i=1}^{n} k_1^{\eta_i}\left[a(s)^{-1}\right]\sum_{\tau=s}^{\infty} q_i(\tau)\right\}^{1/\delta_2}\right)^{1/\delta_1} \tag{13}$$

If we sum both sides of the eq. (13) from t_3 to $t-1$, we obtain:

$$\frac{G(t)}{\Gamma(1-\alpha)} \le \frac{G(t_3)}{\Gamma(1-\alpha)} - \sum_{s=t_3}^{t-1}\left\{\frac{1}{\psi(s)}\sum_{\tau=s}^{\infty}\left[\sum_{i=1}^{n}\frac{k_1^{\eta_i}}{a(\tau)}\sum_{\zeta=\tau}^{\infty} q_i(\zeta)\right]^{1/\delta_2}\right\}^{1/\delta_1}$$

Therefore, by eq. (9), we obtain $\lim_{t\to\infty} G(t) = -\infty$ with contradicts with $G(t) > 0$.
For the Case 4, we have:

$$\frac{G(t)}{\Gamma(1-\alpha)} = \frac{G(t_2)}{\Gamma(1-\alpha)} + \sum_{s=t_2}^{t-1}\frac{\left\{\psi(s)\left[\Delta^{\alpha} x(s)\right]^{\delta_1}\right\}^{1/\delta_1}}{\psi^{1/\delta_1}(s)} > \left\{\psi(t_2)\left[\Delta^{\alpha} x(t_2)\right]^{\delta_1}\right\}^{1/\gamma_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)} > 0$$

That is:

$$\left(\left\{\psi(t_2)\left[\Delta^{\alpha} x(t_2)\right]^{\delta_1}\right\}^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(s)}\right)^{\eta_i} < G^{\eta_i}(t)$$

Then from eq. (2):

$$\sum_{i=1}^{n} q_i(t)\left(\left\{\psi(t_2)\left[\Delta^{\alpha} x(t_2)\right]^{\delta_1}\right\}^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(s)}\right)^{\eta_i} < -\Delta\left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^{\alpha} x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right] \tag{14}$$

If we sum both sides of the eq. (14) from t_3 to $t-1$, we obtain:

$$\sum_{i=1}^{n}\left(\left\{\psi(t_2)\left[\Delta^{\alpha} x(t_2)\right]^{\delta_1}\right\}^{1/\delta_1}\right)^{\eta_i}\sum_{s=t_3}^{t-1} q_i(s)\left(\sum_{\tau=t_2}^{s-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\tau)}\right)^{\eta_i} < a(t_3)\left(\Delta\left\{\psi(t_3)\left[\Delta^{\alpha} x(t_3)\right]^{\delta_1}\right\}\right)^{\delta_2}$$

If we take $t\to\infty$, we get a contradiction with eq. (10). Therefore, the proof of the *Theorem 1* is complete

Theorem 2. Suppose that C2, eqs. (9) and (10) hold. Furthermore, for all sufficiently large *t*:

$$\sum_{s=t_4}^{\infty}\left(\frac{\Gamma(1-\alpha)}{\psi(s)}\sum_{\tau=t_3}^{s-1}\left\{\sum_{i=1}^{n}\frac{1}{a(\tau)}\sum_{\zeta=t_2}^{\tau-1} q_i(\zeta)\left[\sum_{\xi=\zeta}^{\infty}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\xi)}\right]^{\eta_i}\right\}^{1/\delta_2}\right)^{1/\delta_1} = \infty \tag{15}$$

and

$$\sum_{s=t_4}^{\infty}\left[\frac{1}{a(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n} q_i(\tau)\left\{\left[\sum_{\zeta=\tau}^{\infty}\frac{1}{a^{1/\delta_2}(\zeta)}\right]^{1/\delta_1}\sum_{\zeta=t_2}^{\tau-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\zeta)}\right\}^{\eta_i}\right)\right]^{1/\delta_2} = \infty \tag{16}$$

Therefore, each solution of eq. (2) is either $\lim_{t\to\infty} G(t)=0$ or oscillatory.

Proof. Let's the contrary that $x(t)$ is non-oscillatory solution of eq. (2). Then without loss of generality, we assume that there is a solution $x(t)$ of eq. (2) such that $0<x(t)$ on $[t_1,\infty)$, where t_1 is sufficiently large, so that $G(t)>0$ on $[t_1,\infty)$. It appears that all of $q_i(t)$'s are not identically zero on $[t_1,\infty)$ for $i=1,2,...,n$. From eq. (11), we obtained that $a(t)(\Delta\{\psi(t)[\Delta^\alpha x(t)]^{\delta_1}\})^{\delta_2}$ is an eventually non-increasing sequence on $[t_1,\infty)$. For the Case 1, we have:

$$-\frac{G(t)}{\Gamma(1-\alpha)}<\sum_{s=t}^{\infty}\left(\left\{\psi(s)\left[\Delta^\alpha x(s)\right]^{\delta_1}\right\}^{1/\delta_1}\left[\psi^{1/\delta_1}(s)\right]^{-1}\right)<\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}^{1/\delta_1}\sum_{s=t}^{\infty}\left[\psi^{1/\delta_1}(s)\right]^{-1}<$$

$$<\left\{\psi(t_1)\left[\Delta^\alpha x(t_1)\right]^{\delta_1}\right\}^{1/\delta_1}\sum_{s=t}^{\infty}\left[\psi^{1/\delta_1}(s)\right]^{-1}=K_1\sum_{s=t}^{\infty}\left[\psi^{1/\delta_1}(s)\right]^{-1}$$

Then from the last inequality and eq. (2), we obtain:

$$\Delta\left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]<\sum_{i=1}^{n}q_i(t)\left[\Gamma(1-\alpha)K_1\sum_{s=t}^{\infty}\frac{1}{\psi^{1/\delta_1}(s)}\right]^{\eta_i} \tag{17}$$

If we sum both sides of the eq. (17) from t_2 to $t-1$:

$$a(t)\left(\Delta\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}<\sum_{i=1}^{n}\left(\Gamma(1-\alpha)K_1\right)^{\eta_i}\sum_{s=t_2}^{t-1}q_i(s)\left[\sum_{\tau=s}^{\infty}\frac{1}{\psi^{1/\delta_1}(\tau)}\right]^{\eta_i}$$

that is:

$$\Delta\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}<\left\{\sum_{i=1}^{n}\frac{\left[\Gamma(1-\alpha)K_1\right]^{\eta_i}}{a(t)}\sum_{s=t_3}^{t-1}q_i(s)\left[\sum_{\tau=s}^{\infty}\frac{1}{\psi^{1/\delta_1}(\tau)}\right]^{\eta_i}\right\}^{1/\delta_2} \tag{18}$$

If we sum both sides of the eq. (18) from t_3 to $t-1$:

$$\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}<\sum_{s=t_3}^{t-1}\left\{\sum_{i=1}^{n}\frac{\left[\Gamma(1-\alpha)K_1\right]^{\eta_i}}{a(s)}\sum_{\tau=t_2}^{s-1}q_i(\tau)\left[\sum_{\zeta=\tau}^{\infty}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right]^{\eta_i}\right\}^{1/\delta_2}$$

then we get:

$$\Delta G(t)<\left(\frac{\Gamma(1-\alpha)}{\psi(t)}\sum_{s=t_3}^{t-1}\left\{\sum_{i=1}^{n}\frac{\left[\Gamma(1-\alpha)K_1\right]^{\eta_i}}{a(s)}\sum_{\tau=t_2}^{s-1}q_i(\tau)\left[\sum_{\zeta=\tau}^{\infty}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right]^{\eta_i}\right\}^{1/\delta_2}\right)^{1/\delta_1} \tag{19}$$

If we sum both sides of the the eq. (19) from t_4 to $t-1$, we have:

$$G(t)-G(t_4)<\sum_{s=t_4}^{t-1}\left(\frac{\Gamma(1-\alpha)}{\psi(s)}\sum_{\tau=t_3}^{s-1}\left\{\sum_{i=1}^{n}\frac{\left[\Gamma(1-\alpha)K_1\right]^{\eta_i}}{a(\tau)}\sum_{\zeta=t_2}^{\tau-1}q_i(\zeta)\left[\sum_{\xi=\zeta}^{\infty}\frac{1}{\psi^{1/\delta_1}(\xi)}\right]^{\eta_i}\right\}^{1/\delta_2}\right)^{1/\delta_1}$$

By eq. (14), we obtain $\lim_{t\to\infty} G(t) = -\infty$ due to $K_1 < 0$, which conradicts with $0 < G(t)$.

For the Case 2:

$$G(t) > \sum_{s=t_2}^{t-1} \frac{\Gamma(1-\alpha)\left\{\psi(s)\left[\Delta^\alpha x(s)\right]^{\delta_1}\right\}^{1/\delta_1}}{\psi^{1/\delta_1}(s)} > \Gamma(1-\alpha)\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}^{1/\delta_1} \sum_{s=t_2}^{t-1} \frac{1}{\psi^{1/\delta_1}(s)}$$

and

$$-\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1} \le \sum_{s=t}^{\infty} \frac{\left[a(s)\left(\Delta\left\{\psi(s)\left[\Delta^\alpha x(s)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]^{1/\delta_2}}{a^{1/\delta_2}(s)} <$$

$$< \left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]^{1/\delta_2} \sum_{s=t}^{\infty} \frac{1}{a^{1/\delta_2}(s)} <$$

$$< \left[a(t_2)\left(\Delta\left\{\psi(t_2)\left[\Delta^\alpha x(t_2)\right]^{\delta_1}\right\}\right)^{\delta_2}\right]^{1/\delta_2} \sum_{s=t}^{\infty} \frac{1}{a^{1/\delta_2}(s)} = K_2 \sum_{s=t}^{\infty} \frac{1}{a^{1/\delta_2}(s)}$$

Thereore, we have:

$$G(t) > -\Gamma(1-\alpha)\left[K_2 \sum_{s=t}^{\infty} \frac{1}{a^{1/\delta_2}(s)}\right]^{1/\delta_1} \sum_{s=t_2}^{t-1} \frac{1}{\psi^{1/\delta_1}(s)}$$

Thus, from eq. (2), we obtain:

$$\Delta\left[a(t)\left(\Delta\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}\right)^{\delta_2}\right] = \sum_{i=1}^{n} q_i(t)\left\{\Gamma(1-\alpha)\left[K_2 \sum_{s=t}^{\infty} \frac{1}{a^{1/\delta_2}(s)}\right]^{1/\delta_1} \sum_{s=t_2}^{t-1} \frac{1}{\psi^{1/\delta_1}(s)}\right\}^{\eta_i} \tag{20}$$

If we sum two sides of the eq. (20) from t_3 to $t-1$, we have:

$$a(t)\left(\Delta\left\{\psi(t)\left[\Delta^\alpha x(t)\right]^{\delta_1}\right\}\right)^{\delta_2} = \sum_{s=t_3}^{t-1}\left(\sum_{i=1}^{n} q_i(s)\left\{\Gamma(1-\alpha)\left[K_2 \sum_{\tau=s}^{\infty} \frac{1}{a^{1/\delta_2}(\tau)}\right]^{1/\delta_1} \sum_{\tau=t_2}^{s-1} \frac{1}{\psi^{1/\delta_1}(\tau)}\right\}^{\eta_i}\right)$$

Then:

$$\psi(t_4)\left[\Delta^\alpha x(t_4)\right]^{\delta_1} = \sum_{s=t_4}^{t-1}\left[\frac{1}{a(s)} \sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n} q_i(\tau)\left\{\Gamma(1-\alpha)\left[K_2 \sum_{\zeta=\tau}^{\infty} \frac{1}{a^{1/\delta_2}(\zeta)}\right]^{1/\delta_1} \sum_{\zeta=t_2}^{\tau-1} \frac{1}{\psi^{1/\delta_1}(\zeta)}\right\}^{\eta_i}\right)\right]^{1/\delta_2}$$

letting $t \to \infty$, we obtain:

$$\sum_{s=t_4}^{\infty}\left[\frac{1}{a(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n}q_i(\tau)\left\{\Gamma(1-\alpha)\left[K_2\sum_{\zeta=\tau}^{\infty}\frac{1}{a^{1/\delta_2}(\zeta)}\right]^{1/\delta_1}\sum_{\zeta=t_2}^{\tau-1}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right\}^{\eta_i}\right)\right]^{1/\delta_2}<\infty$$

which contradicts with eq. (16). The rest of the proof is made similar to the proof of the *Theorem 1*. Thus the proof of the theorem is completed.

Application

Let as consider the following fractional difference equation as an example:

$$\Delta\left[\sqrt{t}\sqrt{\left(\Delta\left\{\left[\Delta^{\alpha}x(t)\right]^{7}\right\}\right)}\right]+\frac{1}{t^2}\left(\sum_{s=t_0}^{t-1+\alpha}\left[-(s+1-t)\right]^{(-\alpha)}x(s)\right)^{3}=0,\quad 2\le t \qquad (21)$$

This corresponds to eq. (2) with $t_0=2$, $\delta_1=7$, $\delta_2=1/2$, $\alpha\in(0,1]$, $a(t)=t^{1/2}$, $\psi(t)=1$, $q(t)=1/t^2$, $n=1$, and $\eta_1=3$. However,

$$\sum_{s=t_0}^{\infty}\frac{1}{\psi^{1/\gamma_1}(s)}=\sum_{s=t_0}^{\infty}1=\infty$$

and

$$\sum_{s=t_0}^{\infty}\frac{1}{a^{1/\delta_2}(s)}=\sum_{s=t_2}^{\infty}\frac{1}{s}=\infty$$

Then C1 holds. So, we have:

$$\sum_{s=t_3}^{\infty}\left\{\frac{1}{\psi(s)}\sum_{\tau=s}^{\infty}\left[\sum_{i=1}^{n}\frac{1}{a(\tau)}\sum_{\zeta=\tau}^{\infty}q_i(\zeta)\right]^{1/\delta_2}\right\}^{1/\delta_1}=\sum_{s=t_3}^{\infty}\left[\sum_{\tau=s}^{\infty}\left(\frac{1}{\tau^{1/2}}\sum_{\zeta=\tau}^{\infty}\zeta^{-2}\right)^{2}\right]^{1/7}=\infty$$

and

$$\sum_{i=1}^{n}\sum_{s=t_3}^{\infty}q_i(s)\left[\sum_{\tau=t_2}^{s-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\tau)}\right]^{\eta_i}=\sum_{s=t_3}^{\infty}\frac{1}{s^2}\left[\sum_{\tau=2}^{s-1}\Gamma(1-\alpha)\right]^{3}=\sum_{s=t_3}^{\infty}\frac{1}{s^2}\left[\Gamma(1-\alpha)(s-2)\right]^{3}=\infty$$

Therefore, eqs. (9) and (10) holds, and then we say that eq. (21) is $\lim_{t\to\infty}G(t)=0$ or oscillatory by *Theorem 1*.

Conclusion

In this work, we studied the qualitative behavior of solutions of non-linear fractional difference equations (FDE) with fractional Riemann-Liouville difference operator. Because there was a gap for the oscillatory solutions of FDE under the condition (C2) in the literature, we considered the equation with the conditions (C1) and (C2). By using some techniques, we obtained some oscillation results. The obtained results improved the many criteria in the literature.

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