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On a Generalization of the Initial-Boundary Problem for the Vibrating String Equation

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Abstract: In the present paper, we study a generalization of the initial-boundary problem for the inhomogeneous vibrating string equation. The initial conditions include the higher order derivatives of the unknown function. The problem is studied under homogeneous boundary conditions of the first kind. The uniqueness and existence of a regular solution of the problem are proved. To prove the main result we use the spectral decomposition method.

Keywords: vibrating string equation; initial conditions; spectral decomposition; regular solution; the uniqueness of the solution; the existence of a solution

1. Introduction

The differential equations are used to model the real world application problems in science and engineering that involve several parameters as well as the change of variables with respect to others. Most of these problems will require the solution of initial and boundary conditions, that is, the solution to the differential equations are forced to satisfy certain conditions and data. However, to model most of the real world problems is very complicated task and in many forms it is also difficult to find the exact solution. Boundary value problems for the Laplace, Poisson and Helmholtz equations with boundary conditions containing the higher order derivatives were studied in works by Bavrin [1], Karachik [2–5], Sokolovskii [6]. In the papers [7–13], boundary problems, including higher derivatives on the boundary, were studied for the Poisson, Helmholtz, and biharmonic equations. It should be noted that unlike our work, in the mentioned papers [1–6,14], the higher order derivative is given on the entire boundary. For an inhomogeneous heat equation, an initial-boundary problem containing a higher order derivative in the presence of an initial condition was studied in [15]. Now we reconsider the following equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \tag{1}$$

in the domain $\Omega = \{(x,y) \mid 0 < x < p,\ 0 < t < T\}$ where $f(x,t)$ is a given function. Then we try to find a solution of the Equation (1) in the domain Ω which satisfies the following conditions

$$u(0,t) = 0, \quad 0 \le t \le T, \tag{2}$$

$$u(p,t) = 0, \quad 0 \le t \le T, \tag{3}$$

$$\frac{\partial^k u(x,0)}{\partial t^k} = \varphi_k(x), \quad 0 \le x \le p, \tag{4}$$

$$\frac{\partial^{k+1}u(x,0)}{\partial t^{k+1}} = \psi_k(x), \quad 0 \le x \le p, \tag{5}$$

where $k \ge 1$ is a fixed integer number. For the case $k = 0$ and $f(x,t) = 0$, the problem (1)–(5) was studied in [16]. Further Tikhonov in [17] studied homogeneous heat equation with the boundary condition $\sum\limits_{k=0}^{n} a_k \frac{\partial^k u(0,t)}{\partial x^k} = 0$ and the initial condition $u(x,0) = 0$ in the domain $(0 < x < \infty,\ t > 0)$. Similarly, Bitsadze in [14] studied the Laplace equation in an n-dimensional domain D under the condition

$$\frac{d^m u}{d\nu^m} = f(x),\ \ x \in \partial D$$

and proved its Fredholm property. There is also more related literature on the boundary conditions problem, see for example ([18–25]). In the present paper, we study a generalized initial-boundary problem (2)–(5) for the inhomogeneous vibrating string Equation (1). The initial conditions include the higher order derivatives of the unknown function. The problem is studied under homogeneous boundary conditions of the first kind. We prove the uniqueness and existence of a regular solution of the problem. To solve the problem (1)–(5), we apply the spectral decomposition method.

2. The Uniqueness of Solution

Theorem 1. *The solution of the problem (1)–(5) is unique if it exists.*

Proof. Let $f(x,t) = 0$ in $\overline{\Omega}$, $\varphi_k(x) = 0$, $\psi_k(x) = 0$ in $[0,p]$. We show that the homogeneous problem (1)–(5) has only the trivial solution. It is known [26], the functions

$$X_n(x) = \sqrt{\frac{2}{p}}\sin(\lambda_n x),\ \ \lambda_n = \frac{n\pi}{p},\ \ n = 1,2,\ldots \tag{6}$$

form in $L_2(0,p)$ a complete orthonormal system. Following [27], we consider the functions

$$\alpha_n(t) = \int\limits_0^p u(x,t)X_n(x)dx,\ \ 0 \le t \le T, \tag{7}$$

where $u(x,t)$ is the solution of the homogeneous equation corresponding to the Equation (1). Differentiating (7) twice with respect to t, we obtain from the corresponding homogeneous Equation (1)

$$\alpha_n''(t) + \lambda_n^2\alpha_n(t) = 0. \tag{8}$$

The solution of (8) has the form

$$\alpha_n(t) = a_n\cos(\lambda_n t) + b_n\sin(\lambda_n t).$$

To find the unknown coefficients a_n and b_n, we use the homogeneous conditions (4) and (5), which lead to the following equations:

$$\alpha_n^{(k)}(t) = 0,\ \ \alpha_n^{(k+1)}(t) = 0. \tag{9}$$

It is not difficult to verify that

$$\begin{aligned}
\alpha_n^{(k)}(t) &= \lambda_n^k\left[a_n\cos\left(\frac{\pi k}{2} + \lambda_n t\right) + b_n\sin\left(\frac{\pi k}{2} + \lambda_n t\right)\right], \\
\alpha_n^{(k+1)}(t) &= \lambda_n^{k+1}\left[a_n\cos\left(\frac{\pi(k+1)}{2} + \lambda_n t\right) + b_n\sin\left(\frac{\pi(k+1)}{2} + \lambda_n t\right)\right].
\end{aligned}$$

Using (9), we obtain the following system of equations to determine the unknown coefficients a_n and b_n:

$$a_n \cos\frac{\pi k}{2} + b_n \sin\frac{\pi k}{2} = 0,$$
$$a_n \cos\frac{\pi(k+1)}{2} + b_n \sin\frac{\pi(k+1)}{2} = 0,$$

whose determinant of coefficients is 1. Hence, that $\alpha_n(t) = 0$. By completeness of functions $X_n(x)$, the Equation (7) implies that $u(x,t) = 0$ in $\overline{\Omega}$. $\square$

3. The Existence of Solution

We search the solution of (1) in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x). \tag{10}$$

Expand the functions $f(x,t)$, $\varphi_k(x)$, and $\psi_k(x)$ in Fourier series by functions $X_n(x)$:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x), \tag{11}$$

$$\varphi_k(x) = \sum_{n=1}^{\infty} \varphi_{kn} X_n(x), \tag{12}$$

$$\psi_k(x) = \sum_{n=1}^{\infty} \psi_{kn} X_n(x), \tag{13}$$

where

$$f_n(t) = \int_0^p f(x,t) X_n(x) dx, \tag{14}$$

$$\varphi_{kn} = \int_0^p \varphi_k(x) X_n(x) dx, \tag{15}$$

$$\psi_{kn} = \int_0^p \psi_k(x) X_n(x) dx. \tag{16}$$

Substituting (10) and (11) into (1), we obtain

$$u_n''(t) + \lambda_n^2 u_n(t) = f_n(t).$$

It can be shown that the solution of this equation satisfying the conditions

$$u_n^{(k)}(0) = \varphi_{kn}, \;\; u_n^{(k+1)}(0) = \psi_{kn},$$

is

$$
\begin{aligned}
u_n(t) \;=\;& \frac{\varphi_{kn}}{\lambda_n^k}\cos\left(\frac{\pi k}{2}-\lambda_n t\right)-\frac{\psi_{kn}}{\lambda_n^{k+1}}\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&+\sum_{s=0}^{\left[\frac{k+1}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k+1-2s}}f_n^{(k-1-2s)}(0)\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&-\sum_{s=0}^{\left[\frac{k}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-2s}}f_n^{(k-2-2s)}(0)\cos\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&+\frac{1}{\lambda_n}\int_0^t f_n(\tau)\sin(\lambda_n(t-\tau))d\tau.
\end{aligned}
\tag{17}
$$

Hereinafter $\sum\limits_{s=0}^{m}(\dots)=0$ for $m<0$. Substituting (17) into (10), we get

$$
\begin{aligned}
u(x,t) \;=\;& \sum_{n=1}^{\infty}X_n(x)\left\{\frac{\varphi_{kn}}{\lambda_n^k}\cos\left(\frac{\pi k}{2}-\lambda_n t\right)-\frac{\psi_{kn}}{\lambda_n^{k+1}}\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\right.\\
&+\sum_{s=0}^{\left[\frac{k+1}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k+1-2s}}f_n^{(k-1-2s)}(0)\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&-\sum_{s=0}^{\left[\frac{k}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-2s}}f_n^{(k-2-2s)}(0)\cos\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&\left.+\frac{1}{\lambda_n}\int_0^t f_n(\tau)\sin(\lambda_n(t-\tau))d\tau\right\}.
\end{aligned}
\tag{18}
$$

Using (18) we find the following derivatives of $u(x,t)$.

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} \;=\;& \sum_{n=1}^{\infty}X_n(x)\left\{-\frac{\varphi_{kn}}{\lambda_n^{k-2}}\cos\left(\frac{\pi k}{2}-\lambda_n t\right)+\frac{\psi_{kn}}{\lambda_n^{k-1}}\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\right.\\
&-\sum_{s=0}^{\left[\frac{k+1}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-1-2s}}f_n^{(k-1-2s)}(0)\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&+\sum_{s=0}^{\left[\frac{k}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-2-2s}}f_n^{(k-2-2s)}(0)\cos\left(\frac{\pi k}{2}-\lambda_n t\right)+f_n(0)\cos(\lambda_n t)\\
&\left.+\frac{1}{\lambda_n}f_n'(0)\sin(\lambda_n t)+\frac{1}{\lambda_n}\int_0^t f_n''(\tau)\sin(\lambda_n(t-\tau))d\tau\right\}.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} \;=\;& \sum_{n=1}^{\infty}X_n(x)\left\{-\frac{\varphi_{kn}}{\lambda_n^{k-2}}\cos\left(\frac{\pi k}{2}-\lambda_n t\right)+\frac{\psi_{kn}}{\lambda_n^{k-1}}\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\right.\\
&-\sum_{s=0}^{\left[\frac{k+1}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-1-2s}}f_n^{(k-1-2s)}(0)\sin\left(\frac{\pi k}{2}-\lambda_n t\right)\\
&+\sum_{s=0}^{\left[\frac{k}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-2-2s}}f_n^{(k-2-2s)}(0)\cos\left(\frac{\pi k}{2}-\lambda_n t\right)-f_n(t)\\
&\left.+f_n(0)\cos(\lambda_n t)+\frac{1}{\lambda_n}f_n'(0)\sin(\lambda_n t)+\frac{1}{\lambda_n}\int_0^t f_n''(\tau)\sin(\lambda_n(t-\tau))d\tau\right\}.
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\frac{\partial^k u}{\partial t^k} \;=\;& \sum_{n=1}^{\infty}X_n(x)\left\{\varphi_{kn}\cos(\lambda_n t)+\frac{1}{\lambda_n}\psi_{kn}\sin(\lambda_n t)\right.\\
&\left.+\frac{(-1)^k}{\lambda_n}\int_0^t f_n^{(k)}(\tau)\sin\left[\pi k+\lambda_n(t-\tau)\right]d\tau\right\},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\frac{\partial^{k+1} u}{\partial t^{k+1}} \;=\;& \sum_{n=1}^{\infty}X_n(x)\left\{-\lambda_n\varphi_{kn}\sin(\lambda_n t)+\psi_{kn}\cos(\lambda_n t)+\frac{(-1)^k}{\lambda_n}f_n^{(k)}(0)\sin(\lambda_n t)\right.\\
&\left.+\frac{(-1)^{k+1}}{\lambda_n}\int_0^t f_n^{(k+1)}(0)\sin\left[\pi(k+1)+\lambda_n(t-\tau)\right]d\tau\right\}.
\end{aligned}
\tag{22}
$$

Next we need to prove the absolute and uniformly convergence of the series (18)–(22). Below we prove several lemmas that are used in the proof of the existence theorem.

Lemma 1. *Let $f(x,t) \in C^1\left(\overline{\Omega}\right)$, $f(0,t) = f(p,t) = 0$, $\dfrac{\partial f}{\partial x} \in Lip_\alpha[0,p]$ uniformly with respect to t and $0 < \alpha < 1$. Then the series (11) converges absolutely and uniformly in $\overline{\Omega}$.*

Proof. Integrating in parts (14) we find

$$f_n(t) = \frac{\sqrt{2p}}{n\pi} \int_0^p \frac{\partial f(x,t)}{\partial x} \cos(\lambda_n x) dx.$$

Then [28]

$$|f_n(t)| \le \frac{c}{n^{1+\alpha}}, \quad c = \frac{Kp^{\frac{3}{2}+\alpha}}{\pi\sqrt{2}},$$

where K is the Hölder constant. Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+\alpha}}$ converges, therefore the series (11) converges absolutely and uniformly in $\overline{\Omega}$. □

Lemma 2. *Let $\varphi_k(x) \in W_2^1(0,p)$, $\varphi_k(0) = \varphi_k(p) = 0$. Then the series (12) converges absolutely and uniformly in $[0,p]$.*

Proof. Integrating by parts (15) we obtain

$$\varphi_{kn} = \frac{1}{\lambda_n}\varphi_{kn}^{(1)}, \quad \varphi_{kn}^{(1)} = \int_0^p \varphi_k'(x)\sqrt{\frac{2}{p}}\cos(\lambda_n x)dx.$$

Using the Bessel inequality [29], $\sum\limits_{n=1}^{\infty} \left|\varphi_{kn}^{(1)}\right|^2 \le \|\varphi_k'\|^2_{L_2(0,p)}$ and the inequality $\sum\limits_{n=1}^{\infty} |\varphi_{kn}| = \frac{p}{\pi} \sum\limits_{n=1}^{\infty} \frac{1}{n} \left|\varphi_{kn}^{(1)}\right|$ and using the Hölder inequality for the sum [29] yields

$$\sum_{n=1}^{\infty} \frac{1}{n}\left|\varphi_{kn}^{(1)}\right| \le \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty}\left|\varphi_{kn}^{(1)}\right|^2\right)^{\frac{1}{2}} \le \frac{\pi}{\sqrt{6}}\|\varphi_k'\|_{L_2(0,p)}.$$

Here the equality $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ was used. This implies the absolutely and uniformly convergence of the series (12) on $[0,p]$. □

Lemma 3. *Let $\psi_k(x) \in W_2^1(0,p)$, $\psi_k(0) = \psi_k(p) = 0$. Then the series (13) converges absolutely and uniformly on $[0,p]$.*

The proof is similar to the proof of Lemma 3.

Lemma 4. *Let $\varphi_k(x) \in W_2^1(0,p)$, $\varphi_k(0) = \varphi_k(p) = 0$. Then the series*

$$\sum_{n=1}^{\infty} \lambda_n \varphi_{kn} X_n(x)\sin(\lambda_n t) \tag{23}$$

converges absolutely and uniformly in $\overline{\Omega}$.

Proof. Integrating by parts (15) we obtain

$$\varphi_{kn} = \frac{1}{\lambda_n^2}\varphi_{kn}^{(2)}, \quad \varphi_{kn}^{(2)} = \int_0^p \varphi_k''(x)\sqrt{\frac{2}{p}}\sin(\lambda_n x)dx.$$

Using the Parseval equality [29],

$$\sum_{n=1}^{\infty}\left|\varphi_{kn}^{(2)}\right|^2 = \|\varphi_k''\|_{L_2(0,p)}^2,$$

we obtain

$$\sum_{n=1}^{\infty}\lambda_n\left|\varphi_{kn}\right| = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|\varphi_{kn}^{(2)}\right| \le \left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left|\varphi_{kn}^{(2)}\right|^2\right)^{\frac{1}{2}} = \frac{p}{\sqrt{6}}\|\varphi_k''\|_{L_2(0,p)}.$$

Hence, the series (23) converges absolutely and uniformly in $\overline{\Omega}$. □

Lemma 5. *If* $\frac{\partial^{k+1}f(x,t)}{\partial t^{k+1}} \in C\left(\overline{\Omega}\right)$, *then the series*

$$\sum_{n=1}^{\infty}\frac{X_n(x)}{\lambda_n}\int_0^t f_n^{(m)}(\tau)\sin\left[\frac{\pi(k+m)}{2}+\lambda_n(t-m)\right]d\tau, \quad m = 1, 2, \ldots, k+1$$

converges absolutely and uniformly in $\overline{\Omega}$.

Proof. Applying the Hölder inequality for integrals [29], we get

$$\frac{1}{\lambda_n}\left|\int_0^t f_n^{(m)}(\tau)\sin\left[\frac{\pi(k+m)}{2}+\lambda_n(t-m)\right]d\tau\right| \le \frac{\sqrt{T}}{\lambda_n}\left\|f_n^{(m)}\right\|_{L_2(0,T)}.$$

Now applying the Hölder inequality for sums and the Bessel inequality, we find

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\|f_n^{(m)}\right\|_{L_2(0,T)} \le \frac{p}{\sqrt{6}}\left\|\frac{\partial^m f}{\partial t^m}\right\|_{L_2(0,T)}.$$

□

Next, consider the following series

$$\sum_{n=1}^{\infty}X_n(x)\sum_{s=0}^{\left[\frac{k+1}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k+1-2s}}f_n^{(k+1-2s)}(0)\sin\left(\frac{\pi k}{2}-\lambda_n t\right), \tag{24}$$

$$\sum_{n=1}^{\infty}X_n(x)\sum_{s=0}^{\left[\frac{k}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-2s}}f_n^{(k-2-2s)}(0)\cos\left(\frac{\pi k}{2}-\lambda_n t\right), \tag{25}$$

$$\sum_{n=1}^{\infty}X_n(x)\sum_{s=0}^{\left[\frac{k}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-2-2s}}f_n^{(k-2-2s)}(0)\cos\left(\frac{\pi k}{2}-\lambda_n t\right), \tag{26}$$

$$\sum_{n=1}^{\infty}X_n(x)\sum_{s=0}^{\left[\frac{k+1}{2}\right]-1}\frac{(-1)^s}{\lambda_n^{k-1-2s}}f_n^{(k-1-2s)}(0)\sin\left(\frac{\pi k}{2}-\lambda_n t\right). \tag{27}$$

Lemma 6. *If* $\frac{\partial^{k-1} f(x,t)}{\partial t^{k-1}} \in C\left(\overline{\Omega}\right)$, $k \geq 1$, *then the series (24) converges absolutely and uniformly in* $\overline{\Omega}$.

Proof. Consider the series

$$\sum_{n=1}^{\infty} \sum_{s=0}^{\left[\frac{k+1}{2}\right]-1} \frac{\left|f_n^{(k-1-2s)}(0)\right|}{\lambda_n^{k+1-2s}}. \tag{28}$$

For $s = 0$ we have $\sum\limits_{n=1}^{\infty} \frac{\left|f_n^{(k-1)}(0)\right|}{\lambda_n^{k+1}}$. The convergence of this series is obvious. Let $s = \left[\frac{k+1}{2}\right] - 1$. Then

$$(1)\ \ k-1-2s = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases} \qquad (2)\ \ k+1-2s = \begin{cases} 3, & \text{if } k \text{ is even,} \\ 2, & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, the series (28) converges and so the series in (24) converges absolutely and uniformly in $\overline{\Omega}$. □

Lemma 7. *If* $\frac{\partial^{k-2} f(x,t)}{\partial t^{k-2}} \in C\left(\overline{\Omega}\right)$, $k \geq 2$, *then the series (25) converges absolutely and uniformly in* $\overline{\Omega}$.

Proof. Consider the series

$$\sum_{n=1}^{\infty} \sum_{s=0}^{\left[\frac{k}{2}\right]-1} \frac{\left|f_n^{(k-2-2s)}(0)\right|}{\lambda_n^{k-2s}}. \tag{29}$$

If $k = 2$, then $s = 0$, and $\sum\limits_{n=1}^{\infty} \frac{\left|f_n^{(k-2)}(0)\right|}{\lambda_n^{k}}$ converges. It is easy to check that if $s = 0, 1, \ldots, \left(\left[\frac{k}{2}\right] - 1\right)$, then $k - 2 - 2s \geq 0$. Thus, the series in (29) converges. Therefore, the series (25) converges absolutely and uniformly in $\overline{\Omega}$. □

Lemma 8. *Let* $\frac{\partial^{k-2} f(x,t)}{\partial t^{k-2}} \in C\left(\overline{\Omega}\right)$, $k \geq 2$. *If either* k *is odd or* k *is even and*

$$\frac{\partial f(x,t)}{\partial x} \in C\left(\overline{\Omega}\right), \ f(0,t) = f(p,t) = 0, \tag{30}$$

then the series (26) converges absolutely and uniformly in $\overline{\Omega}$.

Proof. The proof is completed by showing that the series

$$\sum_{n=1}^{\infty} \sum_{s=0}^{\left[\frac{k}{2}\right]-1} \frac{\left|f_n^{(k-2-2s)}(0)\right|}{\lambda_n^{k-2-2s}} \tag{31}$$

is the convergent. Indeed, if we let $k \geq 2$ and $s = \left[\frac{k}{2}\right] - 1$, then

$$k-2-2s = \begin{cases} 0, & \text{if } k \text{ is even,} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, the series (31) converges for odd k. Then the series (26) converges absolutely and uniformly in $\overline{\Omega}$. If $k = 2$, then $s = 0$ and the series (31) takes the form

$$\sum_{n=1}^{\infty} |f_n(0)|, \ \ f_n(0) = \int_0^p f(x,0) X_n(x) dx. \tag{32}$$

In general, for any even k, the term of

$$\sum_{s=0}^{\left[\frac{k}{2}\right]-1} \frac{\left|f_n^{k-2-2s}(0)\right|}{\lambda_n^{k-2-2s}}$$

corresponding to $s = \left[\frac{k}{2}\right] - 1$ is $|f_n(0)|$. For $n = 1, 2, \ldots$, these terms in (31) form the series (32). Show that the series (32) converges. Indeed, integrating the last integral, we obtain by virtue of (30) that

$$f_n(0) = \frac{1}{\lambda_n} f_n^{(1,0)}(0), \;\; f_n^{(1,0)} = \int\limits_0^p \frac{\partial f(x,0)}{\partial x} \sqrt{\frac{2}{p}} \cos(\lambda_n x) dx. \tag{33}$$

By using the Bessel inequality

$$\sum_{n=1}^{\infty} \left| f_n^{(1,0)}(0) \right|^2 \leq \left\| \frac{\partial f(x,0)}{\partial x} \right\|^2_{L_2(0,p)},$$

and therefore taking into account (33), we can see that

$$\begin{aligned} \sum_{n=1}^{\infty} |f_n(0)| &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| f_n^{(1,0)}(0) \right| \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left| f_n^{(1,0)}(0) \right|^2 \right)^{\frac{1}{2}} \leq \frac{p}{\sqrt{6}} \left\| \frac{\partial f(x,0)}{\partial x} \right\|_{L_2(0,p)}. \end{aligned}$$

Thus, the series (26) converges absolutely and uniformly in $\overline{\Omega}$ for any even k. □

Lemma 9. *Let* $\frac{\partial^{k-1} f(x,t)}{\partial t^{k-1}} \in C\left(\overline{\Omega}\right)$, $k \geq 1$. *If either* k *is even or* k *is odd and conditions (30) are satisfied, then the series (27) converges absolutely and uniformly in* $\overline{\Omega}$.

Proof. In order to prove this Lemma it is sufficient to prove that the following series

$$\sum_{n=1}^{\infty} \sum_{s=0}^{\left[\frac{k+1}{2}\right]-1} \frac{\left| f_n^{(k-1-2s)}(0) \right|}{\lambda_n^{k-1-2s}}$$

convergent. Indeed, for $s = \left[\frac{k+1}{2}\right] - 1$, we have

$$k - 1 - 2s = \begin{cases} 0, \text{ if } k \text{ is even}, \\ 1, \text{ if } k \text{ is odd}. \end{cases}$$

The rest of the proof runs as the proof of Lemma 8. □

Theorem 2. *Let*

(1) $f(x,t) \in C^1\left(\overline{\Omega}\right)$, $f(0,t) = f(p,t) = 0$,
and $\dfrac{\partial f}{\partial x} \in Lip_\alpha[0,p]$ *uniformly with respect to* t, $0 < \alpha < 1$;
(2) $\dfrac{\partial^k f(x,t)}{\partial t^k} \in C\left(\overline{\Omega}\right)$, $\dfrac{\partial^{k+1} f(x,t)}{\partial t^{k+1}} \in L_2(\Omega)$;
(3) $\varphi_k(x) \in W_2^2(0,p)$, $\varphi_k(0) = \varphi_k(p) = 0$;
(4) $\psi_k(x) \in W_2^1(0,p)$, $\psi_k(0) = \psi_k(p) = 0$.

Then the series (18)–(22) converge absolutely and uniformly in $\overline{\Omega}$, *the solution (18) satisfies the Equation (1), conditions (2)–(5), and* $u(x,t) \in C_{x,t}^{2,k+1}\left(\overline{\Omega}\right)$.

Proof. The fact that the series (18)–(22) converge absolutely and uniformly follows from Lemmas 1–9. Properties of the function $X_n(x)$ imply that (18) satisfies the conditions (2) and (3). Passing to the limit as $t \to 0$ in equalities (21) and (22), we can see that (18) satisfies the conditions (4) and (5). Comparing series (19) and (20), we can see that (18) satisfies Equation (1). The fact that the series (20) and (22) converge uniformly and absolutely in $\overline{\Omega}$ imply that $u(x,t) \in C^{2,k+1}_{x,t}\left(\overline{\Omega}\right)$ and that (18) satisfies the Equation (1). $\square$

4. Conclusions

We have studied and generalized the initial-boundary problem for the inhomogeneous vibrating string equation. The problems studied in the present paper are the first work for hyperbolic equation which contain higher order derivatives of unknown function in initial conditions. This problem generalizes the classic initial-boundary value problems for hyperbolic equation. We have proved the uniqueness and existence of a regular solution of the problem. To prove the main result we have used the spectral decomposition method. In addition, we have explicitly presented the solution in the form of series. We also state that the extension to the multi variables form is an open question.

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