ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2019, Vol. 40, No. 2, pp. 156–165. © Pleiades Publishing, Ltd., 2019.

Mixed Solutions of Monotone Iterative Technique for Hybrid Fractional Differential Equations

Faten H. Damag[1*], Adem Kılıçman[2], and Rabha W. Ibrahim[3***]**

(Submitted by E. K. Lipachev)

[1] *Department of Mathematics, University Taiz, Taiz, Yemen*

[2] *Department of Mathematics and Institute for Mathematical Researchs, University Putra Malaysia, 43400 UPM, Serdang, Selangor Malaysia*

[2] *Istanbul Gelisim University, Avcilar, Turkey*

[3] *Institute of Mathematical Sciences, University Malaya, 50603 Malaysia*

Received July 14, 2018; revised November 16, 2018; accepted November 16, 2018

Abstract—In this present work we concern with mathematical modelling of biological experiments. The fractional hybrid iterative differential equations are suitable for mathematical modelling of biology and also interesting equations since the structure are rich with particular properties. The solution technique is based on the Dhage fixed point theorem that describes the mixed solutions by monotone iterative technique in the nonlinear analysis. In this method we combine two solutions, namely, lower and upper solutions. It is shown an approximate result for the hybrid fractional differential equations in the closed assembly formed by the lower and upper solutions.

DOI: 10.1134/S1995080219020069

Keywords and phrases: *Fractional differential equations, fractional operators, monotone sequences; mixed solutions.*

1. INTRODUCTION

Calculus of fractional order is a field of mathematical analysis (nonlinear part). It follows the traditional definition of derivatives and integrals of calculation operators in the form of fractional order [1–3]. Using the fractional order differential operator in mathematical modeling has become more and more interesting and extended in the last years. Recently, fractional order differential equations have been revisited and become active research area and concentrate on several different studies since having many interesting properties and their occurrence in diverse applications in economics, biology, physics and engineering. Currently, there is a great development in the literature based on the applying nonlinear differential equations of fractional order, see [4].

The class of fractional order differential equations is a generalization of the classical of ordinary differential equations. One can argue that the fractional order differential equations are more appropriate than the ordinary in mathematical modeling of biological, economics and also social systems, see [5–7]. Thus fractional calculus is utilized in biology and medicine to explore the potential of fractional differential equations in order to describe and understand the biological grow of organisms. Moreover, it is also utilized to develop the structure and functional properties of populations. In order to extend this concept we need to evaluate the changes which are associated with the diseases that contribute to the understanding of the pathogenic processes of medicine, see [8]. The researchers have learned how to employ bacteria as well as other microbes to making more mathematical and useful, such as to generate genetically engineered human insulin, see [9].

[*] E-mail: faten_212326@hotmail.com
[**] E-mail: akilic@upm.edu.my
[***] E-mail: rabhaibrahim@yahoo.com

The importance of the differential equations of the hybrid type implies to study a number of dynamical systems which dealt as special cases, [10, 11]. Dhage, Lakshmikantham and Jadhav proved some of the major outcomes for hybrid linear differential equations in the first order and second type disturbances [12–14]. An interesting a mathematical modelling for bacteria growing by the iterative difference equation were also described. Ibrahim [15] established the existence of solution for an iterative fractional differential equation (Cauchy type) by using the technique of nonexpansive operator. Similiar studies are also seen in [16–19].

In this work, we discuss a mathematical model of biological experiments, and how its influence on our lives. The most prominent influence of biological organisms that is affect negative or positive in our lives like a bacteria. Fractional hybrid iterative differential equations are equations that interested in mathematical model of biology. Our technique is based on the Dhage fixed point theorem. This tool describes mixed solutions by monotone iterative technique in the nonlinear analysis. This method is used to combine two solutions: lower and upper. It is shown an approximate result for the hybrid fractional differential equations iterative in the closed assembly formed by the lower and upper solutions.

2. PRELIMINARIES

First of all we need some preliminary results thus recall the following definitions.

Definition 2.1. *The derivative of fractional* (γ) *order for the function* $\phi(s)$, *where* $0 < \gamma < 1$, *is introduced by*

$$D_a^\gamma \phi(s) = \frac{d}{ds}\int_a^s \frac{(s-\beta)^{-\gamma}}{\Gamma(s-\beta)}\phi(\beta)d\beta = \frac{d}{ds}I_a^{1-\gamma}\phi(s), \quad (\kappa-1)\gamma < \kappa,$$

in which κ *is a whole number and* γ *is real number.*

Definition 2.2. *The integral of fractional* (γ) *order for the function* $\phi(s)$, *where* $\gamma > 0$, *is introduced by*

$$I_a^\gamma \phi(s) = \int_a^s \frac{(s-\beta)^{\gamma-1}}{\Gamma(\gamma)}\phi(\beta)d\beta.$$

While $a = 0$, *it becomes* $I_a^\gamma \phi(s) = \phi(s) * \Upsilon_\gamma(s)$, *wherever* $(*)$ *signify the convolution product* $\Upsilon_\gamma(s) = s^{\gamma-1}/\Gamma(\gamma)$ *and* $\Upsilon_\gamma(s) = 0, s \le 0$, $\Upsilon_\gamma \to \delta(s)$ *as* $\gamma \to 0$ *wherever* $\delta(s)$ *is the delta function.*

Further based on the Riemann–Liouville differential operator, we state the following definitions.

Definition 2.3. *Assume the closed period bounded interval* $I = [s_0, s_0 + a]$ *in* $\Re$ ($\Re$ *is the real line), for some* $s_0 \in \Re$, $a \in \Re$. *The problem of initial value problem in fractional iterative hybrid differential equations (FIHDE) which can be formulated as*

$$D^\alpha[v(s) - \psi(s, v(s), v(v(s))] = \aleph(s, v(s), v(v(s))), \quad s \in I, \tag{1}$$

with $v(s_0) = v_0$, *where* $\psi, \aleph : I \times \Re \to \Re$ *are continuous. A solution* $v \in C(I, \Re)$ *of the FIHDE (1) can be problem by*

1. $s \to v - \psi(s, v, v(v)))$ *is a function which is continuous* $\forall v \in \Re$, *and*

2. v *contented the equations in (1), where* $C(I, \Re)$ *is the space of real-valued continuous functions on* I.

The definitions of the lower and upper solutions of (1) as follows.

Definition 2.4. *The function* $\imath \in C(I, \Re)$ *is called a lower solution for the equation introduced on* I *if*

1. $s \mapsto (\imath(s) - \psi(s, \imath(s)), \imath(\imath(s))))$ *is continuous and*

2. $D^\alpha[\imath(s) - \psi(s, v(s), v(v(s)))] \ge \aleph(s, \imath(s), \imath(\imath(s)))$, $s \in I$, $\imath(s_0) \ge v_0$.

Similarly, the function $\tau \in C(I, \mathbb{R})$ *is called an upper solution if*

1. $s \mapsto (\tau(s) - \psi(s, \tau(s), \tau(\tau(s)))$ *is continuous and*

2. $D^\alpha[\tau(s) - \psi(s, v(s), v(v(s)))] \le \aleph(s, \tau(s), \tau(\tau(s)))$, $s \in I$, $\tau(s_0) \le v_0$.

Thus one can build the monotonous sequences of consecutive iterations that converge towards the extremes values among the lower and upper solutions for the differential equation. here we treat the case that if ψ is neither non-decreasing nor non-increasing in the state of the variable v. If the function $\aleph$ can be separated into two components as $\aleph(s, v, v(v))) = \aleph_1(s, v, v(v))) + \aleph_2(s, v, v(v))$, where $\aleph_1(s, v, v(v)))$ is a non-decreasing component while another component is not $\aleph_2(s, v, v(v)))$ increases in the state variables of v, then we may be construct the sequences by iteration which converge to solutions extremal $FIHDE$ (1) on I.

Definition 2.6. *Currently we consider the following initial value problem $FIHDE$*

$$\begin{cases} D^{\alpha}[v(s) - \psi(s, v(s), v(v(s))] = \aleph_1(s, v, v(v))) + \aleph_2(s, v, v(v))), & s \in I, \\ v(s_0) = v_0, \end{cases} \tag{2}$$

where $\psi \in C(I \times R, R)$ and $\aleph_1, \aleph_2 \in \mathfrak{L}(I \times R, R)$.

Thus the lower and upper solutions of (2) can be as defined as follows:

Definition 2.7. *The functions $\sigma, \rho \in C(I, \Re)$ fulfill the following condition: the maps $s \to \sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))$ and $s \to \rho(s) - \psi(s, \rho(s), \rho(\rho(s)))$ are absolute continuous on I. Thus the functions (σ, ρ) are supposed to be of the kind*

(a) which is mixed lower solutions and upper solutions for (2) on I, sa following

$$\begin{cases} D^{\alpha}[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))] \le \aleph_1(s, \sigma, \sigma(\sigma(s))) + \aleph_2(s, \rho(s), \rho(\rho(s)))), & s \in I, \\ \sigma(s_0) \le v_0 \end{cases} \tag{3}$$

and

$$\begin{cases} D^{\alpha}[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] \ge \aleph_1(s, \rho, \rho(\rho(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s))), & s \in I, \\ \rho(s_0) \ge v_0. \end{cases} \tag{4}$$

Whether the sign was of equality achieves in relationships (3) and (4), hence the even of functions (σ, ρ) set is been calling a mixed solution of kind (a) for the $FIHDE$ (2) on I.

(b) which is mixed lower solutions and upper for (2) on I, as follows

$$\begin{cases} D^{\alpha}[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))] \le \aleph_1(s, \rho, \rho(\rho(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s)))), & s \in I, \\ \sigma(s_0) \le v_0 \end{cases} \tag{5}$$

and

$$\begin{cases} D^{\alpha}[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] \ge \aleph_1(s, \sigma, \sigma(\sigma(s))) + \aleph_2(s, \rho(s), \rho(\rho(s))), & s \in I, \\ \rho(s_0) \ge v_0. \end{cases} \tag{6}$$

Whether the sign was of equality achieves in relationships (5) and (6), hence the even of functions (σ, ρ) set is been calling a mixed solution of kind (b) for the (2) on I.

2.1. Assumptions

In the next we consider the function ψ that is important in the studying of Eq. (2).

(a0) The function $v \mapsto (v - \psi(s_0, v, v(v)))$ is injective in $\Re$.

(b0) $\aleph$ is a bounded real-valued function on $I \times \Re$.

(a1) The function $v \mapsto (v - \psi(s, v, v(v)))$ is increasing in $\Re$ for all $s \in I$.

(a2) There is a constant $\ell > 0$ so that

$$|\psi(s, v, v(v)) - \psi(s, z, z(z))| \le \frac{\ell |v - z|}{M + |v - z|}, \quad M > 0, \quad \forall s \in I, \quad v, z \in \Re \quad \text{and} \quad \ell \le M.$$

(b1) There is a constant $\kappa > 0$ so that $|\aleph(s, v, v(v)| \le \kappa \, \forall s \in I$ and $\forall v \in \Re$.

(b2) $\aleph_1(s, v, v(v)))$ is function which is non-decreasing in v function, and $\aleph_2(s, v, v(v))$ is function which is not increasing in v for each $s \in I$.

(b3) (σ_0, ρ_0) is functions which are mixing the lower and upper solutions for (2) kind(a) on I with $\sigma_0 \le \rho_0$.

(b4) The pair is (σ_0, ρ_0), the upper and lower mixing solutions for (2) kinds (b) on I with $\sigma_0 \le \rho_0$.

3. MAIN RESULTS

In the next, we discuss the approximate outcome for (2).

Lemma 3.1 [11]. *Suppose the assumptions* $(a0)-(b0)$ *are achieved. Then the function* v *is a solution for Eq.(1) if and only if the solution of the fractional iterative hybrid type equation satisfies*

$$v(t) = [v_0 - \psi(s_0, v_0, v(v_0))] + \psi(s, v(s), v(v(s))) + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta$$

$$(s \in I, \quad v(0) = v_0). \tag{7}$$

Theorem 3.1 [20]. *Let* ϱ *be a closed convex and bounded subset of the Banach space* A. *Moreover, let* $Q : A \to A$ and $P : \varrho \to A$ *be two operators so that*

(i) Q *is nonlinear D-contraction,*

(ii) P *is compact and continuous,*

(iii) $v = Qv + Pz$ *for all* $v \in \varrho \Rightarrow z \in \varrho$.

Theorem 3.2. *Let the assumptions* $(a1), (a2)$ *and* $(b1)$ *be hold. Then (1) has a solution on* I.

Proof. Let $A = C(I, R)$ be a set and ç $\subseteq A$, such that $\varrho = \{v \in A| ||A|| \le M\}$, where

$$M = |v_0 - \psi(s_0, v_0, v(v(0))| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha+1)} ||\xi||_{\ell^1}$$

and $\Psi_0 = \sup_{s \in I} |\psi(s, 0, 0)|$. Obviously ϱ is a convex, bounded and closed subset of the space A. By using the assumptions (a1) and (b1) together with the help of the Lemma 3.1, we conclude that the $FIHDE$ (1) is tantamount to the nonlinear $FIHIE$ (7). We define two operators $Q : A \to A$ and $P : \varrho \to A$ as follows: $Qy(s) = \psi(s, v(s), v(v(s))), s \in I$, and

$$Pv(s) = [v_0 - \psi(s_0, v_0, v(v_0))] + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta, \quad s \in I.$$

Consequently, the $FIHIE$ (7) is equivalent to the operator equation $Qv(s) + Pv(s) = v(s)$, $s \in I$. We demonstrate that the operators Q and P fulfill all the conditions of Theorem 3.1. Foremost, we examine that Q is a nonlinear Υ-contraction on Q with a Υ function φ. Let $v, z \in A$. In view of assumption $(a2)$, we conclude that

$$|Qv(s) - Qz(s)| = |\psi(s, v(s)) - \psi(s, z(s))| \le \frac{\ell |v(s) - z(s)|}{M + |v(s) - z(s)|} \le \frac{\ell |v - z|}{M + |v - z|}$$

for all $s \in I$. Take the supremum over s yields

$$||Av - Az|| \le \frac{\ell |v - z|}{M + |v - z|}$$

$\forall v, z \in A$. This proves that Q is a nonlinear D-contraction A with the D-function φ defined by $\varphi(r) = \ell r/(M + r)$.

Next, we examine that P is a continuous and compact operator on ϱ into A. Let $\{v_t\}$ be a sequence in ϱ converging to a point $v \in \varrho$, thus we have

$$\lim_{t\to\infty} Pv_t(s) = \lim_{t\to\infty} \left[v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \aleph(\beta, v_t(\beta), v_t(v_t(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \right]$$

$$= v_0 - \psi(s_0, y_0, v(v_0)) + \lim_{t\to\infty} \int_0^s \aleph(\beta, v_t(\beta), v_t(v_t(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta$$

$$= v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \lim_{t\to\infty} \left[\aleph(\beta, v_t(\beta), v_t(v_t(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} \right] d\beta$$

$$= v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta = Pv(s)$$

for all $s \in I$. Now, we proceed to prove that $\{Pv_t\}$ is equi-continuous with respect to v. According to [21], we attain that P is a continuous operator on ϱ. To show that P is a compact operator on ϱ. It suffices to examine that ϱ is a regularly bounded and equi-continuous set in A. Let $v \in \varrho$ be arbitrary, then by the assumption (b1), we have

$$|Pv(s)| \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \int_0^s \left| \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} \right| d\beta$$

$$\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \int_0^s \xi(\beta) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \frac{a^\alpha}{\Gamma(\alpha+1)} ||\xi||_{\ell^1}$$

for all $s \in I$. By taking the supremum over t, we obtain

$$|Pv(s)| \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \frac{a^\alpha}{\Gamma(\alpha+1)} ||\xi||_{\ell^1}$$

$\forall v \in \varrho$. This proves that P is uniformly bounded on ϱ.

Also let $s_1, s_2 \in I$ with $s_1 < s_2$. Then for any $v \in \varrho$, one has

$$|Pv(s_1) - Pv(s_2)|$$

$$= \left| \int_{s_0}^{s_1} \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s_1-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta - \int_{s_0}^{s_2} \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \right|$$

$$\leq \left| \int_{s_0}^{s_1} \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s_1-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta - \int_{s_0}^{s_1} \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \right|$$

$$+ \left| \int_{s_o}^{s_1} \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta - \int_{s_0}^{s_2} \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s_2-\beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \right|$$

$$\leq \frac{||\xi||_{\ell^1}}{\Gamma(\alpha+1)} [|(s_2 - s_2)^\alpha - (s_1 - s_0)^\alpha - (s_2 - s_1)^\alpha| + (s_2 - s_1)^\alpha].$$

Hence, for $\delta > 0$, there exists a $\epsilon > 0$ so that $|s_1 - s_2| < \epsilon \Rightarrow |Pv(s_1) - Pv(s_2)| < \delta \ \forall s_1, s_2 \in I$ and $\forall v \in \varrho$. This examines for $P(\varrho)$ is equi-continuous in A. presently $P(\varrho)$ is bounded and hence it is compact by Arzela–Ascoli Theorem. Resulting, ϱ is a continuous and compact operator on ϱ. Then, we prove that assumptions (iii) of Theorem 3.1 is fulfilled. Let $v \in A$ be fixed and $z \in \varrho$ be arbitrary such that $v = Qv + Pz$. In view of the assumption (a2) yields

$$|v(s)| \leq |Qv(s)| + |Pz(s)|$$

$$\leq |v_0 - \psi(s_0, v_0)| + |\psi(s, v(s), v(v(s))| + \int_0^s \left| \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} \right| d\beta$$

$$\leq |v_0 - \psi(s_0, v_0)| + |\psi(s, v(s), v(v(s))| + \int_0^s \left| \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} \right| d\beta$$

$$\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \ell + \Psi_0 + \int_0^s \left| \xi(\beta) \frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)} \right| d\beta$$

$$\leq |v_0 - \psi(s_0, v_0, v(v_0))| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha+1)} ||\xi||_{\ell^1}.$$

Take the supremum over s, implies

$$||v|| \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha+1)} ||\xi||_{\ell^1} = M.$$

Thus, $v \in \varrho$. Therefore, fulfilled all conditions of the Theorem 3.1 and thus the operator equation $v = Qv + Pz$ has a solution in ϱ. Resulting, the $FIHDE$ (1) has a solution introduced on I. This completes the proof.

Theorem 3.3. *Let $\imath, \tau \in C(I, \Re)$ be lower and upper solutions of $FIHDE$ (1) fulfilling $\imath(s) \leq \tau(s), s \in I$ and further if the assumptions $(a1) - (a2)$ and $(b1)$ are held. Then, there is a solution $v(s)$ of (1), in the closed set $\overline{\mho}$, satisfying $\imath(s) \leq v(s) \leq \tau(s)$, for $s \in I$.*

Proof. Assume that $\Theta : I \times \Re \mapsto \Re$ is a function defined by $\Theta(s, v, v(v)) = \max\{\imath(s)\ , \min v(s), \tau(s)\}$, satisfying $\check{\aleph}(s, v, v(v))) := \aleph(s, \Theta(s, v, v(v))))$. Moreover, define a continuous extension of $\aleph$ on $I \times \Re$ such that

$$|\check{\aleph}(s, v, v(v)))| = |\aleph(u, \Theta(s, v, v(v))))| \leq \kappa, \quad s \in I \quad \forall v \in \Re.$$

In view of Theorem 3.2, the $FIHDE$

$$\begin{cases} D^\alpha[v(s) - \psi(s, v(s), v(v(s))] = \check{\aleph}(s, v, v(v))), & s \in I, \\ v(u_0) = v_0 \in \Re \end{cases}$$

has a solution v defined on I. For any $\delta > 0$, define

$$\imath_\delta(s)\psi(s, \imath_\delta(\imath_\delta(s))) = (\imath(s) - \psi(s, \imath(s), \imath(\imath(s)))\delta(1+s)$$

and

$$\tau_\delta(s)\psi(s, \tau_\delta(\tau_\delta(s))) = (\tau(s) - \psi(s, \tau(s), \tau(\tau(s)))\delta(1+s)$$

for $s \in I$. In virtue of the assumptions $(a1)$, we get $\imath_\delta(s) < \imath(s)$ and $\tau(s) < \tau_\delta(s)$ for $s \in I$. Since $\imath(s_0) \leq v_0 \leq \tau(s_0)$, one has $\imath_\delta(s_0) < v_0 < \tau_\delta(s_0)$. To show that

$$\imath_\delta(s) < v_0 < \tau_\delta(s), \quad s \in I, \tag{8}$$

we define $v(s) = v(s) - \psi(s, v(s), v(v(s))$, $s \in I$. Likewise, we consider

$$\hbar_\delta(s) = \imath_\delta(s) - \psi(s, \imath_\delta(\imath_\delta(s))), \quad \hbar(s) = \imath(u) - \psi(s, \imath(s), \imath(\imath(s))$$

and

$$T_\delta(s) = \tau_\delta(s)\psi(s, \tau_\delta(s), \tau(\tau_\delta(s)), \quad T(s) = \tau(s)\psi(s, \tau(s), \tau(\tau(s))$$

$\forall s \in I$. If Eq. (8) is wrong, then there exists a $s_\varepsilon \in (s_0, s_0 + a]$ such that $v(_\varepsilon) = \tau_\delta(s_\varepsilon)$ and $\imath_\delta(s) < v(s) < \tau_\delta(s), s_0 \leq s < s_\varepsilon$. If $v(s_\varepsilon) > \tau(s_\varepsilon)$, then $\Theta(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) = \tau(s_\varepsilon)$. Furthermore, $\imath(s_\varepsilon) \leq \Theta(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) \leq \tau(s_\varepsilon)$. Now,

$$D^\alpha T(s_\varepsilon) \geq \aleph(s_\varepsilon, \tau(s_\varepsilon), \tau(\tau(s_\varepsilon))) = \check{\aleph}(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon)))) = D^\alpha V(s)$$

$\forall s \in I$. Since $T_\delta(us) > D^\alpha T(s)$, $\forall s \in I$, we have

$$D^\alpha T_\delta(s_\varepsilon) > D^\alpha V(s_\varepsilon). \tag{9}$$

But, $V(s_\varepsilon) = T_\delta(s_\varepsilon)$ also $V(s) = T_\delta(s), s_0 \leq s < s_\varepsilon$, means that together

$$\frac{V(s_\varepsilon + \rho) - V(s_\varepsilon)}{\rho^\alpha} > \frac{T_\delta(s_\varepsilon + \rho) - T_\delta(s_\varepsilon)}{\rho^\alpha},$$

if $\rho < 0$ a small. Take the limit $\rho \to 0$ in the up variance yields $D^\alpha V(s_\varepsilon) \geq D^\alpha T_\delta(s_\varepsilon)$ that is a contradiction to (9). Hence, $v(s) < \tau_\delta(s)$ $\forall s \in I$. Consequently $\imath_\delta(s) < v(s) < \tau_\delta(s)$, $s \in I$. Letting $\delta \to 0$ in the up inequality, we get $\imath(s) \leq v(s) \leq \tau(s)$, $s \in I$. This completes the proof. □

Theorem 3.4 *Let assumptions $(a1) - (a2)$ and $(b2) - (b3)$ are held. Then there are the monotonous sequences $\{\sigma_t\}, \{\rho_t\}$ such that $\sigma_t \to \sigma$ and $\rho_t \to \rho$ uniformly on I in which (σ, ρ) are mixed extremal solutions $FIHDE$ (2) type(a) on I.*

Proof. Note the following a quadratic $FIHDE$

$$\begin{cases} D^\alpha[\sigma_{t+1}(s) - \psi(s, \sigma_{t+1}(s), \sigma(\sigma_{t+1}(s))] \leq \aleph_1(s, \sigma_t(s), \sigma(\sigma_t(s))) + \aleph_2(s, \rho_t(s), \rho(\rho_t(s)))), \\ s \in I, \\ \sigma_{t+1}(s_0) \leq v_0 \end{cases} \tag{10}$$

and

$$\begin{cases} D^\alpha[\rho_{t+1}(s) - \psi(s, \rho_{t+1}(s), \rho(\rho_{t+1}(s))] \geq \aleph_1(s, \rho_t(s), \rho(\rho_t(s))) + \aleph_2(s, \sigma_t(s), \sigma(\sigma_t(s))), \\ s \in I, \\ \rho_{t+1}(s_0) \geq v_0 \end{cases} \tag{11}$$

for $t \in N$. Obviously, the equations (10) and (11) having unique solutions σ_{t+1} and ρ_{t+1} on I respectively given Banach contraction mapping principle. We now want to demonstrate that

$$\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_t \leq \rho_t \leq \ldots \leq \rho_2 \leq \rho_1 \leq \rho_0$$

on I for $t = 0, 1, 2, \ldots$ Let $t = 0$ and set

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s))) -)))\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))$$

for $s \in I$. Next by monotonicity of $\aleph_1$ and $\aleph_2$, we get

$$\begin{aligned} D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^\alpha[(\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s)))] \\ - D^\alpha[\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] &\leq \aleph_1(s_0, \sigma_0(s), \sigma(\sigma_0(s))) + \aleph_2(s, \rho_0(s), \rho(\rho_0(s))) \\ &- \aleph_1(s_0, \rho_0(s), \rho(\rho_0(s))) + \aleph_2(s, \sigma_0(s), \sigma(\sigma_0(s))) = 0 \end{aligned}$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This implies that $\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s))) \leq \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))$ $\forall s \in I$. In view of (a1), one can get $\sigma_0(s) \leq \sigma_1(s)$, $\forall s \in I$. Likewise it can be demonstrated which $\rho_1(s) \leq \rho_0(s)$ on I. Setting

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))) - (\rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s))))$$

$\forall s \in I$. By monotonicity of $\aleph_1$ and $\aleph_2$, we obtain

$$\begin{aligned} D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^\alpha[\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] \\ - D^\alpha[(\rho_1(s)\psi(s, \rho_1(s), \rho(\rho_1(s))))] &\leq \aleph_1(s_0, \sigma_0(s), \sigma(\sigma_0(s))) + \aleph_2(s, \rho_0(s), \rho(\rho_0(s))) \\ &- \aleph_1(s_0, \rho_0(s), \rho(\rho_0(s))) + \aleph_2(s, \sigma_0(s), \sigma(\sigma_0(s))) \leq 0 \end{aligned}$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This leads to $\sigma_1(s)\psi(s, \sigma_1(s), \sigma(\sigma_1(s))) \leq \rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s)))$ $\forall s \in I$. By (a1), we attain to $\sigma_1(s) \leq \rho_1(s)$, $\forall s \in I$. Next, for $\jmath \in N$, yields $\sigma_{\jmath+1} \leq \sigma_\jmath \leq \rho_\jmath \leq \rho_{\jmath-1}$ and hence $\sigma_\jmath \leq \sigma_{\jmath+1} \leq \rho_{\jmath+1} \leq \rho_\jmath$. Setting

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_\jmath(s) - \psi(s, \sigma_\jmath(s), \sigma(\sigma_\jmath(s)))) - (\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s)))).$$

Then the humdrum of $\aleph_1$ and $\aleph_2$, we receive

$$\begin{aligned} D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^\alpha[(\sigma_\jmath(s) - \psi(s, \sigma_\jmath(s), \sigma(\sigma_\jmath(s))))] \\ - D^\alpha[(\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s))))] &\leq \aleph_1(s, \sigma_{\jmath-1}, \sigma(\sigma_{\jmath-1}(s)) \\ &+ \aleph_2(s, \rho_{\jmath-1}, \rho(\rho_{\jmath-1})) - \aleph_1(s, \sigma_\jmath, \sigma(\sigma_\jmath)) - \aleph_2(s, \rho_\jmath, \rho(\rho_\jmath)) \leq 0 \end{aligned}$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This implies that $\sigma_\jmath - \psi(s, \sigma_\jmath(s), \sigma(\sigma_\jmath(s))) \leq \sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s)))$ for every $s \in I$. Since assumption (a1) achieved, we have $\sigma_\jmath(s) \leq \sigma_{\jmath+1}(s)$, $\forall s \in I$. Likewise it can be demonstrated which $\rho_{\jmath+1}(s) \leq \rho_\jmath(s)$ on I . The same way it is assumed that the inequality

$\sigma_{\jmath-1} \leq \sigma_\jmath \leq \rho_\jmath \leq \rho_{\jmath-1}$ achieves on I. We are going to demonstrate that $\sigma_\jmath \leq \sigma_{\jmath+1} \leq \rho_{\jmath+1} \leq \rho_\jmath$ on I. Set

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s)))) - (\rho_{\jmath+1}(s) - \psi(s, \rho_{\jmath+1}, \rho(\rho_{\jmath+1})))$$

for $s \in I$. So by monotonicity of $\aleph_1$ and $\aleph_2$ we get

$$D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] = D^\alpha[(\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s))))]$$
$$- D^\alpha[(\rho_{\jmath+1}(s) - \psi(s, \rho_{\jmath+1}, \rho(\rho_{\jmath+1})))] \leq \aleph_1(s, \sigma_\jmath(s), \sigma(\sigma_\jmath(s))) + \aleph_2(s, \rho_\jmath(s), \rho(\rho_\jmath(s)))$$
$$- \aleph_1(s, \rho_{\jmath+1}, \rho(\rho_{\jmath+1})) - \aleph_2(s, \sigma_\jmath(s), \sigma(\sigma_\jmath(s))) \leq 0$$

for the whole $s \in I$ and $\Theta(s_0) = 0$. This means that

$$\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s)))) \leq \rho_{\jmath+1} - \psi(s, \rho_{\jmath+1}, \rho(\rho_{\jmath+1}))$$

for every $s \in I$. Since assumption (a1) is achieved, we have $\sigma_{\jmath+1}(s) \leq \rho_{\jmath+1}(s)$, $\forall s \in I$.

Presently it is readily shown that the sequence $\{\sigma\}$ and $\{\rho\}$ are bounded uniformly and equi-continuous sequences and have therefore converge uniformly on I. As are monotonous sequences, $\{\sigma_t\}$ and $\{\rho_t\}$ converse uniformly monotonous σ and ρ on I respectively. Course, the pair (σ, ρ) is a mixed solution of these equations (2) on I. Lastly, we establish which (σ, ρ) is a mixed solution of minimum and maximum for the equations (2) on I. Let v whatever solution of the equations (2) on I as $\sigma_0(s) \leq v(s) \leq \rho(s)$ on$I$. Assume that for $\jmath \in N$, $\sigma_\jmath(s) \leq v(s) \leq \rho_\jmath(s)$, $s \in I$. We will demonstrate which $\sigma_{\jmath+1}(s) \leq v(s) \leq \rho_{\jmath+1}(s)$, $s \in I$. Adjustment

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s)))) - (v(s) - \psi(s, v(s), v(v(s))))$$

for every $s \in I$. After, for the monotony of $\aleph_1$ and $\aleph_2$ we get

$$D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] = D^\alpha[(\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s))))]$$
$$- D^\alpha[(v(s) - \psi(s, v(s), v(v(s))))] \leq \aleph_1(s, \sigma_\jmath(s), \sigma(\sigma_\jmath(s))) + \aleph_2(s, \rho_\jmath(s), \rho(\rho_\jmath(s)))$$
$$- \aleph_1(s, v(s), v(v(s))) - \aleph_2(s, v(s), v(v(s))) \leq 0$$

for the whole $s \in I$ and $\Theta(s_0) = 0$. This yields

$$\sigma_{\jmath+1}(s) - \psi(s, \sigma_{\jmath+1}(s), \sigma(\sigma_{\jmath+1}(s))) \leq v(s) - \psi(s, v(s), v(v(s)))$$

for every $s \in I$. Since assumption (a1) is valid, we get $\sigma_{\jmath+1}(s) \leq v(s)$, $\forall s \in I$. Likewise it can be demonstrated which $v(s) \leq \rho_{\jmath+1}(s)$ on I. In principle, the method of induction, $\sigma_t \leq v \leq \rho_t$ for every $s \in I$. By taking $t \to \infty$ limit, we get $\sigma \leq v \leq \rho$ on I. So (σ, ρ) they are mixed type (a) extreme solutions for the equations (2) on I., i.e,

$$\begin{cases} D^\alpha[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))] \leq \aleph_1(s, \sigma(s), \sigma(\sigma(s))) + \aleph_1(s, \rho(s), \rho(\rho(s)))), s \in I, \\ \sigma(s_0) = v_0 \end{cases}$$

and

$$\begin{cases} D^\alpha[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] \geq \aleph_1(s, \rho(s), \rho(\rho(s))) + \aleph_1(s, \sigma(s), \sigma(\sigma(s))), s \in I, \\ \rho(s_0) = v_0. \end{cases}$$

The proof is completed. □

Corollary 3.1. *Suppose the hypothesis of Theorem 3.4 are fulfilled. Assume that for* $\imath_1 \geq \imath_2$, $\imath_1, \imath_2 \in \overline{\mho}$, *then*

$$\aleph_1(s, \imath_1(s), \imath(\imath_1(s))) - \aleph_1(s, \imath_2(s), \imath(\imath_2(s)))$$
$$\leq N_1[\imath_1(s) - \psi(s, \imath_1(s), \imath(\imath_1(s))) - (\imath_2(s) - \psi(s, \imath_2(s), \imath(\imath_2(s))],$$

$N_1 > 0$, *and*

$$\aleph_2(s, \imath_1(s), \imath(\imath_1(s))) - \aleph_2(s, \imath_2(s), \imath(\imath_2(s)))$$
$$\leq N_2[\imath_1(s) - \psi(s, \imath_1(s), \imath(\imath_1(s))) - (\imath_2(s) - \psi(s, \imath_2(s), \imath(\imath_2(s))),$$

$N_2 > 0$, *thus* $\sigma(s) = v(s) = \rho(s)$ *on* I.

Proof. For $\sigma \le \rho$ on I, it suffices to demonstrate that $\rho \le \sigma$ on I. Introduce a function $\Theta \in C(I, \Re)$

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\rho(s) - \psi(s, \rho(s), \rho(\rho(s)))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))).$$

Next, $\Theta(s_0) = 0$ and

$$\begin{aligned} &D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] \\ &= D^\alpha[(\rho(s) - \psi(s, \rho(s), \rho(\rho(s))))] - D^\alpha[(\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))))] \\ &= \aleph_1(s, \rho(s), \rho(\rho(s))) - \aleph_1(s, \sigma(s), \sigma(\sigma(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s))) - \aleph_2(s, \rho(s), \rho(\rho(s))) \\ &\le N_1[(\rho(s) - \psi(s, \rho(s), \rho(\rho(s))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))))] \\ &+ N_2[(\sigma(s) - \sigma(s, \sigma(s), \sigma(\sigma(s)))) - (\rho(s) - \psi(s, \rho(s), \rho(\rho(s))))] \\ &= (N_1 + N_2)[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))]. \end{aligned}$$

This demonstrates that $\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) \le 0$ on I, demonstrating that $\rho \le \sigma$ on I. Therefore $\sigma = \rho = v\ I$, the proof is completed. □

Theorem 3.5. *Let us suppose that the assumption* (a1)−(a2) *and* (b2)−(b4) *achieved. Therefore, for any solution $v(s)$ of (2) with $\sigma_0 \le v \le \rho_0$, and we are an iteration σ_t, ρ_t satisfactory for $s \in I$,*

$$\begin{cases} \sigma_0 \le \sigma_2 \le \ldots \le \sigma_{2t} \le v \le \sigma_{2t+1} \le \ldots \le \sigma_3 \le \sigma_1, \\ \rho_1 \le \rho_3 \le \ldots \le \rho_{2t+1} \le v \le \rho_{2t} \le \ldots \le \rho_2 \le \rho_0, \end{cases}$$

as long as $\sigma_0 \le \sigma_2$ and $\rho_2 \le \rho_0$ on I, in which iterating is given by

$$\begin{cases} D^\alpha[\sigma_{2t+1}(s) - \psi(s, \sigma_{2t+1}(s), \sigma(\sigma_{2t+1}(s))] = \aleph_1(s, \rho_t(s), \rho(\rho_t(s))) + \aleph_2(s, \sigma_t(s), \sigma(\sigma_t(s))), \\ s \in I, \\ \sigma_{2t+1}(s_0) = v_0 \end{cases}$$

and

$$\begin{cases} D^\alpha[\rho_{2t+1}(s) - \psi(s, \rho_{2t+1}(s), \rho(\rho_{2t+1}(s))] = \aleph_1(s, \sigma_t(s), \sigma(\sigma_t(s))) + \aleph_2(s, \rho_t(s), \rho\rho_t(s))), \\ s \in I, \\ \rho_{2t+1}(s_0) = v_0 \end{cases}$$

of $t \in N$. Furthermore, the monotonous sequences $\{\sigma_{2t}\}$, $\{\sigma_{2t+1}\}$, $\{\rho_{2t}\}$, $\{\rho_{2t+1}\}$ converge uniformly to σ, ρ, $\sigma^\diamond$, $\rho^\diamond$, respectively, and fulfilling this assumptions:

$$(1) \quad D^\alpha[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))] = \aleph_1(s, \rho(s), \rho(\rho(s))) + \aleph_2(s, \sigma(s), \sigma(\sigma(s)));$$

$$(2) \quad D^\alpha[\rho(s) - \psi(s, \rho(s), \rho(\rho(s))] = \aleph_1(s, \sigma_t(s), \sigma(\sigma(s))) + \aleph_2(s, \rho(s), \rho\rho(s)));$$

$$(3) \quad D^\alpha[\sigma^\diamond(s) - \psi(s, \sigma^\diamond(s), \sigma(\sigma^\diamond(s)))] = \aleph_1(s, \rho^\diamond(s), \rho(\rho^\diamond(s))) + \aleph_2(s, \sigma^\diamond(s), \sigma(\sigma^\diamond(s)));$$

$$(4) \quad D^\alpha[\rho^\diamond(s) - \psi(s, \rho^\diamond(s), \rho(\rho^\diamond(s))] = \aleph_1(s, \sigma^\diamond(s), \sigma(\sigma^\diamond(s))) + \aleph_2(s, \rho^\diamond(s), \rho\rho^\diamond(s)))$$

for $s \in I$ and $\sigma \le v \le \rho, \sigma^\diamond \le v \le \rho^\diamond, s \in I, \sigma(0) = \sigma(0) = \sigma^\diamond(0) = \rho^\diamond(0) = v_0$.

Proof. By the assumptions of the theorem, we suppose that $\sigma_0 \le \sigma_2$ and $\rho_2 \le \rho_0$, on I. We demonstrate that

$$\begin{cases} \sigma_0 \le \sigma_2 \le v \le \sigma_3 \le \sigma_1, \\ \rho_1 \le \rho_3 \le v \le \rho_2 \le \rho_0 \end{cases} \tag{12}$$

on I. Set

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (v(s) - \psi(s, v(s), v(v(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))$$

utilization that $\sigma_0 \le v \le \rho_0$ on I, as v is any solution of (2) and the monotonous the nature of functions $\aleph_1$ and $\aleph_2$, this yields

$$\begin{aligned} D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] &= D^\alpha[(v(s) - \psi(s, v(s), v(v(s))))] \\ - D^\alpha[(\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] &= \aleph_1(s, v(s), v(v(s)))) \end{aligned}$$

$$+ \aleph_2(s, v(s), v(v(s)))) - \aleph_1(s, \rho_0(s), \rho(\rho_0(s))) - \aleph_2(s, \sigma_0(s), \sigma(\sigma_0(s)))) \leq 0$$

for every $s \in I$ and $\Theta(s_0) = 0$. Thus, we reached the conclusion

$$v(s) - \psi(s, v(s), v(v(s))) \leq \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))$$

or $v(s) \leq \sigma_1(s)$ for every $s \in I$. In the same way, we can show that $\sigma_3 \leq \sigma_1, \rho_1 \leq v$ and $\sigma_2 \leq v$, taking into account differences

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_3(s) - \psi(s, \sigma_3(s), \sigma(\sigma_3(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))),$$

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s)))) - (v(s) - \psi(s, v(s), v(v(s))))$$

and

$$\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_2(s) - \psi(s, \sigma_2(s), \sigma(\sigma_2(s)))) - (v(s) - \psi(s, v(s), v(v(s))))$$

respectively. At each of these cases, we get $\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)) \leq 0$, for all $s \in I$ and representation (12) is established. This completed prove.

Competing interests. The authors declare that they have no competing interests.

Authors′ contributions. All the authors jointly worked together on deriving the results and approved the final case of manuscript.

REFERENCES

1. A. Loverro, "Fractional calculus: history, definitions and applications for the engineer. Rapport technique," Report (Dep. Aerospace Mech. Eng., Univ. Notre Dame, Notre Dame, 2004).
2. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of Their Applications* (Academic, New York, 1998), Vol. 198.
3. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations* (Wiley-Interscience, New York, 1993).
4. V. V. Kulish and J. L. Lage, "Application of fractional calculus to fluid mechanics," J. Fluids Eng. **124**, 803–806 (2002).
5. S. Havlin, S. V. Buldyrev, A. L. Goldberger, R. N. Mantegna, S. M. Ossadnik, C. K. Peng, et al., "Fractals in biology and medicine," Chaos, Solitons Fractals **6**, 171–201 (1995).
6. A. Kılıçman and F. H. M. Damag, "Solution of the fractional iterative integro-differential equations," Malays. J. Math. Sci. **12**, 121–141 (2018).
7. F. H. Damag and A. Kılıçman, "Sufficient conditions on existence of solution for nonlinear fractional iterative integral equation," J. Nonlin. Sci. Appl. **10**, 368–376 (2017).
8. T. F. Nonnenmacher, G. A. Losa, and E. R. Weibel, *Fractals in Biology and Medicine* (Birkhäuser, Boston, 2013).
9. H. L. Smith, Bacterial Growth, 09-15 (2007).
10. B. C. Dhage, "Basic results in the theory of hybrid differential equations with linear perturbations os second type," Tamkang J. Math. **44**, 171–186 (2012).
11. H. Lu, S. Sun, D. Yang, and H. Teng, "Theory of fractional hybrid differential equations with linear perturbations of second type," Boundary Value Probl., No. 1, 1–16 (2013).
12. B. C. Dhage, "Approximation methods in the theory of hybrid differential equations with linear perturbations of second type," Tamkang J. Math. **45**, 39–61 (2014).
13. B. C. Dhage and N. S. Jadhav, "Basic results in the theory of hybrid differential equations with linear perturbations of second type," Tamkang J. Math. **44**, 171–186 (2013).
14. B. C. Dhage and V. Lakshmikantham, "Basic results on hybrid differential equations," Nonlin. Anal.: Hybrid Syst. **4**, 414–424 (2010).
15. R. W. Ibrahim, "Existence of deviating fractional differential equation," CUBO A Math. J. **14**, 127–140 (2012).
16. R. W. Ibrahim, A. Kılıçman, and F. H. Damag, "Existence and uniqueness for a class of iterative fractional differential equations," Adv. Differ. Equat., No. 1, 1–13 (2015).
17. F. H. Damag, A. Kılıçman, and R. W. Ibrahim, "Findings of fractional iterative differential equations involving first order derivative," Int. J. Appl. Comput. Math., 1–10 (2016).
18. F. H. Damag, A. Kılıçman, and R. W. Ibrahim, "Approximate solutions for non-linear iterative fractional differential equations," AIP Conf. Proc. **1739**, 020015 (2016). https://doi.org/10.1063/1.4952495
19. R. W. Ibrahim, A. Kiliçman, and F. H. Damag, "Extremal solutions by monotone iterative technique for hybrid fractional differential equations," Turk. J. Anal. Number Theory **4** (3), 60–66 (2016).
20. B. Dhage and D. O'Regan, "A fixed point theorem in Banach algebras with applications to functional integral equations," Funct. Differ. Equat. **7**, 259 (2004).
21. A. Granas, R. B. Guenther, and J. W. Lee, "Some general existence principles for Caratheodory theory of nonlinear differential equations," J. Math. Pures Appl. **70**, 153–196 (1991).