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Stability, Synchronization Control and Numerical Solution of Fractional Shimizu–Morioka Dynamical System

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Abstract: In this paper we concern with asymptotic stability, synchronization control and numerical solution of incommensurate order fractional Shimizu–Morioka dynamical system. Firstly we prove the existence and uniqueness of the solutions via a new theorem. After finding steady–state points, we obtain necessary and sufficient conditions for the asymptotic stability of the Shimizu–Morioka system. We also study the synchronization control where we employ master–slave synchronization scheme. Finally, employing Adams–Bashforth–Moulton's technique we solve the Shimizu–Morioka system numerically. To the best of our knowledge, there exist not any study about analysis of chaotic dynamics of fractional Shimizu–Morioka system in the literature. In this sense the present paper is going to be a totally new contribution and highly useful research for synthesis of a nonlinear system of fractional equations.

Keywords: Fractional Shimizu–Morioka equation, stability, synchronization control, numerical solution, Adams–Bashforth–Moulton method, Caputo fractional derivative.

1 Introduction

Albeit there exist a vast amount of research work regarding the analysis of chaotic structures and solutions of fractional order Chua, Lorenz, Lü, Chen and Rössler systems, there exists not any study about analysis of fractional Shimizu–Morioka dynamical system. In this paper our major goals are to study asymptotic stability, synchronization control and numerical solution of fractional Shimizu–Morioka dynamical system. Shimizu–Morioka system is defined as

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - ay - xz \\ \frac{dz}{dt} &= -bz + x^2 \end{aligned} \tag{1}$$

where x, y, z are the state variables, $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$ are parameters. Replacing the standard time derivative at 1 with a fractional time derivative of order $\alpha_i \in (0,1]$, $i = 1,2,3$, the incommensurate fractional order Shimizu–Morioka system is defined as

$$\begin{aligned} \frac{d^{\alpha_1} x}{dt^{\alpha_1}} &= y \\ \frac{d^{\alpha_2} y}{dt^{\alpha_2}} &= x - ay - xz \\ \frac{d^{\alpha_3} z}{dt^{\alpha_3}} &= -bz + x^2 \end{aligned}$$

Let us remember that an $n-$dimensional fractional dynamical system

$$\frac{d^{\alpha}\mathbf{X}(t)}{dt^{\alpha}} = H(t, \mathbf{X}(t)), \quad \mathbf{X}(0) = \mathbf{X}_0,$$

where $\mathbf{X}(t) = (x_1(t), \cdots, x_n(t))^T$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i \in (0,1)$ for $i = 1,2,\cdots,n$ is said to be a commensurate order if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, otherwise it is said to be an incommensurate order. In this paper we concern with both of the commensurate and

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incommensurate order fractional Shimizu–Morioka systems.

Structure of this paper is in the following way. Section 2 reviews the fundamental concepts in the fractional calculus. In Section 3 we prove existence and uniqueness of solutions via anew theorem. In the section 4 we study stability analysis of the fractional Shimizu–Morioka system where we find the characteristic function of the Jacobian matrix in terms of parameters a and b. Selecting some different values for a and b, we obtain stability conditions. In the Section 5 we study synchronization control where we employ a technique namely master–slave synchronization controller. In the Section 6 we solve the fractional Shimizu–Morioka dynamical system numerically exploiting Adams–Bashforth–Moulton method. Finally we complete the paper with an overview of the present study.

2 Fractional calculus

Fractional calculus is one of the most popular calculus types having a vast range of applications in many different scientific and engineering disciplines. Order of the derivatives in the fractional calculus might be any reel number which separates the fractional calculus from the ordinary calculus where the derivatives are allowed to be only natural numbers. Fractional calculus is a highly efficient and useful tool in the modeling of many sorts of scientific phenomena including image processing, earthquake engineering, biomedical engineering, computational fluid mechanics and Physics. Fundamental concepts of fractional calculus and applications of it to different research areas can be seen in the references [1]–[2] and [12]–[14].

In this section, we briefly overview the some fundamental concepts of fractional calculus. As we mentioned in the introductory part, orders of derivatives and integrals in fractional calculus might be at any real number. The most popular definitions of a fractional derivative of a function are Riemann–Liouville, Grunwald–Letnikow, Caputo and Generalized functions. In this paper Caputo's definition of fractional differentiation will be employed.

Definition. A real function $f(x), x > 0$, is said to be in the space $C_\rho, \rho \in R$ if there exists a real number $(p > \rho)$, such that $f(x) = x^p f_1(x)$ for a continuous function $f_1(x) \in C[0, \infty)$.

Definition. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\rho, \rho \geq -1$, is defined as

$$J_0^v f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt, \quad v > 0,$$

$$J^0 f(x) = f(x).$$

It has the following properties:

For $f \in C_\rho, \rho \geq -1, \alpha, \beta \geq 0$ and $\gamma > 1$:

$$i.) J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$ii.) J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$

$$iii.) J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Next we present the Caputo sense derivative.

Definition. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^v f(x) = \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{m-v-1} f^{(m)}(t) dt,$$

for $m-1 < v < m,\ m \in N,\ x > 0,\ f \in C_{-1}^m$.

Definition. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as $D_{*t}^\alpha u(x,t) =$

$$\begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N \end{cases}$$

and the space-fractional derivative operator of order $\beta > 0$ is defined as $D_{*x}^\alpha u(x,t) =$

$$\begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^x (x-\theta)^{m-\beta-1} \frac{\partial^m u(\theta,t)}{\partial \theta^m} d\theta, & m-1 < \beta < m, \\ \frac{\partial^m u(x,t)}{\partial x^m}, & \beta = m \in N. \end{cases}$$

Lemma. If $m-1 < \alpha < m,\ m \in N$ and $f \in C_\rho^m, \rho \geq -1$, then

$$D_*^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D_*^v f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!}, \text{ x} > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

3 Existence of solutions

While solving an equation or system of equations, the first question to ask is about the existence and uniqueness of solutions. In this section we will prove that the commensurate order fractional Shimizu–Morioka system has unique solution and point that the same ideas applies for the incommensurate case as well.

Theorem 1 ([6]). Suppose that $D = [0, T^*] \times [x_0 - \rho, x_0 + \rho]$ with some $T^* > 0$ and some $\rho > 0$. Let $f : D \to \mathbb{R}$ be a continuous function. Define

$$T := \min\left\{T^*, \left(\frac{\rho\Gamma(\alpha+1)}{||f||_\infty}\right)^{1/\alpha}\right\}, \quad (2)$$

then there exists a function $x : [0, T] \to \mathbb{R}$ which solves the initial value problem

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t)) \text{ with } x(0) = x_0,\ \alpha \in (0, 1). \quad (3)$$

Theorem 2 ([6]). Assume that $D = [0, T^*] \times [x_0 - \rho, x_0 + \rho]$ with some $T^* > 0$ and some $\rho > 0$. Let $f : D \to \mathbb{R}$ be a bounded function that satisfies the Lipschitz condition with respect to second component. Then, there exist only one function $x : [0, T] \to \mathbb{R}$ which solves the initial value problem (3) where T is defined at (2).

Let $\alpha := \alpha_1 = \alpha_2 = \alpha_3 \in (0, 1]$. Define $\mathbf{X} := (x, y, z)^T$. It is not hard to show that one can express the commensurate order fractional Shimizu–Morioka system as

$$\frac{d^\alpha \mathbf{X}}{dt^\alpha} = A\mathbf{X} + xB\mathbf{X}, \quad (4)$$

where

$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & -a & 0 \\ 0 & 0 & -b \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (5)$$

Theorem 3. Let $0 \le t \le T$ for some $T \in \mathbb{R}_+$. The initial value problem of commensurate order fractional Shimizu–Morioka dynamical system represented as in (4) with $\mathbf{X}(0) = \mathbf{X}_0$ has a unique solution.

Proof. Define $H(\mathbf{X}(t)) := A\mathbf{X}(t) + x(t)B\mathbf{X}(t)$, where A and B are defined at (5). It is clear that $H(\mathbf{X}(t))$ defined in this way is a continuous and bounded function on the interval $[x_0 - \rho, x_0 + \rho]$ for some $\rho \in \mathbb{R}_+$. Now we show that $H(\mathbf{X}(t))$ satisfies the Lipschitz condition with respect to $\mathbf{X}$.

$$\begin{aligned} |H(\mathbf{X}(t)) - H(\mathbf{Y}(t))| = |A\mathbf{X}(t) + x(t)B\mathbf{X}(t) - \\ A\mathbf{Y}(t) - y(t)B\mathbf{Y}(t)| \le ||A|||\mathbf{X}(t) - \mathbf{Y}(t)| + \\ ||B||(|\mathbf{X}(t)| + |y(t)|)|\mathbf{X}(t) - \mathbf{Y}(t)| \\ \le K|\mathbf{X}(t) - \mathbf{Y}(t)|, \end{aligned}$$

s where $K = ||A|| + ||B||(2|\mathbf{X}_0| + \rho) > 0$, $\mathbf{Y}(t) \in \mathbb{R}^3$, $||\cdot||$ and $|\cdot|$ represent some suitable matrix and vector norms, respectively. This proves that $H(\mathbf{X}(t))$ satisfies the Lipschitz condition with respect to $\mathbf{X}(t)$. Therefore, we conclude via Theorem 2 that the solution of the initial value problem of commensurate order fractional Shimizu–Morioka dynamical system uniquely exists.

4 Stability analysis

In this section we study stability analysis of incommensurate order fractional Shimizu–Morioka system. Let us again point that the ideas could be applied for the commensurate order case as well. We can write the incommensurate order fractional Shimizu–Morioka system once again as

$$\begin{aligned} \frac{d^{\alpha_1} x}{dt^{\alpha_1}} &= y \\ \frac{d^{\alpha_2} y}{dt^{\alpha_2}} &= x - ay - xz \\ \frac{d^{\alpha_3} z}{dt^{\alpha_3}} &= -bz + x^2 \end{aligned} \quad (6)$$

Let us notice that the concepts regarding the stability analysis of fractional dynamical systems have some differences with respect to the ones about deterministic systems. Before we present a theorem including the conditions for the asymtotic stability of a fractional system, let us remaind that the steady–state points (or fixed points or equilibrium points) of a fractional system like Shimizu–Morioka system is obtained by letting the right–hand side of the equation equal to the zero. Therefore, we can see that the steady–state solutions of the Shimizu–Morioka system are given by

$$E_1 = (\sqrt{b}, 0, 1),\ E_2 = (-\sqrt{b}, 0, 1) \quad (7)$$

where $b \in \mathbb{R}_+$.

Theorem 4. For $n-$dimensional incommensurate order fractional dynamical systems, if all eigenvalues $(\lambda_1, \cdots, \lambda_n)$ of the Jacobian matrix of a steady–state point satisfy

$$|arg(\lambda_i)| > \frac{\alpha\pi}{2},\ \alpha = \max(\alpha_1, \cdots, \alpha_n),\ i = 1, \cdots, n,$$

then, the fractional dynamical system is locally asymptotically stable at the steady-state point.

This theorem is the most well–known theorem regarding the stability analysis of fractional systems and its similar versions and proofs might be seen, for instance, at the references [3]–[6] amongst many others.

The Jacobian matrix of the fractional Shimizu–Morioka system is given by

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 1 - z & -a & -x \\ 2x & 0 & -b \end{bmatrix}. \quad (8)$$

Characteristic equation of the Jacobian matrix J is obtained by

$$Det(J - \lambda I_{3\times 3}) = 0 \quad (9)$$

where $I_{3\times 3}$ is the identity matrix. Having calculated (9) at both of the equilibrium points given at (7) we obtain the same characteristic equation:

$$CHeq(\lambda) = \lambda^3 + (a+b)\lambda^2 + ab\lambda + 2b. \quad (10)$$

Next our major goals are to make computational discussions about the stability of the Shimizu–Morioka system for different cases of a and b.

For $a = b = 1$, (10) reduces to

$$CHeq(\lambda) = \lambda^3 + 2\lambda^2 + \lambda + 2 = 0. \quad (11)$$

Roots (eigenvalues of Jacobian matrix) of (11) are given by $\lambda_1 = -2$, $\lambda_2 = i$ and $\lambda_3 = -i$. Because $arg(\lambda_1) = \pi$, $arg(\lambda_2) = \pi/2$ and $arg(\lambda_3) = 3\pi/2$, employing Theorem 4, we can say that the system is asymptotically stable at both of the steady-state points (7) for every $\alpha \in (0,1)$.

For $a = b = -1$, (10) reduces to

$$CHeq(\lambda) = \lambda^3 - 2\lambda^2 + \lambda - 2 = 0. \quad (12)$$

Thus, the eigenvalues of Jacobian matrix are $\lambda_1 = 2$, $\lambda_2 = i$ and $\lambda_3 = -i$. Because $arg(\lambda_1) = 0$, $arg(\lambda_2) = \pi/2$ and $arg(\lambda_3) = 3\pi/2$, using Theorem 4, we can say that at the equilibrium points (7), the system will never be stable for any $\alpha \in (0,1)$.

For $a = 0$ and $b = 1$, (10) reduces to

$$CHeq(\lambda) = \lambda^3 + \lambda^2 + +2 = 0. \quad (13)$$

Hence, the eigenvalues of Jacobian matrix are $\lambda_1 = -1.6956$, $\lambda_2 = 0.3478 + 1.0289i$ and $\lambda_3 = 0.3478 - 1.0289i$. Since $arg(\lambda_1) = \pi$, $arg(\lambda_2) = 1.8968$ and $arg(\lambda_3) = -1.8968$, exploiting Theorem 4, we can say that the system is asymptotically stable at both of the equilibrium points (7) for every $\alpha \in (0,1)$.

Let us notice that (10) is a cubic equation (or cubic polynomial equation) having the parameters a and b. The most well–known solution algorithm for cubic equations are due to the G. Cardano and the solution of these types of equations might be seen, for example, at [7]. Either exploiting Cardano's solution method or in the way that we proceeded above, interested readers can keep making computational solutions for different values of a and b, analyse the stability of fractional Shimizu–Morioka system.

5 Synchronization control

Synchronization control is a significant and highly useful concept in the research area of fractional dynamical systems. The underlying idea of the synchronization control for an $n-$dimensional fractional dynamical system is to choose a suitable control function $u = (u_1, u_2, \cdots, u_n)^T$ such that the states of the driving system and response system are synchronized. In the literature there are several different models for synchronization control including master–slave, complete, chaos, robust, Q.–S. schemes. In the present paper for the synchronization of the incommensurate order fractional Shimizu–Morioka system, we adopt master–slave technique that can be briefly outlined as follows.

Let $\alpha \in (0,1)$. Consider

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)), \text{ (namely, master system)}$$

and

$$\frac{d^\alpha y(t)}{dt^\alpha} = g(y(t)) + u, \text{ (slave system)}$$

where $x, y \in R^n$ denote the states and response systems, respectively, $f, g : R^n \to R^n$ are the vector fields of the state and response systems, respectively. As we shortly mentioned above, the main goal of any synchronization method is to choose an appropriate control function $u = (u_1, \cdots, u_n)$ such that

$$lim_{t\to\infty} ||y(t) - x(t)|| = 0.$$

Because the fractional Shimizu–Morioka system is a 3-dimensional system, the master–slave synchronization scheme employed at [8] might be described as follows. The master system is

$$\frac{d^{\alpha_i} x_i(t)}{dt^{\alpha_i}} = f_i(x_1, x_2, x_3), \quad i = 1,2,3. \quad (14)$$

Assuming that the slave system defined by

$$\dot{y_i} = f_i(y_1, y_2, y_3) + u_i$$

is an integer order system with $u_i,\ i = 1,2,3$ are control parameters. Now, having defined the controllers as

$$u_i = v_i + \dot{y_i} - \frac{d^{\alpha_i} y_i(t)}{dt^{\alpha_i}},$$

the slave system is converted into the system

$$\frac{d^{\alpha_i} y_i(t)}{dt^{\alpha_i}} = f_i(y_1, y_2, y_3) + v_i, \quad (15)$$

for $i = 1,2,3$. Defining the error functions $e_i := y_i - x_i,\ i = 1,2,3$ and subtracting (14) from (15), we obtain the error system as

$$\frac{d^{\alpha_i} e_i(t)}{dt^{\alpha_i}} = f_i(y_1, y_2, y_3) - f_i(x_1, x_2, x_3) + v_i, \quad i = 1,2,3. \quad (16)$$

Choosing

$$u_i = f_i(x_1, x_2, x_3) - f_i(y_1, y_2, y_3) + a_i e_1 + b_i e_2 + c_i e_3,$$

for $i = 1,2,3$, the error system (16) can be written as

$$\frac{d^{\alpha_i} e_i(t)}{dt^{\alpha_i}} = a_i e_1 + b_i e_2 + c_i e_3, \;\; i = 1,2,3. \qquad (17)$$

Finally, selecting suitable constants $a_i, b_i, c_i, \;\; i = 1,2,3,$ one can design a stabilizing controller for the synchronization control.

Now we apply this procedure to the incommensurate order fractional Shimizu–Morioka system (6).

The master system is given by

$$\begin{aligned} \frac{d^{\alpha_1} x}{dt^{\alpha_1}} &= y \\ \frac{d^{\alpha_2} y}{dt^{\alpha_2}} &= x - ay - xz \\ \frac{d^{\alpha_3} z}{dt^{\alpha_3}} &= -bz + x^2 \end{aligned} \qquad (18)$$

The slave system is defined by

$$\begin{aligned} \dot{\widetilde{x}} &= \widetilde{y} + u_1 \\ \dot{\widetilde{y}} &= \widetilde{x} - a\widetilde{y} - \widetilde{x}\widetilde{z} + u_2 \\ \dot{\widetilde{z}} &= -b\widetilde{z} + \widetilde{x}^2 + u_3 \end{aligned} \qquad (19)$$

Defining

$$u_i = v_i + \dot{\widetilde{\mathbf{X}}}_i - \frac{d^{\alpha_i} \widetilde{\mathbf{X}}_i(t)}{dt^{\alpha_i}},$$

where $\widetilde{\mathbf{X}} := (\widetilde{x}, \widetilde{y}, \widetilde{z})^T$, the slave system is transformed into fractional system

$$\begin{aligned} \frac{d^{\alpha_1} \widetilde{x}}{dt^{\alpha_1}} &= \widetilde{y} + v_1 \\ \frac{d^{\alpha_2} \widetilde{y}}{dt^{\alpha_2}} &= \widetilde{x} - a\widetilde{y} - \widetilde{x}\widetilde{z} + v_2 \\ \frac{d^{\alpha_3} \widetilde{z}}{dt^{\alpha_3}} &= -b\widetilde{z} + \widetilde{x}^2 + v_3 \end{aligned} \qquad (20)$$

Defining the error function $e = (e_1, e_2, e_3) = (\widetilde{x} - x, \widetilde{y} - y, \widetilde{z} - z)$, and subtracting (18) from (20), we obtain the error system as

$$\begin{aligned} \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} &= e_1 + v_1 \\ \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} &= e_1 - ae_2 - \widetilde{x}\widetilde{z} + xz + v_2 \\ \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}} &= -be_3 + \widetilde{x}^2 + x^2 + v_3 \end{aligned} \qquad (21)$$

The identities (21) are the most crucial steps for the synchronization control. One can design many different types of controllers easily for the stabilizing controllers for synchronization control by selecting suitable parameters at (21) which shows one of the powers of the master–slave synchronization control technique.

6 Numerical solution

In this section we solve the incommensurate fractional order Shimizu–Morioka system numerically employing a highly well-known technique known as Adams–Bashforth–Moulton numerical scheme. Interested reader can read, for instance, [12] in conjunction with the present paper to see furhter applications of this method to some other nonlinear fractional systems. One can employ some other numerical methods such as the ones presented at [9]–[11] as well as Milne's and Adomian decomposition methods.

Now, firstly let us write the incommensurate fractional order Shimizu–Morioka dynamical system once again.

$$\begin{aligned} \frac{d^{\alpha_1} x}{dt^{\alpha_1}} &= y \\ \frac{d^{\alpha_2} y}{dt^{\alpha_2}} &= x - ay - xz \\ \frac{d^{\alpha_3} z}{dt^{\alpha_3}} &= -bz + x^2 \end{aligned} \qquad (22)$$

where $\alpha_i \in (0,1]$, $i = 1,2,3$.

Having applied the Adams–Bashforth–Moulton scheme to the Shimizu–Morioka system (22), we obtain the following system of discrete equations.

$$\begin{aligned} x_{n+1} &= x_0 + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} y^p_{n+1} + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \sum_{j=0}^{n} \beta_{1,j,n+1} y_j \\ y_{n+1} &= y_0 + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \left(x^p_{n+1} - a y^p_{n+1} - x^p_{n+1} z^p_{n+1} \right) \\ &\quad + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \sum_{j=0}^{n} \beta_{2,j,n+1} (x_j - a y_j - x_j z_j) \\ z_{n+1} &= z_0 + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+2)} \left(-b z^p_{n+1} + x^{2p}_{n+1} \right) \\ &\quad + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+2)} \sum_{j=0}^{n} \beta_{3,j,n+1} (-b z_j + x_j^2), \end{aligned}$$

where

$$x_{n+1}^p = x_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} \gamma_{1,j,n+1} y_j$$

$$y_{n+1}^p = y_0 + \frac{1}{\Gamma(\alpha_2)} \sum_{j=0}^{n} \gamma_{2,j,n+1} (x_j - a y_j - x_j z_j)$$

$$z_{n+1}^p = z_0 + \frac{1}{\Gamma(\alpha_3)} \sum_{j=0}^{n} \gamma_{3,j,n+1} (-b z_j + x_j^2),$$

where $\beta_{i,j,n+1} =$

$$\begin{cases} n^{\alpha_i+1} - (n-\alpha_i)(n+1)^{\alpha_i}, & j=0; \\ (n-j+2)^{\alpha_i+1} + (n-j)^{\alpha_i+1} - 2(n-j+1)^{\alpha_i+1}, & 1 \le j \le n; \\ 1, & j = n+1. \end{cases}$$

$$\gamma_{i,j,n+1} = \frac{h^{\alpha_i}}{\alpha_i} \left((n-j+1)^{\alpha_i} - (n-j)^{\alpha_i} \right), \ 0 \le j \le n,$$

for $i = 1, 2, 3$.

7 Conclusion

In this paper we studied asymptotic stability, synchronization control and numerical solution of incommensurate fractional order Shimizu–Morioka dynamical system. Firstly we proved the existence and uniqueness of the solutions via a new theorem. After finding steady–state points, we obtained necessary and sufficient conditions for the asymptotic stability of the Shimizu–Morioka system. We also concern with the synchronization control where we employed master–slave synchronization method. Finally, employing Adams–Bashforth–Moulton's scheme we solve the Shimizu–Morioka system numerically.

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