Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 11 NO: 2 PAGE: 586-593 YEAR: 2022
DOI: 10.17798/bitlisfen.1063550

Algebraic Construction for Dual Quaternions with $\mathcal{GCN}$

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Keywords:
Generalized complex number, Dual quaternion, Matrix form.

Abstract
In this paper, we explain how dual quaternion theory can be extended to dual quaternions with generalized complex number $(\mathcal{GCN})$ components. More specifically, we algebraically examine this new type dual quaternion and give several matrix representations both as a dual quaternion and as a $\mathcal{GCN}$.

1. Introduction

A real quaternion, as an extension of complex number in four dimensions, is defined as

$$q = \mathrm{a}_0 + \mathrm{a}_1 e_1 + \mathrm{a}_2 e_2 + \mathrm{a}_3 e_3,$$

where $\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3$ are real components and e_1, e_2, e_3 are non-real quaternionic units with the following multiplication schema [1-3]:

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_1 e_2 &= -e_2 e_1 = e_3, \\ e_2 e_3 &= -e_3 e_2 = e_1, \\ e_3 e_1 &= -e_1 e_3 = e_2. \end{aligned}$$

The set of real quaternions, which is isomorphic to Euclidean 4-space, forms a non-commutative and an associative algebra under addition and multiplication. The real quaternions have many applications such as describing rotations in robotics and computer animation with rotation axis and angle.

A dual quaternion, as an extension of dual number in four dimensions, is defined by the same form with different multiplication conditions for quaternionic units as [4-10]:

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0, \\ \mathbf{ij} = \mathbf{ji} = \mathbf{jk} = \mathbf{kj} = \mathbf{ki} = \mathbf{ik} = 0. \end{aligned} \tag{1}$$

The set of dual quaternions $\mathcal{Q}_{\mathcal{D}}$, which is isomorphic to Galilean 4-space, forms a commutative division algebra under addition and multiplication [7]. Furthermore, using the dual quaternions, one can express the Galilean transformation in one quaternionic equation.

From a different viewpoint, the set of generalized complex numbers ($\mathcal{GCN}$), the general bidimensional hypercomplex system, is denoted by $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ and defined by the ring [11-16]:

$$\frac{\mathbb{R}[X]}{\langle X^2 - \mathfrak{q}X - \mathfrak{p}\rangle} \cong \left\{ \begin{array}{c} z = x_1 + x_2 I : I^2 = I\mathfrak{q} + \mathfrak{p}, I \notin \mathbb{R}, \\ \mathfrak{p}, \mathfrak{q}, x_1, x_2 \in \mathbb{R} \end{array} \right\},$$

where I is the generalized complex unit. It is isomorphic (as ring) to the following types considering the sign of $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$: for $\Delta > 0$ *hyperbolic system*, for $\Delta < 0$ *elliptic system* and for

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Received: 26.01.2022, Accepted: 26.06.2022

$\Delta = 0$ *parabolic system*. The canonical forms of these systems are given by, respectively,

- hyperbolic (perplex, split complex, double) numbers $\mathbb{C}_{0,1}$ [17-20],
- complex (ordinary) numbers $\mathbb{C}_{0,-1}$ [20, 21],
- dual numbers $\mathbb{C}_{0,0}$ [20, 22, 23].

Specially, dual numbers have been widely used for the search of closed form solutions in the fields of displacement analysis, kinematic synthesis, and dynamic analysis of spatial mechanisms.

In $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, the value

$$\mathcal{D}_z = z\overline{z} = (x_1 + x_2 I)(x_1 - x_2 I) = x_1^{\,2} - \mathfrak{p}x_2^{\,2} + \mathfrak{q}x_1x_2$$

is referred to as the characteristic determinant of z. Considering this characteristic value, z is called timelike for $\mathcal{D}_z < 0$, spacelike for $\mathcal{D}_z > 0$ and null for $\mathcal{D}_z = 0$ [12].

Number systems play a special role in defining different types of quaternions. Combining fundamental properties of numbers and quaternions enables to determine new features. Considering the numbers mentioned above, the quaternions with different number components have been studied by several authors in many points of view [24-30]. One can see the combination of the dual numbers and the real quaternions in the studies [27, 28, 30]. Moreover, as an application, the representational method based on quaternions with dual number coefficients related to electromagnetism can be seen in [31, 32].

In this paper, we are interested in the combination of dual quaternions and $\mathcal{GCN}$. In Section 2, we extend definitions and some universal known results of dual quaternions to dual quaternions with $\mathcal{GCN}$ components. Finally, we provide a complete classification in conclusion.

2. Dual Quaternions with $\mathcal{GCN}$ Components

This original section discusses an algebraic behavior of dual quaternions with $\mathcal{GCN}$ components. Also it proceeds with the examination of several matrix representations.

Definition 1. The dual quaternion with $\mathcal{GCN}$ components is of the form:

$$\tilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where the dual quaternion units satisfy equations in (1). The set of these quaternions is denoted by $\tilde{\mathcal{Q}}_{\mathcal{D}}$.

Here, I commutes with the three dual quaternion units. One can see that, the usual dual operator distinct from the dual quaternion units for $\mathfrak{q} = 0, \mathfrak{p} = 0$.

Throughout the paper, $\tilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\tilde{p} = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\tilde{r} = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ are taken.

We firstly define the basic algebraic operations on dual quaternions with $\mathcal{GCN}$ components. For any $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$, $S_{\tilde{q}} = a_0$ is the scalar part and $V_{\tilde{q}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is the vector part. Equality is as follows: $\tilde{p} = \tilde{q} \Leftrightarrow S_{\tilde{p}} = S_{\tilde{q}},\ V_{\tilde{p}} = V_{\tilde{q}}$. The addition of $\tilde{q}$ and $\tilde{p}$ is defined as:

$$\begin{aligned}\tilde{q} + \tilde{p} &= (a_0 + b_0) + (a_1 + b_1)\mathbf{i} \\ &\quad + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \\ &= S_{\tilde{p}} + S_{\tilde{q}} + V_{\tilde{p}} + V_{\tilde{q}} \\ &= S_{\tilde{p}+\tilde{q}} + V_{\tilde{p}+\tilde{q}}.\end{aligned}$$

The quaternion $\overline{\tilde{q}} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k} = S_{\tilde{q}} - V_{\tilde{q}}$ is called the conjugate of $\tilde{q}$. Furthermore, for $c \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, the scalar multiplication of c and $\tilde{q}$ is defined as:

$$\begin{aligned}c\tilde{q} &= ca_0 + ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k} \\ &= cS_{\tilde{q}} + cV_{\tilde{q}}.\end{aligned}$$

The multiplication of $\tilde{q}$ and $\tilde{p}$ is defined as:

$$\begin{aligned}\tilde{q}\,\tilde{p} &= a_0b_0 + (a_0b_1 + a_1b_0)\mathbf{i} \\ &\quad + (a_0b_2 + a_2b_0)\mathbf{j} + (a_0b_3 + a_3b_0)\mathbf{k} \\ &= S_{\tilde{q}}S_{\tilde{p}} + S_{\tilde{q}}V_{\tilde{p}} + S_{\tilde{p}}V_{\tilde{q}} \\ &= \tilde{p}\tilde{q}.\end{aligned} \tag{2}$$

Additionally, $N_{\tilde{q}} = \tilde{q}\overline{\tilde{q}} = \overline{\tilde{q}}\tilde{q} = a_0^2$ is called the norm of $\tilde{q}$. Hence, the quaternion $(\tilde{q})^{-1} = \dfrac{\overline{\tilde{q}}}{N_{\tilde{q}}}$ is called the inverse of $\tilde{q}$ for non-null $N_{\tilde{q}}$ that is $\mathcal{D}_{N_{\tilde{q}}} \neq 0$. As it is seen many properties of dual quaternions with $\mathcal{GCN}$ components are familiar with the usual dual quaternions.

Standard elementary conjugate properties establish the following proposition.

Proposition 1. For any $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_D$ and $c_1, c_2 \in \mathbb{R}$, the followings hold:

- **i)** $\overline{\overline{\tilde{q}}} = \tilde{q}$,
- **ii)** $\overline{c_1\tilde{p} + c_2\tilde{q}} = c_1\overline{\tilde{p}} + c_2\overline{\tilde{q}}$,
- **iii)** $\overline{\tilde{q}\tilde{p}} = \overline{\tilde{p}}\overline{\tilde{q}} = \overline{\tilde{q}}\overline{\tilde{p}}$,
- **iv)** $N_{c_1\tilde{q}} = c_1^2 N_{\tilde{q}}$,
- **v)** $N_{\tilde{q}\tilde{p}} = N_{\tilde{q}} N_{\tilde{p}} = N_{\tilde{p}} N_{\tilde{q}}$.

Proof: Considering the conjugate, properties i) and ii) are quickly obvious.

iii) By using equation (2), we have:

$$\overline{\tilde{q}\tilde{p}} = a_0b_0 - \left(a_0b_1 + a_1b_0\right)\mathbf{i} \\ -\left(a_0b_2 + a_2b_0\right)\mathbf{j} - \left(a_0b_3 + a_3b_0\right)\mathbf{k}.$$

So, it is verified that $\overline{\tilde{q}\tilde{p}} = \overline{\tilde{p}}\overline{\tilde{q}} = \overline{\tilde{q}}\overline{\tilde{p}}$.

iv) By having property ii) and the norm, we have: $N_{c_1\tilde{q}} = \left(c_1\tilde{q}\right)\overline{\left(c_1\tilde{q}\right)} = c_1^2 N_{\tilde{q}}$.

v) From property iii), we get: $N_{\tilde{q}\tilde{p}} = \left(\tilde{q}\tilde{p}\right)\overline{\left(\tilde{q}\tilde{p}\right)} = \tilde{q}\tilde{p}\overline{\tilde{p}}\overline{\tilde{q}} = N_{\tilde{q}}N_{\tilde{p}} = N_{\tilde{p}}N_{\tilde{q}}$.

Proposition 2. $\tilde{\mathcal{Q}}_D$ is a 4-dimensional module over $\mathbb{C}_{\mathrm{q,p}}$ and an 8-dimensional vector space over $\mathbb{R}$ with bases $\left\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\right\}$ and $\left\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\right\}$, respectively.

Definition 2. For any $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_D$, the scalar product is given by:

$$\tilde{\mathcal{Q}}_D \times \tilde{\mathcal{Q}}_D \to \mathbb{C}_{\mathrm{q,p}} \\ \left(\tilde{q}, \tilde{p}\right) \mapsto \left\langle \tilde{q}, \tilde{p}\right\rangle = S_{\tilde{q}}S_{\tilde{p}} = a_0b_0 = S_{\tilde{p}\tilde{q}}$$

and the vector product is defined by:

$$\tilde{\mathcal{Q}}_D \times \tilde{\mathcal{Q}}_D \to \tilde{\mathcal{Q}}_D \\ \left(\tilde{q}, \tilde{p}\right) \mapsto \tilde{q} \times \tilde{p} = S_{\tilde{q}}V_{\bar{\tilde{p}}} + S_{\bar{\tilde{p}}}V_{\tilde{q}} = V_{\tilde{q}\bar{\tilde{p}}}.$$

More specially, we examine some identities for the scalar product.

Proposition 3. For any $\tilde{q}, \tilde{p}$ and $\tilde{r} \in \tilde{\mathcal{Q}}_D$, the followings hold:

- **i)** $\langle \tilde{q}\tilde{r}, \tilde{p}\tilde{r}\rangle = \langle \tilde{r}\tilde{q}, \tilde{r}\tilde{p}\rangle = \langle \tilde{r}\tilde{q}, \tilde{p}\tilde{r}\rangle = \langle \tilde{q}\tilde{r}, \tilde{r}\tilde{p}\rangle$, $= N_{\tilde{r}}\langle \tilde{q}, \tilde{p}\rangle$,
- **ii)** $\langle \tilde{r}\tilde{q}, \tilde{p}\rangle = \langle \tilde{q}, \bar{\tilde{r}}\tilde{p}\rangle = \langle \tilde{q}, \tilde{r}\tilde{p}\rangle$.

Proof: Considering the scalar product and the norm, the following proofs can be conducted:

- **i)** $\langle \tilde{q}\tilde{r}, \tilde{p}\tilde{r}\rangle = \left(a_0c_0\right)\left(b_0c_0\right) = \left(c_0^2\right)\left(a_0b_0\right)$ $= N_{\tilde{r}}\langle \tilde{q}, \tilde{p}\rangle$,
- **ii)** $\langle \tilde{q}\tilde{r}, \tilde{p}\rangle = \left(a_0c_0\right)b_0 = a_0\left(b_0c_0\right) = \langle \tilde{q}, \tilde{p}\bar{\tilde{r}}\rangle$.

We are now ready to prove the results based on matrix approach.

Theorem 1. Every element $\tilde{q}$ of $\tilde{\mathcal{Q}}_D$ can be represented by a quaternionic matrix:

$$\mathsf{E}_{\tilde{q}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix}. \tag{3}$$

Hence $\tilde{\mathcal{Q}}_D \subseteq \mathbb{M}_2(\tilde{\mathcal{Q}}_D)$.

Proof: For $\tilde{q} \in \tilde{\mathcal{Q}}_D$, $\mathcal{L} : \tilde{\mathcal{Q}}_D \to \mathcal{E}$, $\tilde{q} \mapsto \mathsf{E}_{\tilde{q}}$ is a linear map, where

$$\mathcal{E} := \left\{ \mathsf{E}_{\tilde{q}} \in \mathbb{M}_2(\tilde{\mathcal{Q}}_D):\ \mathsf{E}_{\tilde{q}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix} \right\}$$

is a subset of $\mathbb{M}_2(\tilde{\mathcal{Q}}_D)$. So, one can realize the correspondence between $\tilde{\mathcal{Q}}_D$ and $\mathcal{E}$ by the map $\mathcal{L}$. So it is no surprised that 2×2 representation of $\tilde{q}$ is $\mathsf{E}_{\tilde{q}}$. The proof is completed.

Corollary 1. For all $\tilde{q} \in \tilde{\mathcal{Q}}_D$, $\mathcal{L}(\tilde{q})$ can also be written as follows:

$$\mathcal{L}(\tilde{q}) = a_0 I_2 + a_1\mathsf{I} + a_2\mathsf{J} + a_3\mathsf{K},$$

where $\mathcal{L}(\mathbf{i}) = \mathsf{I}$, $\mathcal{L}(\mathbf{j}) = \mathsf{J}$, $\mathcal{L}(\mathbf{k}) = \mathsf{K}$ satisfy equations in (1).

Theorem 2. For $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ and $\lambda \in \mathbb{R}$, then the followings hold:

- **i)** $\tilde{q} = \tilde{p} \Leftrightarrow \mathsf{E}_{\tilde{q}} = \mathsf{E}_{\tilde{p}}$,
- **ii)** $\mathsf{E}_{\tilde{q}+\tilde{p}} = \mathsf{E}_{\tilde{q}} + \mathsf{E}_{\tilde{p}}$,
- **iii)** $\mathsf{E}_{\lambda\tilde{q}} = \lambda\left(\mathsf{E}_{\tilde{q}}\right)$,
- **iv)** $\mathsf{E}_{\tilde{q}\tilde{p}} = \mathsf{E}_{\tilde{q}}\mathsf{E}_{\tilde{p}}$.

Proof: iv) For $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$, using equations (2) and (3), we can write:

$$\mathsf{E}_{\tilde{q}\tilde{p}} = \begin{bmatrix} a_0b_0 + (a_0b_3 + a_3b_0)\mathbf{k} & (a_0b_1 + a_1b_0)\mathbf{i} + (a_0b_2 + a_2b_0)\mathbf{j} \\ (a_0b_1 + a_1b_0)\mathbf{i} + (a_0b_2 + a_2b_0)\mathbf{j} & a_0b_0 + (a_0b_3 + a_3b_0)\mathbf{k} \end{bmatrix}$$

Moreover, we obtain:

$$\mathsf{E}_{\tilde{q}}\mathsf{E}_{\tilde{p}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix}\begin{bmatrix} b_0 + b_3\mathbf{k} & b_1\mathbf{i} + b_2\mathbf{j} \\ b_1\mathbf{i} + b_2\mathbf{j} & b_0 + b_3\mathbf{k} \end{bmatrix}$$
$$= \begin{bmatrix} a_0b_0 + (a_0b_3 + a_3b_0)\mathbf{k} & (a_0b_1 + a_1b_0)\mathbf{i} + (a_0b_2 + a_2b_0)\mathbf{j} \\ (a_0b_1 + a_1b_0)\mathbf{i} + (a_0b_2 + a_2b_0)\mathbf{j} & a_0b_0 + (a_0b_3 + a_3b_0)\mathbf{k} \end{bmatrix}.$$

It is clear that $\mathsf{E}_{\tilde{q}\tilde{p}} = \mathsf{E}_{\tilde{q}}\mathsf{E}_{\tilde{p}}$. The other properties can be proved similarly.

Theorem 3. Every element $\tilde{q}$ of $\tilde{\mathcal{Q}}_{\mathcal{D}}$ can be represented by the following matrix:

$$\mathcal{F}_{\tilde{q}} = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix}. \qquad (4)$$

So, $\tilde{\mathcal{Q}}_{\mathcal{D}}$ is subset of $\mathbb{M}_4(\mathbb{C}_{\mathsf{q,p}})$. Moreover, for $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ and $\lambda \in \mathbb{R}$,

- **i)** $\tilde{p} = \tilde{q} \Leftrightarrow \mathcal{F}_{\tilde{p}} = \mathcal{F}_{\tilde{q}}$,
- **ii)** $\mathcal{F}_{\tilde{p}+\tilde{q}} = \mathcal{F}_{\tilde{p}} + \mathcal{F}_{\tilde{q}}$,
- **iii)** $\mathcal{F}_{\lambda\tilde{p}} = \lambda(\mathcal{F}_{\tilde{p}})$,
- **iv)** $\mathcal{F}_{\tilde{p}\tilde{q}} = \mathcal{F}_{\tilde{p}}\mathcal{F}_{\tilde{q}} = \mathcal{F}_{\tilde{q}}\mathcal{F}_{\tilde{p}}$,
- **v)** $\delta\mathcal{F}_{\tilde{p}}\delta = \mathcal{F}_{\bar{\tilde{p}}}$ where $\delta = \operatorname{diag}(1,-1,-1,-1)$,

and $\det(\mathcal{F}_{\tilde{p}}) = N_{\tilde{p}}^2$.

Proof: iv) For $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$, using equations (2) and (4), we obtain:

$$\mathcal{F}_{\tilde{q}\tilde{p}} = \begin{bmatrix} a_0b_0 & 0 & 0 & 0 \\ a_0b_1 + a_1b_0 & a_0b_0 & 0 & 0 \\ a_0b_2 + a_2b_0 & 0 & a_0b_0 & 0 \\ a_0b_3 + a_3b_0 & 0 & 0 & a_0b_0 \end{bmatrix}.$$

Also, we have:

$$\mathcal{F}_{\tilde{q}}\mathcal{F}_{\tilde{p}} = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{bmatrix}\begin{bmatrix} b_0 & 0 & 0 & 0 \\ b_1 & b_0 & 0 & 0 \\ b_2 & 0 & b_0 & 0 \\ b_3 & 0 & 0 & b_0 \end{bmatrix}$$
$$= \begin{bmatrix} a_0b_0 & 0 & 0 & 0 \\ a_0b_1 + a_1b_0 & a_0b_0 & 0 & 0 \\ a_0b_2 + a_2b_0 & 0 & a_0b_0 & 0 \\ a_0b_3 + a_3b_0 & 0 & 0 & a_0b_0 \end{bmatrix}$$
$$= \mathcal{F}_{\tilde{p}}\mathcal{F}_{\tilde{q}}.$$

It is obvious that $\mathcal{F}_{\tilde{q}\tilde{p}} = \mathcal{F}_{\tilde{p}}\mathcal{F}_{\tilde{q}} = \mathcal{F}_{\tilde{q}}\mathcal{F}_{\tilde{p}}$.

v) Considering equation (4) and $\delta = \operatorname{diag}(1,-1,-1,-1)$, we get:

$$\delta\mathcal{F}_{\tilde{p}}\delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} b_0 & 0 & 0 & 0 \\ b_1 & b_0 & 0 & 0 \\ b_2 & 0 & b_0 & 0 \\ b_3 & 0 & 0 & b_0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$
$$= \begin{bmatrix} b_0 & 0 & 0 & 0 \\ -b_1 & b_0 & 0 & 0 \\ -b_2 & 0 & b_0 & 0 \\ -b_3 & 0 & 0 & b_0 \end{bmatrix}.$$

One can see that the final matrix is the matrix $\mathcal{F}_{\bar{\tilde{p}}}$, so we can write $\delta\mathcal{F}_{\tilde{p}}\delta = \mathcal{F}_{\bar{\tilde{p}}}$.

The proofs of the other properties are straightforward by considering 4×4 real matrix representation of the dual quaternions.

Definition 3. The column matrix form of $\tilde{p}$ with respect to $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is $\tilde{p} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}^T$.

Corollary 2. Using the above definition, the multiplication of $\tilde{q}$ and $\tilde{p}$ is also calculated as: $\tilde{q}\tilde{p} = \mathcal{F}_{\tilde{q}}\,\tilde{p} = \tilde{p}\tilde{q}$.

Corollary 3. For $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$,

$$\mathcal{F}_{\tilde{q}} = a_0 I_4 + a_1 \Im + a_2 \xi + a_3 \mathfrak{K},$$

where

$$\Im = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \xi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathfrak{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The following theorem indicates how to calculate the formula for matrix representation of the inverse of $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$.

Theorem 4. Let $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ and $\tilde{q}^{-1}$ be the inverse of $\tilde{q}$. Then, $\mathcal{F}_{\tilde{q}^{-1}} = \dfrac{1}{\sqrt{\det(\mathcal{F}_{\tilde{q}})}} \mathcal{F}_{\bar{\tilde{q}}}$ for non-null $\det\left(\mathcal{F}_{\tilde{q}}\right)$ that is $\mathcal{D}_{\det\left(\mathcal{F}_{\tilde{q}}\right)} \neq 0$.

Theorem 5. According to $\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\}$, the real matrix representation of $\tilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is:

$$\mathcal{G}_{\tilde{q}} = \begin{bmatrix} x_{01} & \mathfrak{p}x_{02} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{02} & x_{01}+\mathfrak{q}x_{02} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{11} & \mathfrak{p}x_{12} & x_{01} & \mathfrak{p}x_{02} & 0 & 0 & 0 & 0 \\ x_{12} & x_{11}+\mathfrak{q}x_{12} & x_{02} & x_{01}+\mathfrak{q}x_{02} & 0 & 0 & 0 & 0 \\ x_{21} & \mathfrak{p}x_{22} & 0 & 0 & x_{01} & \mathfrak{p}x_{02} & 0 & 0 \\ x_{22} & x_{21}+\mathfrak{q}x_{22} & 0 & 0 & x_{02} & x_{01}+\mathfrak{q}x_{02} & 0 & 0 \\ x_{31} & \mathfrak{p}x_{32} & 0 & 0 & 0 & 0 & x_{01} & \mathfrak{p}x_{02} \\ x_{32} & x_{31}+\mathfrak{q}x_{32} & 0 & 0 & 0 & 0 & x_{02} & x_{01}+\mathfrak{q}x_{02} \end{bmatrix}, \quad (5)$$

where $a_i = x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, $0 \leq i \leq 3$. Moreover, for $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ and $\lambda \in \mathbb{R}$,

- **i)** $\tilde{p} = \tilde{q} \Leftrightarrow \mathcal{G}_{\tilde{p}} = \mathcal{G}_{\tilde{q}}$,
- **ii)** $\mathcal{G}_{\tilde{p}+\tilde{q}} = \mathcal{G}_{\tilde{p}} + \mathcal{G}_{\tilde{q}}$,
- **iii)** $\mathcal{G}_{\lambda\tilde{p}} = \lambda(\mathcal{G}_{\tilde{p}})$,
- **iv)** $\mathcal{G}_{\tilde{q}\tilde{p}} = \mathcal{G}_{\tilde{q}}\mathcal{G}_{\tilde{p}}$.

Proof: Let us define the linear map $f_{\tilde{q}}$ from $\tilde{\mathcal{Q}}_{\mathcal{D}}$ to subset of $\mathbb{M}_8(\mathbb{R})$ such that $f_{\tilde{q}}\left(\tilde{p}\right) = \tilde{q}\tilde{p}$ for every $\tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$. By taking $a_i = x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}, 0 \leq i \leq 3$, we have the following equations:

$$\begin{aligned}
f_{\tilde{q}}(1) &= \tilde{q} &&= x_{01} + x_{02}I + x_{11}\mathbf{i} + x_{12}I\mathbf{i} + x_{21}\mathbf{j} \\
& && \quad + x_{22}I\mathbf{j} + x_{31}\mathbf{k} + x_{32}I\mathbf{k}, \\
f_{\tilde{q}}(I) &= \tilde{q}I &&= \mathfrak{p}x_{02} + (x_{01} + \mathfrak{q}x_{02})I + \mathfrak{p}x_{12}\mathbf{i} + \left(x_{11} + \mathfrak{q}x_{12}\right)I\mathbf{i} + \mathfrak{p}x_{22}\mathbf{j} \\
& && \quad + \left(x_{21} + \mathfrak{q}x_{22}\right)I\mathbf{j} + \mathfrak{p}x_{32}\mathbf{k} + \left(x_{31} + \mathfrak{q}x_{32}\right)I\mathbf{k}, \\
f_{\tilde{q}}(\mathbf{i}) &= \tilde{q}\mathbf{i} &&= x_{01}\mathbf{i} + x_{02}I\mathbf{i}, \\
f_{\tilde{q}}(I\mathbf{i}) &= \tilde{q}I\mathbf{i} &&= \mathfrak{p}x_{02}\mathbf{i} + (x_{01} + \mathfrak{q}x_{02})I\mathbf{i}, \\
f_{\tilde{q}}(\mathbf{j}) &= \tilde{q}\mathbf{j} &&= x_{01}\mathbf{j} + x_{02}I\mathbf{j}, \\
f_{\tilde{q}}(I\mathbf{j}) &= \tilde{q}I\mathbf{j} &&= \mathfrak{p}x_{02}\mathbf{j} + (x_{01} + \mathfrak{q}x_{02})I\mathbf{j}, \\
f_{\tilde{q}}(\mathbf{k}) &= \tilde{q}\mathbf{k} &&= x_{01}\mathbf{k} + x_{02}I\mathbf{k}, \\
f_{\tilde{q}}(I\mathbf{k}) &= \tilde{q}I\mathbf{k} &&= \mathfrak{p}x_{02}\mathbf{k} + (x_{01} + \mathfrak{q}x_{02})I\mathbf{k}.
\end{aligned}$$

Hence, by concerning the standard basis $\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\}$, we have 8×8 real matrix representation of $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ is calculated as in equation (5). The proof of the properties can be conducted by considering the above linear map. Specially, for property iv), by taking

$$a_i = x_{i1} + x_{i2}I,\ b_i = y_{i1} + y_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}},\ 0 \leq i \leq 3$$

for $\tilde{q}, \tilde{p} \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ and using equations (2) and (5), the multiplication of $\mathcal{G}_{\tilde{q}}$ and $\mathcal{G}_{\tilde{p}}$ gives the matrix $\mathcal{G}_{\tilde{q}\tilde{p}}$ quickly.

With an alternative thought, $\tilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $a_i = x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, can be written as $\tilde{q} = q_0 + q_1 I$ in $\tilde{\mathcal{Q}}_{\mathcal{D}}$ where $q_{j-1} = x_{0j} + x_{1j}\mathbf{i} + x_{2j}\mathbf{j} + x_{3j}\mathbf{k} \in \mathcal{Q}_{\mathcal{D}}$ for $0 \leq i \leq 3$, $1 \leq j \leq 2$. So, $\tilde{\mathcal{Q}}_{\mathcal{D}}$ is a 2-dimensional module over $\mathcal{Q}_{\mathcal{D}}$ with base $\{1, I\}$. This consideration provides a reformulation of the previous results.

Theorem 6. Let $\tilde{q} = q_0 + q_1 I, \tilde{p} = p_0 + p_1 I \in \tilde{\mathcal{Q}}_{\mathcal{D}}$ and $\lambda \in \mathbb{R}$. Every element of $\tilde{\mathcal{Q}}_{\mathcal{D}}$ is written by a 2×2 dual quaternion matrix:

$$\mathcal{H}_{\tilde{q}} = \begin{bmatrix} q_0 & \mathfrak{p}q_1 \\ q_1 & q_0 + \mathfrak{q}q_1 \end{bmatrix}.$$

It means that $\tilde{\mathcal{Q}}_{\mathcal{D}}$ is subset of $\mathbb{M}_2(\mathcal{Q}_{\mathcal{D}})$. So, we have:

- **i)** $\tilde{p} = \tilde{q} \Leftrightarrow \mathcal{H}_{\tilde{p}} = \mathcal{H}_{\tilde{q}}$,
- **ii)** $\mathcal{H}_{\tilde{p}+\tilde{q}} = \mathcal{H}_{\tilde{p}} + \mathcal{H}_{\tilde{q}}$,
- **iii)** $\mathcal{H}_{\lambda\tilde{p}} = \lambda(\mathcal{H}_{\tilde{p}})$,
- **iv)** $\mathcal{H}_{\tilde{p}\tilde{q}} = \mathcal{H}_{\tilde{p}}\mathcal{H}_{\tilde{q}}$,

and $\det(\mathcal{H}_{\tilde{q}}) = q_0^2 + \mathfrak{q} q_0 q_1 - \mathfrak{p} q_1^2$, where $\det$ corresponds the determinant of the quaternion matrix[2]. Moreover, $\mathcal{H}_{\tilde{q}} = q_0 I_2 + q_1 \mathsf{I}$, where $\mathsf{I} = \begin{bmatrix} 0 & \mathfrak{p} \\ 1 & \mathfrak{q} \end{bmatrix}$. (It is worth noting that there exists different ways to take I, for instance: $\mathsf{I} = \begin{bmatrix} \mathfrak{q} & 1 \\ \mathfrak{p} & 0 \end{bmatrix}$, see in [34]).

Definition 4. The column matrix form of $\tilde{p}$ with respect to $\{1, I\}$ is $\tilde{p} = \begin{bmatrix} p_0 & p_1 \end{bmatrix}^T$.

Corollary 4. By using above definition, we obtain $\tilde{q}\,\tilde{p} = \mathcal{H}_{\tilde{q}} \tilde{p} = \tilde{p}\tilde{q}$.

Definition 5. The matrix

$$\vec{\tilde{q}} = \begin{bmatrix} \vec{q}_0^{\,T} & \vec{q}_1^{\,T} \end{bmatrix}^T = \begin{bmatrix} \vec{q}_0 \\ \vec{q}_1 \end{bmatrix} \in \mathbb{M}_{8\times 1}(\mathbb{R})$$

is called as the vector form of $\tilde{q}$, where

$$q_{j-1} = x_{0i} + x_{1j}\mathbf{i} + x_{2j}\mathbf{j} + x_{3j}\mathbf{k} \in \mathcal{Q}_{\mathcal{D}}$$

and $\vec{q}_{j-1} = \left(x_{0j}, x_{1j}, x_{2j}, x_{3j}\right)^T = \begin{bmatrix} x_{0j} & x_{1j} & x_{2j} & x_{3j} \end{bmatrix}^T$ are vectors (matrices) for $1 \le j \le 2$.

4. Conclusion

Quaternions ([1-3]) have a deep mathematical meaning with a long history dating back and are used in physics to clarify the formulation of physical laws. A milestone moment in the use of quaternions in theoretical physic is the creation of special relativity, which unifies space and time to form a 4-dimensional space-time. By replacing real quaternions with complex ones offers a valuable tool in creating classical physical laws. Complex quaternions having several properties allow the desirable theorems of modern algebra to be applied. Furthermore, an important extension of real quaternions are the hyperbolic quaternions and the dual quaternions. Using the different types of quaternions are the way to description of the classical and quantum fields and reasonable to express space-time transformations. In terms of the hyperbolic quaternion, the general Lorentz space-time transformation can be discussed. With similar thought, the dual quaternions can be expressed for discussing the Galilean transformation. In terms of the dual quaternions this transformation with underlying algebraic features enables an efficient form [7-9].

With the leading of the above discussions, considering $\mathcal{GCN}$ as components of dual quaternions is the main motivation of this study. For this purpose, we construct dual quaternions with $\mathcal{GCN}$ coefficients for real $\mathfrak{p}, \mathfrak{q}$. Moreover, we examine the basic structures and algebraic properties by writing them in two forms: a $\mathcal{GCN}$ and a quaternion. Additionally, we established 2×2, 4×4 and 8×8 matrix representations.

With this approach, we can easily write the dual quaternions with elliptic, parabolic and hyperbolic number components considering $\Delta < 0, \Delta = 0$ and $\Delta > 0$, respectively, where $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$. Bearing in mind the special values $\mathfrak{p}$ and $\mathfrak{q}$, we have several types of dual quaternions with $\mathcal{GCN}$ components. For $\mathfrak{q} = 0,\ \mathfrak{p} = -1$ dual quaternions with complex number components, for $\mathfrak{q} = \mathfrak{p} = 0$ dual quaternions with dual number components and for $\mathfrak{q} = 0,\ \mathfrak{p} = 1$ dual quaternions with hyperbolic number components are obtained.

Contributions of the authors

Every author contributed equally to this work.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

[2] $\det\left(\begin{bmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{bmatrix}\right) = \mathsf{da} - \mathsf{cb}$, [35].

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