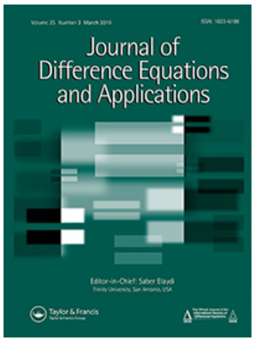

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Asymptotic and boundedness behaviour of a rational difference equation

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ABSTRACT
In this work, we investigate the asymptotic behaviour and examine boundedness of the solutions for the following difference equation

$$x_{n+1} = \frac{\alpha\lambda^{-(nx_n+(n-k)x_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}}, \quad n = 0, 1, 2, \dots \tag{1}$$

where $\lambda \geq 1$ and $\alpha, \beta \geq 0$ and $x_{-k}, x_{-(k-1)}, \dots, x_{-1}, x_0$ are arbitrary numbers.

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1. Introduction

Difference equations are the discrete form of differential equations and they are very important in many applications. For example, the differential equations involved with exponential terms have applications in biology and have interesting properties. The equation

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}}, \quad L_{t+1} = \frac{L_t^2}{L_t + d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}} \tag{2}$$

has oscillatory and chaotic nature and was discussed in [12] where B represents the living biomass, L the litter mass, N the total soil nitrogen, t the time and constants $a,b,c,d > 0$ and $0 < k < 1$.

In [5], El-Metwally et al. studied the global stability, boundedness and periodicity of the positive solution of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}, \quad n = 0, 1, 2, \dots$$

where $\alpha > 0$ and $\beta > 0$ are the immigration rate and population growth, respectively, x_{-1} and x_0 are arbitrary nonnegative numbers.

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Similarly, the boundedness and global asymptotic behaviour of the solution for

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-1}}, \quad n = 0, 1, 2, \ldots$$

and

$$x_{n+1} = \frac{\alpha e^{-(nx_n+(n-k)x_{n-k})}}{\beta + x_n + (n-k)x_{n-k}}, \quad n = 0, 1, 2, \ldots \tag{3}$$

were studied by Ozturk et al. [9,10], where α and β are positive numbers $k \in \{1, 2, 3, \ldots\}$ and the $x_{-k}, x_{-(k-1)}, \ldots, x_{-1}, x_0$ are arbitrary numbers.

Similar properties concerning the biological model

$$x_{n+1} = \frac{ax_n^2}{x_n + b} + c\frac{e^{k-dx_n}}{1 + e^{k-dx_n}}$$

was established in [11], where $0 < a < 1$,b,c,d,k are positive constants and x_0 is a real number. The stability analysis of a nonlinear difference equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, \quad n = 0, 1, 2, \ldots$$

was established in [6], where α, β and initial conditions are arbitrary positive numbers.

Properties of solutions of various types of second and third order several types of difference equations were discussed in [2,7], such as the stability properties and conditions for boundedness of nonlinear difference equation $x_{n+1} = f(x_n)g(x_{n-k})$ was studied in [8] and asymptotic properties of solutions of the difference equation $y_n = (f(y_{n-1}, \ldots, y_{n-k}))/(g(y_{n-1}, \ldots, y_{n-k})), n = 0, 1, 2, \ldots$ was studied in [1].

Motivated by above studies, we generalize (3) by considering $\lambda \geq 1$ and investigate many properties including the asymptotic stability and boundedness of the solutions.

2. Preliminaries

Definition 2.1: [2] Let

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{4}$$

where I is an interval and $f : I^{k+1} \to I$ is a map. Then a solution of (4) is a sequence $\{x_n\}_{n=-k}^{\infty}$ and satisfies (4) for all $n \geq 0$. A solution of (4) if constant for all $n \geq -k$ then it is called an equilibrium solution of (4). Further if $x_n = \bar{x}$, for all $n \geq -k$ is an equilibrium solution then $\bar{x}$ is called an equilibrium point of (4). Note that for discrete case I can also be the subset of integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$. Further

(1) An equilibrium point $\bar{x}$ of (4) is locally stable if given any $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$(|x_{-k} - \bar{x}| + |x_{-(k-1)} - \bar{x}| + \cdots + |x_0 - \bar{x}|) < \delta$$

then $|x_n - \bar{x}| < \epsilon$ for all $n \geq -k$.

(2) The equilibrium $\bar{x}$ of (4) is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that $x_{-k}, x_{-(k-1)}, \ldots, x_0 \in I$ with $(|x_{-k} - \bar{x}| + |x_{-(k-1)} - \bar{x}| + \cdots + |x_0 - \bar{x}|) < \gamma$ then $\lim_{n\to\infty} x_n = \bar{x}$.
(3) The equilibrium $\bar{x}$ of (4) is called a global attractor if for every $x_{-k}, x_{-(k-1)}, \ldots, x_0 \in I$ we have $\lim_{n\to\infty} x_n = \bar{x}$.
(4) The equilibrium $\bar{x}$ of (4) is called globally asymptotically stable if it is locally stable and a global attractor.
(5) The equilibrium $\bar{x}$ of (4) is called unstable if it is not stable.

see [4].

Definition 2.2 ([4]): Consider (4). Then the corresponding linearized equation about $\bar{x}$ is given by

$$y_{n+1} = P_0 y_n + P_1 y_{n-1} + P_2 y_{n-2} + \cdots + P_k y_{n-k} \tag{5}$$

where $P_i = \partial f / \partial x_i(\bar{x}, \bar{x}, \ldots, \bar{x})$. The characteristic equation of (5) is given by

$$\lambda^{k+1} - P_0 \lambda^k - P_1 \lambda^{k-1} - \cdots - P_{k-1}\lambda - P_k = 0. \tag{6}$$

Theorem 2.3 ([3]): *Let $a, b \in \mathbb{R}$ and $k \in \{1, 2, 3, \ldots\}$. Then,*

$$|a| + |b| < 1 \tag{7}$$

is a sufficient condition for the asymptotic stability for (8)

$$y_{n+1} - a y_n + b y_{n-k} = 0, \quad n = 0, 1, 2, \ldots \tag{8}$$

The condition is still valid if k odd and $b < 0$, or k even and $ab < 0$.

Theorem 2.4 ([3]): *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots \tag{9}$$

where $k \in \{1, 2, 3, \ldots\}$. Now assume $f : [a, b] \times [a, b] \to [a, b]$ is a continuous map and having the following properties,

(a) *$f(u, v)$ is non-increasing in each argument.*
(b) *If $(c_1, c_2) \in [a, b]$ is a solution of the system satisfy $c_1 = f(c_2, c_2)$ and $c_2 = f(c_1, c_1)$ then implies that $c_1 = c_2$,*

then (9) *has a unique positive equilibrium which converges to $\bar{x}$.*

3. Main results

In this section, we discuss the stability and boundedness of the solutions for Equation (1).

The equilibrium points of Equation (1) are the solutions of the equation

$$\bar{x} = \frac{\alpha \lambda^{-(2n-k)\bar{x}}}{\beta + (2n-k)\bar{x}}. \tag{10}$$

Set $g(x) = (\alpha\lambda^{-(2n-k)x})/(\beta + (2n-k)x) - x$. Then we get $g(0) = \alpha/\beta > 0$, $\lim_{x\to\infty} g(x) = -\infty$ and

$$g'(x) = \frac{-\alpha(2n-k)\lambda^{-(2n-k)x}\{\ln\lambda[\beta+(2n-k)x]+1\}}{[\beta+(2n-k)x]^2} - 1.$$

This gives us the equilibrium solution for (1).

The associated linearized equation in $\bar{x}$ is given by

$$y_{n+1} + n\bar{x}\left[\ln\lambda + \frac{1}{\beta+(2n-k)\bar{x}}\right]y_n + (n-k)\bar{x}\left[\ln\lambda + \frac{1}{\beta+(2n-k)\bar{x}}\right]y_{n-k} = 0. \tag{11}$$

The characteristic equation of (1) is,

$$\lambda^2 + n\bar{x}\left[\ln\lambda + \frac{1}{\beta+(2n-k)\bar{x}}\right]\lambda + (n-k)\bar{x}\left[\ln\lambda + \frac{1}{\beta+(2n-k)\bar{x}}\right]\lambda^{-k+1} = 0. \tag{12}$$

Theorem 3.1: (1) *Let $n > k$. The positive equilibrium point of* (1) *is locally asymptotically stable if,*

$$\bar{x} \in \left(0, \frac{-\beta\ln\lambda + \sqrt{\beta^2(\ln\lambda)^2 + 4\beta\ln\lambda}}{2(2n-k)\ln\lambda}\right). \tag{13}$$

(2) *Let $k > 2n$ and $\beta - (k-2n)\bar{x} > 0$. The positive equilibrium point $\bar{x}$ is locally asymptotically stable if*

$$\bar{x} \in \left(0, \frac{[2(k-n)+k\beta\ln\lambda] - \sqrt{[2(k-n)+k\beta\ln\lambda]^2 - 4\beta k\ln\lambda(k-2n)}}{2k(k-2n)\ln\lambda}\right). \tag{14}$$

Proof: (1) Using Theorem 2.3, we have

$$\left|-n\bar{x}\left[\ln\lambda + \frac{1}{\beta+(2n-k)\bar{x}}\right]\right| + \left|(n-k)\bar{x}\left[\ln\lambda + \frac{1}{\beta+(2n-k)\bar{x}}\right]\right| < 1. \tag{15}$$

$$\Rightarrow (2n-k)^2(\ln\lambda)(\bar{x})^2 + [(2n-k)\beta\ln\lambda]\bar{x} - \beta < 0. \tag{16}$$

Since $\bar{x}$ is positive, $\bar{x}$ satisfies the equation (16) if,

$$\bar{x} \in \left(0, \frac{-\beta\ln\lambda + \sqrt{\beta^2(\ln\lambda)^2 + 4\beta\ln\lambda}}{2(2n-k)\ln\lambda}\right).$$

If k is odd, the asymptotic stability condition does not hold.

Now let k be even. By Theorem 2.3 and since

$$\left[-nx\left(\ln\lambda + \frac{1}{\beta-(k-2n)\bar{x}}\right)\right]\left[(n-k)\bar{x}\left(\ln\lambda + \frac{1}{\beta-(k-2n)\bar{x}}\right)\right] < 0$$

(15) is a necessary condition for the asymptotic stability of (11).

(2) Since $k > 2n$, using Theorem 2.3 and equation (11) we get,

$$\left| -n\bar{x}\left(\ln\lambda + \frac{1}{\beta-(k-2n)\bar{x}}\right)\right| + \left| -(k-n)\bar{x}\left(\ln\lambda + \frac{1}{\beta-(k-2n)\bar{x}}\right)\right| < 1. \tag{17}$$

Using the hypothesis $\beta-(k-2n)\bar{x} > 0$, the inequality (17) gives us,

$$(k-2n)k(\ln\lambda)(\bar{x})^2 - [2(k-n)+k\beta\ln\lambda]\,\bar{x} + \beta > 0. \tag{18}$$

Since $\bar{x} > 0$ and $\beta-(k-2n)\bar{x} > 0$, (18) holds if

$$\bar{x} \in \left(0, \frac{[2(k-n)+k\beta\ln\lambda] - \sqrt{[2(k-n)+k\beta\ln\lambda]^2 - 4\beta k(k-2n)\ln\lambda}}{2k(k-2n)\ln\lambda}\right).$$

By Theorem 2.3, we get the following results.

(a) If k is odd, since

$$-(k-n)\bar{x}\left(\ln\lambda + \frac{1}{\beta-(k-2n)\bar{x}}\right) < 0$$

then (17) holds as the necessary condition for the asymptotic stability.

(b) Now if k is even and since

$$\left(-n\bar{x}\left(\ln\lambda + \frac{1}{\beta-(k-2n)\bar{x}}\right)\right)\left(-(k-n)\bar{x}\left(1 + \frac{1}{\beta-(k-2n)\bar{x}}\right)\right) > 0$$

then (17) does not hold as the necessary condition for asymptotic stability.

■

Example 3.2: (a) Taking $n=2$, $k=1$ and for $\alpha=3, \beta=5, \lambda=2$, the equilibrium point of (1) is 0.284008. By (13), $\bar{x} \in (0, 0.38975)$ and by computation we see that it is locally asymptotically stable.

(b) Suppose let $\beta=1$ and n, k, α, λ as said in (a), then $\bar{x} = 0.46939$ which does not belongs to $(0, 0.26701)$. Then by computation we see that the equilibrium point is unstable.

Example 3.3: Taking $n=1, k=3, \alpha=300, \beta=20,000$ and $\lambda=25$, we get two equilibrium points 0.0157817 and 1.41184. By (14), we see that 0.0157817 is in $(0, 0.10355)$ and it is locally asymptotically stable but 0.141184 does not belongs to $(0, 0.10355)$ and by computation we see that it is unstable.

Theorem 3.4: *Let $n=k$ and $\beta + k\bar{x} > 0$. The equilibrium point of* (1) *is locally asymptotically stable if and only if*

$$\bar{x} \in \left(-\frac{\beta}{k}, \frac{-[2+\beta\ln\lambda] + \sqrt{\beta^2(\ln\lambda^2)+4}}{2k\ln\lambda}\right). \tag{19}$$

Proof: Using Theorem 2.3, we can write (11) in the form

$$y_{n+1} - \left(-k\bar{x}\left[\ln\lambda + \frac{1}{\beta + k\bar{x}}\right]\right) y_n = 0, \quad n = 0, 1, 2, \dots \tag{20}$$

and from Theorem 2.3 we get

$$\begin{aligned}
&\left|-k\bar{x}\left[\ln\lambda + \frac{1}{\beta + k\bar{x}}\right]\right| < 1 \\
&\Rightarrow -1 < -k\bar{x}\left(\ln\lambda + \frac{1}{\beta + k\bar{x}}\right) < 1 \\
&\Rightarrow k^2(\ln\lambda)(\bar{x})^2 + k\beta(\ln\lambda)\bar{x} - \beta < 0 \\
\text{and} \quad & k^2(\ln\lambda)(\bar{x})^2 + (2 + \beta\ln\lambda)k\bar{x} + \beta > 0.
\end{aligned}$$

Both the inequalities must hold for $\bar{x}$. Furthermore, from $\beta + k\bar{x} > 0$, we get $\bar{x} > \frac{-\beta}{k}$. This gives us the interval

$$\bar{x} \in \left(\frac{-\beta}{k}, \frac{-[2 + \beta\ln\lambda] + \sqrt{\beta^2(\ln\lambda)^2 + 4}}{2k\ln\lambda}\right).$$

Here k has no important role for proving asymptotic stability since the characteristic roots of (20) lie in the unit disk $|\delta| < 1$ with the condition (19). ■

Example 3.5: When $n = k = 2$ and $\alpha = 30, \beta = 10, \lambda = 20$, we see that the equilibrium point 0.348204 does not satisfy (19) and by computation we see that the equilibrium point is unstable.

Theorem 3.6: *Let $n \geq k$ and $\{x_n\}_{n=1}^{\infty}$ be a positive solution of* (1).

(i) *Then every solution of* (1) *is bounded.*
(ii) *Then the equilibrium point of* (1) *is bounded if $\bar{x} > 0$.*

Proof: (i) Since $x_n > 0$ for all n and $\lambda \geq 1$ we have

$$0 < x_{n+1} = \frac{\alpha\lambda^{-(nx_n + (n-k)x_{n-k})}}{\beta + nx_n + (n-k)x_{n-k}} < \frac{\alpha}{\beta}.$$

Therefore, every positive solution is bounded.

(ii) Assume that $\bar{x} > 0$ and $\lambda \geq 1$. Then

$$0 < \bar{x} = \frac{\alpha\lambda^{-(2n-k)\bar{x}}}{\beta + (2n-k)\bar{x}} < \frac{\alpha}{\beta}.$$

Therefore, the equilibrium point is bounded. ■

Example 3.7: Let α, β, λ be as in Example 3.2 (a) and let $k = n - 1$. When $x_{-1} = 0.3$ and $x_0 = 0.5$, we see that every solution of (1) is less than 0.6.

Lemma 3.8: *Let* $n \geq k$ *and* $f(u,v) = (\alpha\lambda^{-(nu+(n-k)v)})/(\beta + nu + (n-k)v),\ n = 1, 2, \ldots$ *The following conditions holds.*

(i) *In* $[0, \infty)$ *the function* $f(x,x)$ *is decreasing.*
(ii) $f(u,v)$ *is decreasing for each* $u, v \in [0, \infty)$.

Proof: (i) We have,

$$f(x,x) = \frac{\alpha\lambda^{-(2n-k)x}}{\beta + (2n-k)x}$$
$$f'(x,x) = \frac{-(2n-k)\alpha\lambda^{-(2n-k)x}[\beta\ln\lambda + (2n-k)x\ln\lambda + 1]}{[\beta + (2n-k)x]^2}.$$

So (i) is trivial.
(ii) The proof is obvious.

■

Theorem 3.9: *Let* $n > k$ *and* $\beta > \alpha$. *If positive equilibrium point is locally asymptotically stable then it is globally asymptotically stable.*

Proof: Let us have $\mu = \lim_{n\to\infty} \inf x_n$, $M = \lim_{n\to\infty} \sup x_n$ and $\epsilon > 0$ such that $\epsilon < \min\{\frac{\alpha}{\beta} - M, \mu\}$. There exists $n_0 \in \mathbb{N}$ such that $\mu - \epsilon \leq x_n \leq M + \epsilon$. Using part (i) of Lemma 3.8, we have

$$\frac{\alpha\lambda^{-(2n-k)(M+\epsilon)}}{\beta + (2n-k)(M+\epsilon)} \leq x_n \leq \frac{\alpha\lambda^{-(2n-k)(\mu-\epsilon)}}{\beta + (2n-k)(\mu+\epsilon)}, \quad n \geq n_0 + 1$$
$$\Rightarrow \frac{\alpha\lambda^{-(2n-k)(M+\epsilon)}}{\beta + (2n-k)(M+\epsilon)} \leq \mu \leq M \leq \frac{\alpha\lambda^{-(2n-k)(\mu-\epsilon)}}{\beta + (2n-k)(\mu+\epsilon)}.$$

Since $\epsilon > 0$ is arbitrary, this inequality yields

$$\frac{\alpha\lambda^{-(2n-k)M}}{\beta + (2n-k)M} \leq \mu \leq M \leq \frac{\alpha\lambda^{-(2n-k)\mu}}{\beta + (2n-k)\mu}$$
$$\Rightarrow \alpha\lambda^{-(2n-k)M} - \beta\mu \leq \alpha\lambda^{-(2n-k)\mu} - \beta M$$
$$\Rightarrow \beta(M - \mu) \leq \alpha(\lambda^{-(2n-k)\mu} - \lambda^{-(2n-k)M}).$$

From our assumption $M, \mu > 0$ and from the hypothesis we arrive at $M \leq \mu$.

Hence $M = \mu = \bar{x}$. From Theorem 2.4, (1) has a unique equilibrium point and every positive solution of (1) converges to $\bar{x}$. The proof of the theorem is complete. ■

Example 3.10: It is clear from Example 3.2 that the equilibrium solution is also globally asymptotically stable but in Example 3.3 the equilibrium point 0.0157817. This example shows us that the locally asymptotically stability need not necessarily be globally asymptotically stable.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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