Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 3, 399–410
DOI: 10.7546/nntdm.2022.28.3.399-410

On linear algebra of one type of symmetric matrices with harmonic Fibonacci entries

Mücahit Akbıyık[1], Seda Yamaç Akbıyık[2] and Fatih Yılmaz[3]

[1] Department of Mathematics, Beykent University
Istanbul, Turkey
e-mail: mucahitakbiyik@beykent.edu.tr

[2] Department of Computer Engineering, Istanbul Gelisim University
Istanbul, Turkey
e-mail: syamac@gelisim.edu.tr

[3] Department of Mathematics, Ankara Hacı Bayram Veli University
Ankara, Turkey
e-mail: fatih.yilmaz@hbv.edu.tr

Received: 17 February 2022 **Revised:** 1 July 2022
Accepted: 8 July 2022 **Online First:** 9 July 2022

Abstract: This paper focuses on a specially constructed matrix whose entries are harmonic Fibonacci numbers and considers its Hadamard exponential matrix. A lot of admiring algebraic properties are presented for both of them. Some of them are determinant, inverse in usual and in the Hadamard sense, permanents, some norms, etc. Additionally, a MATLAB-R2016a code is given to facilitate the calculations and to further enrich the content.

Keywords: Harmonic Fibonacci numbers, Norm, Determinant, Permanent.
2020 Mathematics Subject Classification: 11B39, 15A09, 15A15, 65F35.

1 Introduction

The n-th harmonic number H_n has the usual definition

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \sum_{j=1}^{\infty} \frac{n}{j(j+n)} \qquad (H_0 = 0)$$

for $n \in \mathbb{N}$. It can be described as a sum of the areas of certain rectangles. In other words, each term of the sum can be interpreted as the area of a rectangle with width equal to 1 and height equal to $\frac{1}{k}$. The Fibonacci numbers have the following recurrence relation for $n \geq 0$:

$$F_{n+2} = F_{n+1} + F_n$$

with $F_0 = 0$, $F_1 = 1$. Inspired by the definition of the harmonic number and Fibonacci number, Tuğlu et al. introduced in [11] harmonic Fibonacci numbers as below:

$$\mathbb{F}_n = \sum_{k=1}^{n} \frac{1}{F_k}$$

and gave various identities for these numbers by using the difference operator. They also studied the theory of the harmonic and the hyperharmonic Fibonacci numbers and got some combinatoric identities. Moreover, the authors obtained in [10] norms of some circulant matrices and some special matrices, whose entries consist of harmonic Fibonacci numbers.

Assume that $A = (a_{ij})$ is an $n \times n$ matrix. The maximum column norm is found by

$$||A||_{c_1} = \max_j \sqrt{\sum_i |a_{ij}|^2} \tag{1.1}$$

and the maximum row norm is calculated by

$$||A||_{r_1} = \max_i \sqrt{\sum_j |a_{ij}|^2}. \tag{1.2}$$

The Frobenius or Euclidean norm of A is found by

$$\|A\|_{\mathbb{E}} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}. \tag{1.3}$$

Let A^H be the conjugate transpose of matrix A, then the spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i}, \tag{1.4}$$

where λ_i is the eigenvalue of matrix AA^H. The following well known inequality can be written

$$\frac{1}{\sqrt{n}} \|A\|_{\mathbb{E}} \leq \|A\|_2 \leq \|A\|_{\mathbb{E}}. \tag{1.5}$$

In the literature, there are many papers dealing with some kind of matrices and some type of norms, (see [1–3, 6, 7, 9, 12, 13]). Suppose that $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are $m \times n$ matrices. The Hadamard product of A and B is defined by $A \circ B = (a_{ij} b_{ij})$. If $C = A \circ B$, then there exists the following relation

$$\|C\|_2 \leq ||A||_{r_1} ||B||_{c_1}. \tag{1.6}$$

The Hadamard exponential of the matrix $A = (a_{ij})_{m \times n}$ is defined by $e^{\circ A} = (e^{a_{ij}})$, [8]. The permanent of an $n \times n$ matrix A is defined as:

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where the sum here extends over all elements σ of the symmetric group S_n over all permutations of the numbers $1, 2, \ldots, n$. The matrices whose permanents arise in various applications often have special structure. Usually these are nonnegative matrices whose nonzero entries are distributed regularly over planes. In this content, Brualdi et al. defined in [4] a new method which is called *contraction method* to calculate the permanents of the matrices. The authors in [7] considered a square matrix M as below:

$$M = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}, \tag{1.7}$$

where A is an $n \times n$ nonsigular matrix and b is an $n \times 1$ matrix, also c is a real number. Then, they obtained the inverse of M as following

$$N = \begin{bmatrix} A^{-1} + \frac{1}{l} A^{-1} b b^T A^{-1} & -\frac{1}{l} A^{-1} b \\ -\frac{1}{l} b^T A^{-1} & c \end{bmatrix},$$

where $l = c - b^T A^{-1} b$.

In this paper, we consider an $n \times n$ matrix $\mathfrak{F} = [\mathbb{F}_{k_{ij}}]_{i,j=1}^{n}$ and its Hadamard exponential matrix $e^{\circ \mathfrak{F}} = [e^{\mathbb{F}_k}]$, where $k_{i,j} = \min(i,j)$ and $\mathbb{F}_n$ is the n-th harmonic Fibonacci number. In other words, these matrices are represented as below:

$$\mathfrak{F} = \begin{bmatrix} \mathbb{F}_1 & \mathbb{F}_1 & \mathbb{F}_1 & \cdots & \mathbb{F}_1 \\ \mathbb{F}_1 & \mathbb{F}_2 & \mathbb{F}_2 & \cdots & \mathbb{F}_2 \\ \mathbb{F}_1 & \mathbb{F}_2 & \mathbb{F}_3 & \cdots & \mathbb{F}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{F}_1 & \mathbb{F}_2 & \mathbb{F}_3 & \cdots & \mathbb{F}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & \frac{5}{2} & \cdots & \frac{5}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \frac{5}{2} & \cdots & \sum_{k=1}^{n} \frac{1}{F_k} \end{bmatrix}, \tag{1.8}$$

and

$$e^{\circ \mathfrak{F}} = [e^{\mathbb{F}_{\min(i,j)+1}}]_{i,j=1}^{n} = \begin{bmatrix} e^{\mathbb{F}_1} & e^{\mathbb{F}_1} & e^{\mathbb{F}_1} & \cdots & e^{\mathbb{F}_1} \\ e^{\mathbb{F}_1} & e^{\mathbb{F}_2} & e^{\mathbb{F}_2} & \cdots & e^{\mathbb{F}_2} \\ e^{\mathbb{F}_1} & e^{\mathbb{F}_2} & e^{\mathbb{F}_3} & \cdots & e^{\mathbb{F}_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{\mathbb{F}_1} & e^{\mathbb{F}_2} & e^{\mathbb{F}_3} & \cdots & e^{\mathbb{F}_n} \end{bmatrix}, \tag{1.9}$$

where $\mathbb{F}_k$ is the k-th harmonic Fibonacci number. Some algebraic properties of these matrices are investigated such as determinant, inversion, some norms and permanents. Moreover, some illustrative examples are given and a MATLAB-R2016a code is presented.

2 Main results

Theorem 2.1. *Let $\mathfrak{F}$ be an n by n matrix as in the matrix from (1.8), then*

$$\det(\mathfrak{F}) = \frac{1}{F_1 F_2 \ldots F_n}.$$

Proof. By using elementary row operations on the matrix (1.8), we calculate:

$$\det(\mathfrak{F}) = \det \begin{bmatrix} \mathbb{F}_1 & \mathbb{F}_1 & \mathbb{F}_1 & \cdots & \mathbb{F}_1 \\ 0 & \mathbb{F}_2 - \mathbb{F}_1 & \mathbb{F}_2 - \mathbb{F}_1 & \cdots & \mathbb{F}_2 - \mathbb{F}_1 \\ 0 & 0 & \mathbb{F}_3 - \mathbb{F}_2 & \cdots & \mathbb{F}_3 - \mathbb{F}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbb{F}_n - \mathbb{F}_{n-1} \end{bmatrix}.$$

So, we have

$$\det(\mathfrak{F}) = \mathbb{F}_1 \prod_{i=2}^{n} (\mathbb{F}_{i+1} - \mathbb{F}_i) = \prod_{i=1}^{n} \frac{1}{\mathbb{F}_i} = \frac{1}{F_1 F_2 ... F_n}.$$ □

Corollary 2.1. *Suppose that $\mathfrak{F}$ is a matrix as in (1.8) and the leading principal minor of $\mathfrak{F}$ is denoted by Δ_n, then we have*

i. $\Delta_n = \frac{1}{F_n} \Delta_{n-1}$,

ii. $\Delta_1 \Delta_2 \Delta_3 \cdots \Delta_n = \frac{1}{F_1^n F_2^{n-1} ... F_{n-1}^2 F_n}$.

Proof. It can be calculated by using the Theorem 2.1. □

Note that the matrix $\mathfrak{F}$ is a positive definite matrix and all eigenvalues of $\mathfrak{F}$ are positive.

Theorem 2.2. *Assume that $\mathfrak{F}$ is a matrix as in the matrix (1.8). The inverse of $\mathfrak{F}$ is*

$$\mathfrak{F}^{-1} = \begin{bmatrix} F_3 & -F_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -F_1 & F_4 & -F_3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -F_3 & F_5 & -F_4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -F_4 & F_6 & -F_5 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -F_{n-1} & F_{n+1} & -F_n \\ 0 & 0 & 0 & \cdots & 0 & 0 & -F_n & F_n \end{bmatrix}.$$

Proof. The inverse of the matrix $\mathfrak{F}$ can be calculated by Principle Mathematical Induction, on n. It verifies for $n = 2$, i.e.: if

$$\mathfrak{F} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

then we find

$$\mathfrak{F}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Assume that the result provides for n, that is $A = \mathfrak{F} = [\mathbb{F}_{k_{ij}}]_{n \times n}$, $A^{-1} = [\mathbb{F}_{k_{ij}}]_{n \times n}^{-1}$.

So, we have $b = (\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_n)^T$, $\quad b^T = (\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_n)$. By taking $c = \mathbb{F}_{n+1}$ and by the help of the equation (1.7), it verifies for $n + 1$. □

Definition 1. *Let us define a second order recurrence relation, for $i \geq 2$, as below:*

$$w^{[i]} = w^{[i-1]} F_{i+5} + w^{[i-2]} F_{i+3}^2$$

where $w^{[1]} = w^{[0]} F_6 + A F_4^2$, $w^{[0]} = A F_5 + F_3^3$ and $A = F_3 F_4 + 1$. Here, F_n is the n-th Fibonacci number.

The first few values of the sequence can be given as following:

$$\begin{array}{rcl} w^{[1]} & = & w^{[0]}F_6 + AF_4^2 \\ w^{[2]} & = & w^{[1]}F_7 + w^{[0]}F_5^2 \\ w^{[3]} & = & w^{[2]}F_8 + w^{[1]}F_6^2 \\ & & \vdots \end{array}$$

where $i = 2, 3, 4, \ldots$.

Here, we construct a new recurrence relation whose permanents are related to inverse of the matrix $\mathfrak{F}$.

Theorem 2.3. *The permanents of the matrix $\mathfrak{F}^{-1}$ are:*

$$\operatorname{per}(\mathfrak{F}^{-1}) = F_n(w^{[n-4]}F_{n-1}^2 + w^{[n-5]}F_{n+2}).$$

Proof. Let us consider $A = F_3F_4+1$ and $w^{[0]} = AF_5+F_3^3$, then by using the contraction method, we get:

$$\left[\mathfrak{F}^{-1}\right]^{(1)} = \begin{bmatrix} A & -F_3 & & & & & \\ -F_3 & -F_5 & -F_4 & & & & \\ & -F_4 & -F_6 & \ddots & & & \\ & & \ddots & \ddots & -F_{n-1} & & \\ & & & -F_{n-1} & -F_{n+1} & -F_n \\ & & & & -F_n & F_n \end{bmatrix}_{(n-1)\times(n-1)}$$

and going on with this method, we obtain:

$$\left[\mathfrak{F}^{-1}\right]^{(2)} = \begin{bmatrix} w^{[0]} & -AF_4 & & & \\ -F_4 & -F_6 & \ddots & & \\ & \ddots & \ddots & -F_{n-1} & \\ & & -F_{n-1} & -F_{n+1} & -F_n \\ & & & -F_n & F_n \end{bmatrix}_{(n-2)\times(n-2)}$$

and we can mention these steps with a general statement, for $n-2 > r \geq 3$, as below:

$$\left[\mathfrak{F}^{-1}\right]^{(r)} = \begin{bmatrix} w^{[r-2]} & -w^{[r-3]}F_{r+2} & & & \\ -F_{r+2} & -F_{r+4} & -F_{r+3} & & \\ & \ddots & \ddots & \ddots & \\ & & -F_{n-1} & -F_{n+1} & -F_n \\ & & & -F_n & F_n \end{bmatrix}_{(n-r)\times(n-r)}$$

Consequently, we get

$$\left[\mathfrak{F}^{-1}\right]^{(n-2)} = \begin{bmatrix} w^{[n-4]} & -w^{[n-5]}F_n \\ -F_n & F_n \end{bmatrix} = F_n(w^{[n-4]}F_{n-1}^2 + w^{[n-5]}F_{n+2})$$

which is desired. □

Theorem 2.4. *Assume that $\mathfrak{F}$ is a matrix as in (1.8). Then, the determinant of Hadamard inverse of $\mathfrak{F}$ is*

$$\det(\mathfrak{F}^{\circ(-1)}) = \frac{(-1)^{n-1}}{F_n\mathbb{F}_n\prod_{i=1}^{n-1}F_i\mathbb{F}_i^2}.$$

Proof. We can write

$$\mathfrak{F}^{\circ(-1)} = \begin{bmatrix} \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_1} & \cdots & \frac{1}{\mathbb{F}_1} \\ \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_2} & \frac{1}{\mathbb{F}_2} & \cdots & \frac{1}{\mathbb{F}_2} \\ \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_2} & \frac{1}{\mathbb{F}_3} & \cdots & \frac{1}{\mathbb{F}_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_2} & \frac{1}{\mathbb{F}_3} & \cdots & \frac{1}{\mathbb{F}_n} \end{bmatrix}.$$

By the elementary row operations, we have

$$\det(\mathfrak{F}^{\circ(-1)}) = \det\begin{bmatrix} \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_1} & \cdots & \frac{1}{\mathbb{F}_1} \\ 0 & \frac{1}{\mathbb{F}_2}-\frac{1}{\mathbb{F}_1} & \frac{1}{\mathbb{F}_2}-\frac{1}{\mathbb{F}_1} & \cdots & \frac{1}{\mathbb{F}_2}-\frac{1}{\mathbb{F}_1} \\ 0 & 0 & \frac{1}{\mathbb{F}_3}-\frac{1}{\mathbb{F}_2} & \cdots & \frac{1}{\mathbb{F}_3}-\frac{1}{\mathbb{F}_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\mathbb{F}_n}-\frac{1}{\mathbb{F}_{n-1}} \end{bmatrix}.$$

Thus, we obtain

$$\det(\mathfrak{F}^{\circ(-1)}) = \frac{1}{\mathbb{F}_1}\prod_{k=2}^{n}\left(\frac{1}{\mathbb{F}_k}-\frac{1}{\mathbb{F}_{k-1}}\right) = \frac{1}{\mathbb{F}_1}\prod_{k=2}^{n}\frac{-1}{F_k\mathbb{F}_k\mathbb{F}_{k-1}} = \frac{(-1)^{n-1}}{F_1F_2...F_n}\frac{1}{\mathbb{F}_1^2\mathbb{F}_2^2...\mathbb{F}_{n-1}^2\mathbb{F}_n}. \qquad \square$$

Theorem 2.5. *If $\mathfrak{F}$ is a matrix which is given in (1.8), then*

$$\|\mathfrak{F}\|_{\mathbb{E}} = \sqrt{(n+1)^2\mathbb{F}_{n+1}^2-(2n+1)+\sum_{k=1}^{n}\frac{(k+1)(k-(2n+1))}{F_{k+1}}(\mathbb{F}_k+\mathbb{F}_{k+1})}.$$

Proof. The Euclidean norm of $\mathfrak{F}$ is

$$\|\mathfrak{F}\|_{\mathbb{E}}^2 = \left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|\mathbb{F}_{ij}|^2\right)^{\frac{1}{2}}\right]^2.$$

Thus,

$$\|\mathfrak{F}\|_{\mathbb{E}}^2 = \sum_{k=1}^{n}(2n-2k+1)\mathbb{F}_k^2 = (2n+1)\sum_{k=1}^{n}\mathbb{F}_k^2 - 2\sum_{k=1}^{n}k\mathbb{F}_k^2.$$

Also, help of the reference [5], we get

$$\sum_{k=1}^{n}\mathbb{F}_k^2 = \mathbb{F}_{n+1}^2(n+1)-1-\sum_{k=1}^{n}\frac{(k+1)}{F_{k+1}}(\mathbb{F}_k+\mathbb{F}_{k+1}), \tag{2.1}$$

and

$$\sum_{k=1}^{n}k\mathbb{F}_k^2 = \mathbb{F}_{n+1}^2\frac{n(n+1)}{2}-\sum_{k=1}^{n}\frac{(k+1)k}{2F_{k+1}}(\mathbb{F}_k+\mathbb{F}_{k+1}). \tag{2.2}$$

According to (2.1) and (2.2), we obtain

$$\|\mathfrak{F}\|_{\mathbb{E}} = \sqrt{(n+1)^2\mathbb{F}_{n+1}^2 - (2n+1) + \sum_{k=1}^{n} \frac{(k+1)(k-(2n+1))}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})}. \qquad \square$$

Corollary 2.2. *Let $\mathfrak{F}$ be a matrix as in the matrix form (1.8) Then,*

$$\frac{1}{\sqrt{n}}\sqrt{(n+1)^2\mathbb{F}_{n+1}^2 - (2n+1) + \sum_{k=1}^{n} \frac{(k+1)(k-(2n+1))}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})} \leq \|\mathfrak{F}\|_2$$

$$\leq \sqrt{(n+1)^2\mathbb{F}_{n+1}^2 - (2n+1) + \sum_{k=1}^{n} \frac{(k+1)(k-(2n+1))}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})}.$$

Proof. The proof can be seen easily by using Theorem 2.5 and the inequality (1.5). $\square$

Theorem 2.6. *Let $\mathfrak{F}$ be a matrix as in the matrix form (1.8), then*

$$\|\mathfrak{F}\|_2 \leq \sqrt{\left[(n+1)\mathbb{F}_{n+1}^2 - 1 - \sum_{k=1}^{n} \frac{(k+1)}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})\right]\left[n\mathbb{F}_n^2 - \sum_{k=1}^{n-1} \frac{(k+1)}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})\right]}.$$

Proof. We can write

$$\mathfrak{F} = A \circ B,$$

where

$$A = \begin{bmatrix} \mathbb{F}_1 & 1 & 1 & \cdots & 1 \\ \mathbb{F}_1 & \mathbb{F}_2 & 1 & \cdots & 1 \\ \mathbb{F}_1 & \mathbb{F}_2 & \mathbb{F}_3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{F}_1 & \mathbb{F}_2 & \mathbb{F}_3 & \cdots & \mathbb{F}_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & \mathbb{F}_1 & \mathbb{F}_1 & \cdots & \mathbb{F}_1 \\ 1 & 1 & \mathbb{F}_2 & \cdots & \mathbb{F}_2 \\ 1 & 1 & 1 & \cdots & \mathbb{F}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

So, we have

$$r_1(A) = \sqrt{\sum_{i=1}^{n} \mathbb{F}_i^2} = \sqrt{(n+1)\mathbb{F}_{n+1}^2 - 1 - \sum_{k=1}^{n} \frac{(k+1)}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})},$$

$$c_1(B) = \sqrt{\sum_{i=1}^{n-1} \mathbb{F}_i^2 + 1} = \sqrt{n\mathbb{F}_n^2 - \sum_{k=1}^{n-1} \frac{(k+1)}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})}.$$

Consequently, we obtain

$$\|\mathfrak{F}\|_2 \leq \sqrt{\left[(n+1)\mathbb{F}_{n+1}^2 - 1 - \sum_{k=1}^{n} \frac{(k+1)}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})\right]\left[n\mathbb{F}_n^2 - \sum_{k=1}^{n-1} \frac{(k+1)}{F_{k+1}}(\mathbb{F}_k + \mathbb{F}_{k+1})\right]}. \qquad \square$$

Theorem 2.7. *Assume that $e^{\circ\mathfrak{F}}$ is a matrix as in the matrix form* (1.9)*, then*

$$\det(e^{\circ\mathfrak{F}}) = e^{\mathbb{F}_1}\prod_{k=2}^{n}(e^{\mathbb{F}_k} - e^{\mathbb{F}_{k-1}}).$$

Proof. The proof can be done easily by the elementary row operations. ☐

Theorem 2.8. *Let $e^{\circ\mathfrak{F}}$ be a matrix as in (1.9) and Δ_n denotes the leading principal minor of $e^{\circ\mathfrak{F}}$, then we have for $n > 1$*

i. $\Delta_n = (e^{\mathbb{F}_n} - e^{\mathbb{F}_{n-1}})\Delta_{n-1}$,

ii. $\Delta_1\Delta_2\Delta_3\cdots\Delta_n = (e^{\mathbb{F}_1})^n(e^{\mathbb{F}_2} - e^{\mathbb{F}_1})^{n-1}(e^{\mathbb{F}_3} - e^{\mathbb{F}_2})^{n-2}...(e^{\mathbb{F}_n} - e^{\mathbb{F}_{n-1}})$.

Proof. It can be easy calculated by the help of Theorem 2.7 and the followings:

$$\begin{aligned}
\Delta_1 &= e,\\
\Delta_2 &= e(e^2 - e),\\
\Delta_3 &= e(e^2 - e)(e^{5/2} - e^2)\\
&\vdots\\
\Delta_n &= e^{\mathbb{F}_1}(e^{\mathbb{F}_2} - e^{\mathbb{F}_1})(e^{\mathbb{F}_3} - e^{\mathbb{F}_2})\dots(e^{\mathbb{F}_n} - e^{\mathbb{F}_{n-1}}).
\end{aligned}$$

☐

Note that the matrix $e^{\circ\mathfrak{F}}$ is a positive definite matrix and all eigenvalues of $e^{\circ\mathfrak{F}}$ are positive.

Theorem 2.9. *Suppose that $e^{\circ\mathfrak{F}}$ is a matrix as in the matrix form* (1.9)*, then:*

$$(e^{\circ\mathfrak{F}})^{-1} = \begin{bmatrix}
\frac{1}{e^{\mathbb{F}_1-1}} & \frac{1}{e^{\mathbb{F}_1}-e^{\mathbb{F}_2}} & 0 & \cdots & 0 & 0 & 0\\
\frac{1}{e^{\mathbb{F}_1}-e^{\mathbb{F}_2}} & A & \frac{1}{e^{\mathbb{F}_2}-e^{\mathbb{F}_3}} & \cdots & 0 & 0 & 0\\
0 & \frac{1}{e^{\mathbb{F}_2}-e^{\mathbb{F}_3}} & \ddots & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \ddots & \frac{1}{e^{\mathbb{F}_{n-2}}-e^{\mathbb{F}_{n-1}}} & 0\\
0 & 0 & 0 & \cdots & \frac{1}{e^{\mathbb{F}_{n-2}}-e^{\mathbb{F}_{n-1}}} & B & \frac{1}{e^{\mathbb{F}_{n-1}}-e^{\mathbb{F}_n}}\\
0 & 0 & 0 & \cdots & 0 & \frac{1}{e^{\mathbb{F}_{n-1}}-e^{\mathbb{F}_n}} & \frac{-1}{e^{\mathbb{F}_{n-1}}-e^{\mathbb{F}_n}}
\end{bmatrix},$$

where

$$A = (1 - \frac{e^{\mathbb{F}_1}}{e^{\mathbb{F}_1} - e^{\mathbb{F}_2}} - \frac{e^{\mathbb{F}_2}}{e^{\mathbb{F}_2} - e^{\mathbb{F}_3}})\frac{1}{e^{\mathbb{F}_2}},$$

and

$$B = (1 - \frac{e^{\mathbb{F}_{n-2}}}{e^{\mathbb{F}_{n-2}} - e^{\mathbb{F}_{n-1}}} - \frac{e^{\mathbb{F}_{n-1}}}{e^{\mathbb{F}_{n-1}} - e^{\mathbb{F}_n}})\frac{1}{e^{\mathbb{F}_{n-1}}}.$$

Proof. It can be proven in a similar way in the proof of Theorem 2.2. ☐

Theorem 2.10. *Assume that $e^{\mathfrak{F}}$ is a matrix as in (1.9). Then,*

$$\det(e^{\mathfrak{F}\circ(-1)}) = \frac{1}{e^{\mathbb{F}_1}}\prod_{k=2}^{n}\frac{1}{e^{\mathbb{F}_{k-1}}}\frac{(1 - e^{\frac{1}{F_k}})}{e^{\frac{1}{F_k}}}.$$

Proof. By the definition of the Hadamard inverse, we get

$$e^{\mathfrak{F}\circ(-1)} = \begin{bmatrix} \frac{1}{e^{\mathbb{F}_1}} & \frac{1}{e^{\mathbb{F}_1}} & \frac{1}{e^{e\mathbb{F}_1}} & \cdots & \frac{1}{e^{\mathbb{F}_1}} \\ \frac{1}{e^{\mathbb{F}_1}} & \frac{1}{e^{\mathbb{F}_2}} & \frac{1}{e^{\mathbb{F}_2}} & \cdots & \frac{1}{e^{\mathbb{F}_2}} \\ \frac{1}{e^{\mathbb{F}_1}} & \frac{1}{e^{\mathbb{F}_2}} & \frac{1}{\mathbb{F}_3} & \cdots & \frac{1}{e^{\mathbb{F}_3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{e^{\mathbb{F}_1}} & \frac{1}{e^{\mathbb{F}_2}} & \frac{1}{\mathbb{F}_3} & \cdots & \frac{1}{e^{\mathbb{F}_n}} \end{bmatrix}.$$

By the elementary row operations,

$$\det(e^{\mathfrak{F}\circ(-1)}) = \frac{1}{e^{\mathbb{F}_1}} \prod_{k=2}^{n} (\frac{1}{e^{\mathbb{F}_k}} - \frac{1}{e^{\mathbb{F}_{k-1}}}) = \frac{1}{e^{\mathbb{F}_1}} \prod_{k=2}^{n} \frac{1}{e^{\mathbb{F}_{k-1}}} \frac{(1 - e^{\frac{1}{F_k}})}{e^{\frac{1}{F_k}}}.$$ □

Theorem 2.11. *Assume that $e^{\circ\mathfrak{F}}$ is a matrix which is given in (1.9). Then,*

$$\|e^{\circ\mathfrak{F}}\|_{\mathbb{E}} = \sqrt{(2n+1)\sum_{k=1}^{n} e^{2\mathbb{F}_k} - 2\sum_{k=1}^{n} k e^{2\mathbb{F}_k}}.$$

Proof. The Euclidean norm of $e^{\circ\mathfrak{F}}$ can be written as

$$\left\|e^{\circ\mathfrak{F}}\right\|_{\mathbb{E}}^2 = \sum_{k=1}^{n} (2n - 2k + 1) e^{2\mathbb{F}_k} = (2n+1)\sum_{k=1}^{n} e^{2\mathbb{F}_k} - 2\sum_{k=1}^{n} k e^{2\mathbb{F}_k}.$$

Thus, the proof is clear. □

Corollary 2.3. *Suppose that $e^{\circ\mathfrak{F}}$ is a matrix as in the matrix form (1.9). Then,*

$$\frac{1}{\sqrt{n}}\sqrt{(2n+1)\sum_{k=1}^{n} e^{2\mathbb{F}_k} - 2\sum_{k=1}^{n} k e^{2\mathbb{F}_k}} \le \|e^{\circ\mathfrak{F}}\|_2 \le \sqrt{(2n+1)\sum_{k=1}^{n} e^{2\mathbb{F}_k} - 2\sum_{k=1}^{n} k e^{2\mathbb{F}_k}}.$$

Proof. The proof can be seen easily by using theorem above and the inequality (1.5). □

Theorem 2.12. *Suppose that $e^{\circ\mathfrak{F}}$ is a matrix as in the matrix form* (1.9). *Then,*

$$\|e^{\circ\mathfrak{F}}\|_2 \le \sqrt{(\sum_{k=1}^{n} e^{2\mathbb{F}_k})(\sum_{k=1}^{n-1} e^{2\mathbb{F}_k} + 1)}.$$

Proof. We can write

$$e^{\circ\mathfrak{F}} = \mathfrak{A} \circ \mathfrak{B},$$

where

$$\mathfrak{A} = \begin{bmatrix} e^{\mathbb{F}_1} & 1 & 1 & \cdots & 1 \\ e^{\mathbb{F}_1} & e^{\mathbb{F}_2} & 1 & \cdots & 1 \\ e^{\mathbb{F}_1} & e^{\mathbb{F}_2} & e^{\mathbb{F}_3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{\mathbb{F}_1} & e^{\mathbb{F}_2} & e^{\mathbb{F}_3} & \cdots & e^{\mathbb{F}_n} \end{bmatrix} \quad \text{and} \quad \mathfrak{B} = \begin{bmatrix} 1 & e^{\mathbb{F}_1} & e^{\mathbb{F}_1} & \cdots & e^{\mathbb{F}_1} \\ 1 & 1 & e^{\mathbb{F}_2} & \cdots & e^{\mathbb{F}_2} \\ 1 & 1 & 1 & \cdots & e^{\mathbb{F}_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

So, we have

$$||\mathfrak{A}||_{r_1} = \sqrt{\sum_{k=1}^{n} e^{2\mathbb{F}_k}}, \quad \text{and} \quad ||\mathfrak{B}||_{c_1} = \sqrt{\sum_{k=1}^{n-1} e^{2\mathbb{F}_k} + 1}.$$

According to (1.6), we obtain

$$\|e^{\circ\mathfrak{F}}\|_2 \leq \sqrt{\left(\sum_{k=1}^{n} e^{2\mathbb{F}_k}\right)\left(\sum_{k=1}^{n-1} e^{2\mathbb{F}_k}+1\right)}. \qquad \square$$

3 Numerical examples

In this section, to verify the obtained some results, we give a MATLAB-R2016a code for the matrix given by (1.8). Moreover, we present an illustrative example.

```
clc
clear all
n=input('n=?');

f(1) = 1;
f(2) = 1;
g(1)=1;
g(2)=1;
for i = 3 : n
    f(i) = f(i-1) + f(i-2);
    g(i) =1/f(i);
end
b = g(1:n);
t=cumsum(b);
for i=1:n
     for j=1:n
        if i==j
             a(i,j)=t(i);
        elseif i<j
            a(i,j)=t(i);
        elseif i>j
            a(i,j)=t(j);
        end
    end
end
A = rats(a)
c(1)=exp(2*1);
c(2)=exp(2*2);
for i = 3 : n
    c(i)=exp(2*t(i));
end
d = c(1:n);
rownorm_1 = cumsum(d);
HEMrownnorm=rownorm_1(n)
e = c(1:n-1);
columnnorm_2 = cumsum(e)+1;
HEMcolumnnorm=columnnorm_2(n-1)
HEMl_2normlessthan=(columnnorm_2(n-1)*rownorm_1(n))^(1/2)
for i = 1 : n
```

```
    f(i)=i*exp(2*t(i));
end
g = f(1:n);
x=cumsum(g);
HEMEuclidNorm=((2*n+1)*rownorm_1(n)-2*x(n))^(1/2)
```

Example 1. *Let $\mathfrak{F}$ is a matrix as in matrix form (1.8) for $n = 5$. Then, the followings can be calculated:*

$$\det(\mathfrak{F}) = \frac{1}{30}$$
$$\det(\mathfrak{F}^{\circ(-1)}) = \frac{36}{657475}$$
$$||\mathfrak{F}||_E = \sqrt{\frac{91381}{900}} \approx 10.076$$
$$4.506 \approx \frac{1}{\sqrt{5}}\sqrt{\frac{91381}{900}} \leq ||\mathfrak{F}||_2 \leq \sqrt{\frac{91381}{900}} \approx 10.076$$
$$||\mathfrak{F}||_2 \leq \sqrt{\frac{1871063}{3240}} \approx 24.031$$
$$\det(e^{\circ\mathfrak{F}}) \approx 1103.985$$
$$\det(e^{\mathfrak{F}\circ(-1)}) \approx 1.1191 \times 10^{-6}$$
$$\|e^{\circ\mathfrak{F}}\|_{\mathbb{E}} \approx 49.891$$
$$22.312 \approx \frac{1}{\sqrt{5}}\sqrt{2.4892 \times 10^3} \leq \|e^{\circ\mathfrak{F}}\|_{\mathbb{E}} \leq \sqrt{2.4892 \times 10^3} \approx 49.891$$
$$\|e^{\circ\mathfrak{F}}\|_2 \leq 682.489$$
$$||\mathfrak{A}||_{r_1} \approx 930.710$$
$$||\mathfrak{B}||_{c_1} \approx 500.4697$$

4 Conclusion

In this paper, we construct a special symmetric matrix $\mathfrak{F}$ whose entries are the harmonic Fibonacci numbers and its Hadamard exponential matrix $e^{\circ\mathfrak{F}}$. Then, we obtain some interesting linear algebraic properties, such as determinants, permanents, some norms, etc., for the constructed matrix family. Also, we give some summation formulas for the harmonic Fibonacci numbers. Moreover, we present a MATLAB-R2016a code for the matrix $\mathfrak{F}$ and for the norm calculations of $e^{\circ\mathfrak{F}}$. For the value n inputted to the code given:

1. writes the matrix $\mathfrak{F}$,
2. for the matrix $e^{\circ\mathfrak{F}}$,
   - i calculates the row norm $\|\mathfrak{A}\|_{r_1}$,
   - ii calculates the column norm $\|\mathfrak{B}\|_{c_1}$,
   - iii gives an upper bound for the spectral norm $\|e^{\circ\mathfrak{F}}\|_2$,
   - iv obtains the Euclidean norm $\|e^{\circ\mathfrak{F}}\|_{\mathbb{E}}$.

Acknowledgements

The authors are thankful for the reviewers' valuable remarks that considerably improved the quality of the paper.

References

[1] Bahsi, M., & Solak, S. (2010). On the matrices with Harmonic numbers. *Gazi Journal of Science*, 23(4), 445–448.

[2] Bapat, R. B. (1988). Multinomial probabilities, permanents and a conjecture of Karlin and Rmou. *Proceedings of the American Mathematical Society*, 102(3), 467–472.

[3] Bozkurt, D. (2011). *A note on the spectral norms of the matrices connected integer numbers sequence*. arXiv. Available online at: `https://arxiv.org/abs/1105.1724v1`

[4] Brualdi, R. A., Gibson, P.M. (1977). Convex polyhedra of doubly stochastic matrices. I. Applications of the permanent function. *Journal of Combinatorial Theory, Series A*, 22, 194–230.

[5] Graham, R., Knuth, D., & Patashnik, K. (1989). *Concrete Mathematics*. Addison-Wesley, Reading.

[6] Ipek, A., & Akbulak, M. (2012). Hadamard exponential Hankel Matrix, its eigenvalues and some norms. *Mathematical Sciences Letters*, 1, 81–87.

[7] Jafari Petrudi, S. H., & Pirouz, B. (2015). A particular matrix, its inversion and some norms. *Applied and Computational Mathematics*, 4(2), 47–52.

[8] Reams, R. (1999). Hadamard inverses, square roots and products of almost semidefinite matrices. *Linear Algebra and Its Applications*, 288, 35–43.

[9] Solak, S., & Bahsi, M. (2013). On the spectral norms of Toeplitz matrices with Fibonacci and Lucas numbers, Nonlinear Analysis. *Hacettepe Journal of Mathematics and Statistics*, 42(1), 15–19.

[10] Tuğlu, N., & Kızılateş, C. (2015). On the Norms of Some Special Matrices with the Harmonic Fibonacci Numbers. *Gazi University Journal of Science*, 28(3), 497–501.

[11] Tuğlu, N., Kızılateş, C., & Kesim, S. (2015). On the harmonic and hyperharmonic Fibonacci numbers. *Advances in Difference Equations*, 1–12.

[12] Yılmaz, F., & Bozkurt, D. (2011). Hessenberg matrices and the Pell and Perrin numbers. *Journal of Number Theory*, 131, 1390–1396.

[13] Zhang, F. (1999). *Matrix Theory: Basic Results and Techniques*. Springer-Verlag, New York.