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Construction of dual-generalized complex Fibonacci and Lucas quaternions

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The aim of this paper is to construct dual-generalized complex Fibonacci and Lucas quaternions. It examines the properties both as dual-generalized complex number and as quaternion. Additionally, general recurrence relations, Binet's formulas, Tagiuri's (or Vajda's like), Honsberger's, d'Ocagne's, Cassini's and Catalan's identities are obtained. A series of matrix representations of these special quaternions is introduced. Finally, the multiplication of dual-generalized complex Fibonacci and Lucas quaternions are also expressed as their different matrix representations.

Key words and phrases: quaternion, dual-generalized complex number, Fibonacci number, Lucas number.

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Introduction

Quaternions (Hamiltonian quaternions) are introduced by W.R. Hamilton [17] as a hypercomplex numbers and denoted by $\mathbb{H}:=\{Q=a+be_1+ce_2+de_3 : a,b,c,d\in\mathbb{R}\}$, where $1=(1,0,0,0)$, $e_1=(0,1,0,0)$, $e_2=(0,0,1,0)$, $e_3=(0,0,0,1)$ that satisfy

$$\begin{aligned} e_1e_2 &= -e_2e_1 = e_3, \\ e_2e_3 &= -e_3e_2 = e_1, \\ e_3e_1 &= -e_1e_3 = e_2, \end{aligned}$$

and

$$e_1^2=e_2^2=e_3^2=e_1e_2e_3=-1.$$

$\mathbb{H}$ is an associative but non commutative algebra [17]. Quaternion is a geometric operator to represent the relationship (relative length and orientation) between two vectors in three and four dimensional spaces. The rotation and the screw (spatial) displacement by quaternions are usually simpler, cheaper and functional than other methods. For instance, the dual quaternion algebra is applied to the kinematic modeling of different types of robots, such as mobile robots, robot manipulators, cooperative systems, mobile manipulators, and humanoids.

Hamilton's ideas related to quaternions are extended in another way by A.F. Horadam in [19]. He defined the nth Fibonacci and Lucas quaternions using respectively nth Fibonacci and Lucas numbers (see Appendix A) as the coefficients. These quaternions are also examined in [15, 20–22].

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Complex numbers are buildings blocks of several commutative number systems [34]. As a generalization of complex numbers, the most known 2-component number system is generalized complex numbers $\mathbb{C}_{\mathfrak{p}}$ (see [18,23]) and defined by

$$\mathbb{C}_{\mathfrak{p}} := \left\{ z = a_1 + a_2 J : a_1, a_2 \in \mathbb{R},\ J^2 = \mathfrak{p} \in \mathbb{R}, J \notin \mathbb{R} \right\},$$

which analogue to complex numbers for $\mathfrak{p} = -1$, hyperbolic numbers for $\mathfrak{p} = 1$ and dual numbers for $\mathfrak{p} = 0$ (see details in [27,30,31,34,35]). Moreover, by using Cayley-Dickson doubling procedure, the set of dual-generalized complex ($\mathcal{DGC}$) numbers defined in [13] as

$$\mathbb{DC}_{\mathfrak{p}} := \left\{ w = z_1 + z_2 \varepsilon : z_1, z_2 \in \mathbb{C}_{\mathfrak{p}},\ \varepsilon^2 = 0, \varepsilon \neq 0, \varepsilon \notin \mathbb{R} \right\}.$$

Reconstructing new numbers with the different combination of above number systems is an attractive area for researchers (see [1,2,4,6–11,23–25,28,29,33]). For the special real values $\mathfrak{p} = -1$, $\mathfrak{p} = 0$ and $\mathfrak{p} = 1$, dual-complex, hyper-dual, dual-hyperbolic numbers are obtained from $\mathcal{DGC}$ numbers, respectively (see [13]). If the literature is examined considering Fibonacci/Lucas extension, the nth complex and dual Fibonacci quaternions are defined in [16] and [26], respectively. Besides, dual-complex Fibonacci/Lucas numbers and their properties are presented in [12]. In [5], dual hyperbolic Fibonacci and Lucas numbers are examined. In [14], the nth $\mathcal{DGC}$ Fibonacci and Lucas numbers are defined and the recurrence relations are introduced (see Appendix B). Additionally, in [32], hyper-dual Horadam quaternions are investigated.

In this paper, the $\mathcal{DGC}$ Fibonacci and Lucas quaternions are defined as an extension of the papers [16, 19, 26]. This construction is adapted from [19]. The recurrence relations, Binet's formula, Tagiuri's (or Vajda's like), Honsberger's, d'Ocagne's, Catalan's and Cassini's identities are obtained. Several matrix representations of these special quaternions are presented. Furthermore, the multiplication of $\mathcal{DGC}$ Fibonacci and Lucas quaternions are also calculated as their different matrix representations.

1 $\mathcal{DGC}$ Fibonacci and Lucas Quaternions

Definition 1. *The $\mathcal{DGC}$ numbers with Fibonacci and Lucas quaternion coefficient are respectively defined as follows*

$$\widetilde{Q}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon \quad \text{and} \quad \widetilde{K}_n = K_n + K_{n+1}J + K_{n+2}\varepsilon + K_{n+3}J\varepsilon,$$

where Q_n and K_n are the nth Fibonacci and Lucas quaternions.

$\widetilde{Q}_n$ and $\widetilde{K}_n$ can also be expressed as

$$\widetilde{Q}_n = \widetilde{F}_n + \widetilde{F}_{n+1}e_1 + \widetilde{F}_{n+2}e_2 + \widetilde{F}_{n+3}e_3 \quad \text{and} \quad \widetilde{K}_n = \widetilde{L}_n + \widetilde{L}_{n+1}e_1 + \widetilde{L}_{n+2}e_2 + \widetilde{L}_{n+3}e_3,$$

respectively, where $\widetilde{F}_n$ and $\widetilde{L}_n$ are nth $\mathcal{DGC}$ Fibonacci and Lucas numbers (see Appendix B and [14] for the $\mathcal{DGC}$ Fibonacci and Lucas numbers).

Axiomatically, J and ε commutes with quaternion versors, that is $e_iJ = Je_i$, $e_i\varepsilon = \varepsilon e_i$, $i = 1,2,3$. It is evident that for $\mathfrak{p} = -1$ usual complex operator distinct from quaternion versors. The base elements $\{1, J, \varepsilon, J\varepsilon\}$ satisfy the properties given in Table 1.

	1	J	ε	$J\varepsilon$	e_1	e_2	e_3
1	1	J	ε	$J\varepsilon$	e_1	e_2	e_3
J	J	$\mathfrak{p}$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	Je_1	Je_2	Je_3
ε	ε	$J\varepsilon$	0	0	εe_1	εe_2	εe_3
$J\varepsilon$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	0	0	$J\varepsilon e_1$	$J\varepsilon e_2$	$J\varepsilon e_3$
e_1	e_1	Je_1	εe_1	$J\varepsilon e_1$	-1	e_3	$-e_2$
e_2	e_2	Je_2	εe_2	$J\varepsilon e_2$	$-e_3$	-1	e_1
e_3	e_3	Je_3	εe_3	$J\varepsilon e_3$	e_2	$-e_1$	-1

Table 1. Multiplication scheme.

For convenience, the sets used in this paper is given in Table 2.

Fibonacci numbers	$\mathbb{F} := \{F_n \| F_n = F_{n-1} + F_{n-2},\ n \geq 1,\ F_0 = 0, F_1 = 1\}$
Lucas numbers	$\mathbb{L} := \{L_n \| L_n = L_{n-1} + L_{n-2},\ n \geq 1,\ L_0 = 2, L_1 = 1\}$
$\mathcal{DGC}$ Fibonacci numbers	$\mathbb{DC}_\mathfrak{p}\mathbb{F} := \{\widetilde{F}_n = F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon \| F_n \in \mathbb{F}\}$
$\mathcal{DGC}$ Lucas numbers	$\mathbb{DC}_\mathfrak{p}\mathbb{L} := \{\widetilde{L}_n = L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon \| L_n \in \mathbb{L}\}$
Fibonacci quaternions	$\mathbb{Q} := \{Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 \| F_n \in \mathbb{F}\}$
Lucas quaternions	$\mathbb{K} := \{K_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 \| L_n \in \mathbb{L}\}$
$\mathcal{DGC}$ numbers with Fibonacci quaternion	$\mathbb{DC}_\mathfrak{p}\mathbb{Q} := \{\widetilde{Q}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon \| Q_n \in \mathbb{Q}\}$
Quaternions with $\mathcal{DGC}$ Fibonacci number	$\mathbb{QDC}_\mathfrak{p} := \{\widetilde{Q}_n = \widetilde{F}_n + \widetilde{F}_{n+1}e_1 + \widetilde{F}_{n+2}e_2 + \widetilde{F}_{n+3}e_3 \| \widetilde{F}_n \in \mathbb{DC}_\mathfrak{p}\mathbb{F}\}$
$\mathcal{DGC}$ numbers with Lucas quaternion	$\mathbb{DC}_\mathfrak{p}\mathbb{K} := \{\widetilde{K}_n = K_n + K_{n+1}J + K_{n+2}\varepsilon + K_{n+3}J\varepsilon \| K_n \in \mathbb{K}\}$
Quaternions with $\mathcal{DGC}$ Lucas number	$\mathbb{KDC}_\mathfrak{p} := \{\widetilde{K}_n = \widetilde{L}_n + \widetilde{L}_{n+1}e_1 + \widetilde{L}_{n+2}e_2 + \widetilde{L}_{n+3}e_3 \| \widetilde{L}_n \in \mathbb{DC}_\mathfrak{p}\mathbb{L}\}$

Table 2. The notations of the sets.

The algebraic structures of $\widetilde{Q}_n$ (also for $\widetilde{K}_n$) as a $\mathcal{DGC}$ number and as a quaternion can be seen in Table 3 and Table 4, respectively.

Number	$\widetilde{Q}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon$
Addition, subtraction	$\widetilde{Q}_n \pm \widetilde{Q}_m = (Q_n \pm Q_m) + (Q_{n+1} \pm Q_{m+1})J$ $+ (Q_{n+2} \pm Q_{m+2})\varepsilon + (Q_{n+3} \pm Q_{m+3})J\varepsilon$
Scalar multiplication	$\lambda\widetilde{Q}_n = (\lambda Q_n) + (\lambda Q_{n+1})J + (\lambda Q_{n+2})\varepsilon + (\lambda Q_{n+3})J\varepsilon, \lambda \in \mathbb{R}$
Multiplication	$\widetilde{Q}_n\widetilde{Q}_m = (Q_nQ_m + \mathfrak{p}Q_{n+1}Q_{m+1}) + (Q_nQ_{m+1} + Q_{n+1}Q_m)J$ $+ (Q_nQ_{m+2} + \mathfrak{p}Q_{n+1}Q_{m+3} + Q_{n+2}Q_m + \mathfrak{p}Q_{n+3}Q_{m+1})\varepsilon$ $+ (Q_nQ_{m+3} + Q_{n+1}Q_{m+2} + Q_{n+2}Q_{m+1} + Q_{n+3}Q_m)J\varepsilon$
Equality	$\widetilde{Q}_n = \widetilde{Q}_m \Leftrightarrow Q_n = Q_m, Q_{n+1} = Q_{m+1}, Q_{n+2} = Q_{m+2}, Q_{n+3} = Q_{m+3}$
Generalized complex conjugate	$\widetilde{Q}_n^{\dagger_1} = Q_n - Q_{n+1}J + Q_{n+2}\varepsilon - Q_{n+3}J\varepsilon$
Dual conjugate	$\widetilde{Q}_n^{\dagger_2} = Q_n + Q_{n+1}J - Q_{n+2}\varepsilon - Q_{n+3}J\varepsilon$
Coupled conjugate	$\widetilde{Q}_n^{\dagger_3} = Q_n - Q_{n+1}J - Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon$
$\mathcal{DGC}$ conjugate	$\widetilde{Q}_n^{\dagger_4} = (Q_n - Q_{n+1}J)\left(1 - \frac{Q_{n+2}+Q_{n+3}J}{Q_n+Q_{n+1}J}\varepsilon\right)$
Anti-dual conjugate	$\widetilde{Q}_n^{\dagger_5} = Q_{n+2} + Q_{n+3}J - Q_n\varepsilon - Q_{n+1}J\varepsilon$
Module	$N_{\widetilde{Q}_n}^{\dagger_i} = \widetilde{Q}_n\widetilde{Q}_n^{\dagger_i}, \quad 1 \leq i \leq 3$

Table 3. Structures for $\widetilde{Q}_n$ as a $\mathcal{DGC}$ number.

Quaternion	$\widetilde{Q}_n = \widetilde{F}_n + \widetilde{F}_{n+1}e_1 + \widetilde{F}_{n+2}e_2 + \widetilde{F}_{n+3}e_3$
Addition, subtraction	$\widetilde{Q}_n \pm \widetilde{Q}_m = \left(\widetilde{F}_n \pm \widetilde{F}_m\right) + \left(\widetilde{F}_{n+1} \pm \widetilde{F}_{m+1}\right)e_1 + \left(\widetilde{F}_{n+2} \pm \widetilde{F}_{m+2}\right)e_2 + \left(\widetilde{F}_{n+3} \pm \widetilde{F}_{m+3}\right)e_3$
Scalar Multiplication	$\lambda\widetilde{Q}_n = (\lambda\widetilde{F}_n) + (\lambda\widetilde{F}_{n+1})e_1 + (\lambda\widetilde{F}_{n+2})e_2 + (\lambda\widetilde{F}_{n+3})e_3, \lambda \in \mathbb{R}$
Multiplication	$\widetilde{Q}_n\widetilde{Q}_m = \widetilde{F}_n\widetilde{F}_m - \widetilde{F}_{n+1}\widetilde{F}_{m+1} - \widetilde{F}_{n+2}\widetilde{F}_{m+2} - \widetilde{F}_{n+3}\widetilde{F}_{m+3} + \left(\widetilde{F}_n\widetilde{F}_{m+1} + \widetilde{F}_{n+1}\widetilde{F}_m + \widetilde{F}_{n+2}\widetilde{F}_{m+3} - \widetilde{F}_{n+3}\widetilde{F}_{m+2}\right)e_1 + \left(\widetilde{F}_n\widetilde{F}_{m+2} + \widetilde{F}_{n+2}\widetilde{F}_m + \widetilde{F}_{n+3}\widetilde{F}_{m+1} - \widetilde{F}_{n+1}\widetilde{F}_{m+3}\right)e_2 + \left(\widetilde{F}_n\widetilde{F}_{m+3} + \widetilde{F}_{n+3}\widetilde{F}_m + \widetilde{F}_{n+1}\widetilde{F}_{m+2} - \widetilde{F}_{n+2}\widetilde{F}_{m+1}\right)e_3$
Equality	$\widetilde{Q}_n = \widetilde{Q}_m \Leftrightarrow \widetilde{F}_n = \widetilde{F}_m, \widetilde{F}_{n+1} = \widetilde{F}_{m+1}, \widetilde{F}_{n+2} = \widetilde{F}_{m+2}, \widetilde{F}_{n+3} = \widetilde{F}_{m+3}$
Conjugate	$\overline{\widetilde{Q}}_n = \widetilde{F}_n - \widetilde{F}_{n+1}e_1 - \widetilde{F}_{n+2}e_2 - \widetilde{F}_{n+3}e_3$
Module	$N_{\widetilde{Q}_n} = \widetilde{Q}_n\overline{\widetilde{Q}}_n = 3\left(\widetilde{F}_{2n+3} + \mathfrak{p}F_{2n+5} + F_{2n+4}J + (F_{2n+5} + 2\mathfrak{p}F_{2n+7})\,\varepsilon + 3F_{2n+6}J\varepsilon\right)$
Inverse	$\widetilde{Q}_n^{-1} = \frac{\overline{\widetilde{Q}}_n}{N_{\widetilde{Q}_n}}$, where $N_{\widetilde{Q}_n}$ is not a null number (null numbers are characterized by zero norm in $\mathbb{DC}_\mathfrak{p}$)

Table 4. Structures for $\widetilde{Q}_n$ as a quaternion.

Theorem 1. *Let $\widetilde{Q}_n \in \mathbb{QDC}_\mathfrak{p}$ and $\widetilde{K}_n \in \mathbb{KDC}_\mathfrak{p}$. Then the following additional recurrence relations hold for $n, r \geq 0$:*

i) $\widetilde{Q}_n + \widetilde{Q}_{n+1} = \widetilde{Q}_{n+2}$;

ii) $\widetilde{Q}_{n+r} + \widetilde{Q}_{n-r} = \begin{cases} L_r\widetilde{Q}_n, & r = 2k, \\ F_r\widetilde{K}_n, & r = 2k+1; \end{cases}$

iii) $\widetilde{Q}_{n+r} - \widetilde{Q}_{n-r} = \begin{cases} F_r\widetilde{K}_n, & r = 2k, \\ L_r\widetilde{Q}_n, & r = 2k+1; \end{cases}$

iv) $\widetilde{Q}_n - \widetilde{Q}_{n+1}e_1 - \widetilde{Q}_{n+2}e_2 - \widetilde{Q}_{n+3}e_3 = 3\widetilde{L}_{n+3}$;

v) $\widetilde{K}_n + \widetilde{K}_{n+1} = \widetilde{K}_{n+2}$;

vi) $\widetilde{K}_{n+r} + \widetilde{K}_{n-r} = \begin{cases} L_r\widetilde{K}_n, & r = 2k, \\ 5F_r\widetilde{Q}_n, & r = 2k+1; \end{cases}$

vii) $\widetilde{K}_{n+r} - \widetilde{K}_{n-r} = \begin{cases} 5F_r\widetilde{Q}_n, & r = 2k, \\ L_r\widetilde{K}_n, & r = 2k+1; \end{cases}$

viii) $\widetilde{K}_n - \widetilde{K}_{n+1}e_1 - \widetilde{K}_{n+2}e_2 - \widetilde{K}_{n+3}e_3 = 15\widetilde{L}_{n+3}$.

Proof. Using the recurrence relations for the $\mathcal{DGC}$ Fibonacci and Lucas numbers (see equations (3)–(5), (9)–(11) and [14]), the proof is completed. □

Theorem 2. *Let $\widetilde{Q}_n \in \mathbb{QDC}_\mathfrak{p}$, $\widetilde{K}_n \in \mathbb{KDC}_\mathfrak{p}$, $\widetilde{F}_n \in \mathbb{DC}_\mathfrak{p}\mathbb{F}$ and $\widetilde{L}_n \in \mathbb{DC}_\mathfrak{p}\mathbb{L}$. Then the following relations hold for $n \geq 0$:*

i) $\widetilde{Q}_n + \overline{\widetilde{Q}}_n = 2\widetilde{F}_n$;

ii) $\widetilde{Q}_n\overline{\widetilde{Q}}_n = 3(\widetilde{F}_{2n+3} + \mathfrak{p}F_{2n+5} + F_{2n+4}J + (F_{2n+5} + 2\mathfrak{p}F_{2n+7})\,\varepsilon + 3F_{2n+6}J\varepsilon)$;

iii) $\widetilde{Q}_n\overline{\widetilde{Q}}_n + \widetilde{Q}_{n+1}\overline{\widetilde{Q}}_{n+1} = 3(\widetilde{L}_{2n+4} + \mathfrak{p}L_{2n+6} + L_{2n+5}J + (L_{2n+6} + 2\mathfrak{p}L_{2n+8})\,\varepsilon + 3L_{2n+7}J\varepsilon)$;

iv) $\widetilde{Q}_n^2 = 2\widetilde{F}_n\widetilde{Q}_n - 3(\widetilde{F}_{2n+3} + \mathfrak{p}F_{2n+5} + F_{2n+4}J + (F_{2n+5} + 2\mathfrak{p}F_{2n+7})\,\varepsilon + 3F_{2n+6}J\varepsilon)$;

v) $\widetilde{K}_n + \overline{\widetilde{K}}_n = 2\widetilde{L}_n$;

vi) $\widetilde{K}_n\overline{\widetilde{K}}_n = 3(\widetilde{L}_{2n+3} + \mathfrak{p}L_{2n+5} + L_{2n+4}J + (L_{2n+5} + 2\mathfrak{p}L_{2n+7})\,\varepsilon + 3L_{2n+6}J\varepsilon)$;

vii) $\widetilde{K}_n\overline{\widetilde{K}}_n + \widetilde{K}_{n+1}\overline{\widetilde{K}}_{n+1} = 15(\widetilde{F}_{2n+4} + \mathfrak{p}F_{2n+6} + L_{2n+5}J + (F_{2n+6} + 2\mathfrak{p}F_{2n+8})\,\varepsilon + 3F_{2n+7}J\varepsilon)$;

viii) $\widetilde{K}_n^2 = 2\widetilde{L}_n\widetilde{K}_n - 3(\widetilde{L}_{2n+3} + \mathfrak{p}L_{2n+5} + F_{2n+4}J + (L_{2n+5} + 2\mathfrak{p}L_{2n+7})\,\varepsilon + 3L_{2n+6}J\varepsilon)$.

Proof. i) Using definition of $\widetilde{Q}_n$ and its conjugate, the proof is clear.

ii) A simple calculation gives us that $\widetilde{Q}_n\overline{\widetilde{Q}}_n = \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 + \widetilde{F}_{n+2}^2 + \widetilde{F}_{n+3}^2$. Then, using equations (4), (1) and (6), we get the result.

iii) With the aid of equality *ii)*, we write

$$\begin{aligned}\widetilde{Q}_n\overline{\widetilde{Q}}_n + \widetilde{Q}_{n+1}\overline{\widetilde{Q}}_{n+1} = 3\Big[&\widetilde{F}_{2n+3} + \widetilde{F}_{2n+5} + \mathfrak{p}(F_{2n+5} + F_{2n+7}) + (F_{2n+4} + F_{2n+6})J \\ &+ ((F_{2n+5} + F_{2n+7}) + 2\mathfrak{p}(F_{2n+7} + F_{2n+9}))\varepsilon + 3(F_{2n+6} + F_{2n+8})J\varepsilon\Big].\end{aligned}$$

Using the identities (4) and (1), the proof is clear.

iv) Using part *i)*, we can write $\widetilde{Q}_n^2 = \widetilde{Q}_n\left(2\widetilde{F}_n - \overline{\widetilde{Q}}_n\right)$. Then from equality *ii)*, the proof is straightforward.

Other parts can be calculated similarly using equations in Appendices A and B. □

Theorem 3. *Let $\widetilde{Q}_n, \widetilde{Q}_m \in \mathbb{QDC}_\mathfrak{p}$ and $\widetilde{K}_n, \widetilde{K}_m \in \mathbb{KDC}_\mathfrak{p}$. Then the following multiplication recurrence relations hold for $n, m, r \geq 0$:*

i)

$$\begin{aligned}\widetilde{Q}_m\widetilde{Q}_n - \widetilde{Q}_{m+r}\widetilde{Q}_{n-r} = (-1)^{n-r}F_r\ &[(1-\mathfrak{p}) + J + 3\,(1-\mathfrak{p})\,\varepsilon + 3J\varepsilon] \\ &\times [2F_{m-n+r} - 2F_{m-n+r-1}e_1 + 2F_{m-n+r-2}e_2 + (4F_{m-n+r} + L_{m-n+r})\,e_3];\end{aligned}$$

ii)

$$\begin{aligned}&\widetilde{Q}_n\widetilde{Q}_m + \widetilde{Q}_{n+1}\widetilde{Q}_{m+1} = \\ &\quad -\left(\widetilde{F}_{n+m+2} + \widetilde{L}_{n+m+6}\right) - \mathfrak{p}\left(F_{n+m+4} + L_{n+m+8}\right) - \left(F_{n+m+3} + L_{n+m+7}\right)J \\ &\quad - \left[\left(F_{n+m+4} + L_{n+m+8}\right) + \mathfrak{p}\left(F_{n+m+6} + L_{n+m+10}\right)\right]\varepsilon - 3\left(F_{n+m+5} + L_{n+m+9}\right)J\varepsilon \\ &\quad + \left[2\left(\widetilde{F}_{n+m+2} + \mathfrak{p}F_{n+m+4} + F_{n+m+3}J + \left(F_{n+m+4} + 2\mathfrak{p}F_{n+m+6}\right)\varepsilon + 3F_{n+m+5}J\varepsilon\right)\right]e_1 \\ &\quad + \left[2\left(\widetilde{F}_{n+m+3} + \mathfrak{p}F_{n+m+5} + F_{n+m+4}J + \left(F_{n+m+5} + 2\mathfrak{p}F_{n+m+7}\right)\varepsilon + 3F_{n+m+6}J\varepsilon\right)\right]e_2 \\ &\quad + \left[2\left(\widetilde{F}_{n+m+4} + \mathfrak{p}F_{n+m+6} + F_{n+m+5}J + \left(F_{n+m+6} + 2\mathfrak{p}F_{n+m+8}\right)\varepsilon + 3F_{n+m+7}J\varepsilon\right)\right]e_3;\end{aligned}$$

iii)

$$\begin{aligned}\widetilde{Q}_n^2+\widetilde{Q}_{n+1}^2 =& -\left(\widetilde{F}_{2n+2}+\widetilde{L}_{2n+6}\right)-\mathfrak{p}\left(F_{2n+4}+L_{2n+8}\right)-\left(F_{2n+3}+L_{2n+7}\right)J\\ &-\left[\left(F_{2n+4}+L_{2n+8}\right)+\mathfrak{p}\left(F_{2n+6}+L_{2n+10}\right)\right]\varepsilon-3\left(F_{2n+5}+L_{2n+9}\right)J\varepsilon\\ &+\left[2\left(\widetilde{F}_{2n+2}+\mathfrak{p}F_{2n+4}+F_{2n+3}J+\left(F_{2n+4}+2\mathfrak{p}F_{2n+6}\right)\varepsilon+3F_{2n+5}J\varepsilon\right)\right]e_1\\ &+\left[2\left(\widetilde{F}_{2n+3}+\mathfrak{p}F_{2n+5}+F_{2n+4}J+\left(F_{2n+5}+2\mathfrak{p}F_{2n+7}\right)\varepsilon+3F_{2n+6}J\varepsilon\right)\right]e_2\\ &+\left[2\left(\widetilde{F}_{2n+4}+\mathfrak{p}F_{2n+6}+F_{2n+5}J+\left(F_{2n+6}+2\mathfrak{p}F_{2n+8}\right)\varepsilon+3F_{2n+7}J\varepsilon\right)\right]e_3;\end{aligned}$$

iv) $\widetilde{K}_m\widetilde{K}_n-\widetilde{K}_{m+r}\widetilde{K}_{n-r}=-5(\widetilde{Q}_m\widetilde{Q}_n-\widetilde{Q}_{m+r}\widetilde{Q}_{n-r})$;

v) $\widetilde{K}_n\widetilde{K}_m+\widetilde{K}_{n+1}\widetilde{K}_{m+1}=5(\widetilde{Q}_n\widetilde{Q}_m+\widetilde{Q}_{n+1}\widetilde{Q}_{m+1})$;

vi) $\widetilde{K}_n^2+\widetilde{K}_{n+1}^2=5(\widetilde{Q}_n^2+\widetilde{Q}_{n+1}^2)$.

Proof. To prove the part *i)* we use the equations (1), (2), (7). To prove the part *ii)* we use the equations (4), (5), (1), (2), (8). For part *iii)* we apply $m\to n$ in part *ii)*. Other parts can be calculated similarly using equations in Appendices A and B. □

It should be noted that parts *i)* (analogue to *iv)*) and *ii)* (analogue to *v)*) in Theorem 3 are often referred to as Tagiuri's (or Vajda's like) and Honsberger's identities, respectively. It is known that d'Ocagne's, Catalan's and Cassini's identities are special cases of the Vajda's identity.

Corollary 1. *Let* $\widetilde{Q}_n,\widetilde{Q}_m\in\mathbb{QDC}_{\mathfrak{p}}$ *and* $\widetilde{K}_n,\widetilde{K}_m\in\mathbb{KDC}_{\mathfrak{p}}$. *For* $n,m\geq 0$,

d'Ocagne's identities can be given as follows:

$$\begin{aligned}\widetilde{Q}_m\widetilde{Q}_{n+1}-\widetilde{Q}_{m+1}\widetilde{Q}_n =& (-1)^n\left[(1-\mathfrak{p})+J+3(1-\mathfrak{p})\varepsilon+3J\varepsilon\right]\\ &\times\left[2F_{m-n}-2F_{m-n-1}e_1+2F_{m-n-2}e_2+\left(4F_{m-n}+L_{m-n}\right)e_3\right],\\ \widetilde{K}_m\widetilde{K}_{n+1}-\widetilde{K}_{m+1}\widetilde{K}_n =& 5(-1)^{n+1}\left[(1-\mathfrak{p})+J+3(1-\mathfrak{p})\varepsilon+3J\varepsilon\right]\\ &\times\left[2F_{m-n}-2F_{m-n-1}e_1+2F_{m-n-2}e_2+\left(4F_{m-n}+L_{m-n}\right)e_3\right];\end{aligned}$$

Catalan's identities are as below:

$$\begin{aligned}\widetilde{Q}_n^2-\widetilde{Q}_{n+r}\widetilde{Q}_{n-r} =& (-1)^{n-r}F_r\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right]\\ &\times\left[2F_r-2F_{r-1}e_1+2F_{r-2}e_2+\left(4F_r+L_r\right)e_3\right],\\ \widetilde{K}_n^2-\widetilde{K}_{n+r}\widetilde{K}_{n-r} =& 5(-1)^{n-r+1}F_r\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right]\\ &\times\left[2F_r-2F_{r-1}e_1+2F_{r-2}e_2+\left(4F_r+L_r\right)e_3\right];\end{aligned}$$

Cassini's identities are given such that:

$$\begin{aligned}\widetilde{Q}_n^2-\widetilde{Q}_{n+1}\widetilde{Q}_{n-1} &= (-1)^{n-1}\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right]\left[2-2e_2+5e_3\right],\\ \widetilde{K}_n^2-\widetilde{K}_{n+1}\widetilde{K}_{n-1} &= 5(-1)^n\left[(1-\mathfrak{p})+J+3\left(1-\mathfrak{p}\right)\varepsilon+3J\varepsilon\right]\left[2-2e_2+5e_3\right].\end{aligned}$$

Proof. By writing $n \to n+1$, $r \to 1$ in parts *i)* and *iv)* in Theorem 3, respectively, the proof of d'Ocagne's identities is completed. By writing $m \to n$ in parts *i)* and *iv)* in Theorem 3, respectively, the proof of Catalan's identities is completed. Substituting $r = 1$ in Catalan's identity completes the proof of the Cassini's identities. □

Theorem 4. *Let $\widetilde{Q}_{-n}$ and $\widetilde{K}_{-n}$ be nega$\mathcal{DGC}$ Fibonacci and Lucas quaternions. Then the following identities can be given for $n \geqslant 0$:*

$$\widetilde{Q}_{-n} = (-1)^{n+1}\widetilde{Q}_n + (-1)^n\widetilde{L}_n\,(e_1+e_2+2e_3) + (-1)^{n+1}K_{n-3}\,(J+\varepsilon+2J\varepsilon)\ e_3,$$
$$\widetilde{K}_{-n} = (-1)^n\widetilde{K}_n + 5(-1)^{n-1}\widetilde{F}_n\,(e_1+e_2+2e_3) + 5(-1)^nQ_{n-3}\,(J+\varepsilon+2J\varepsilon)\ e_3.$$

Proof. Using the nega$\mathcal{DGC}$ Fibonacci and Lucas numbers (see in (12) and (13), [14]), the proof is obvious. □

Theorem 5. *Let $\widetilde{Q}_n \in \mathbb{QDC}_{\mathfrak{p}}$ and $\widetilde{K}_n \in \mathbb{KDC}_{\mathfrak{p}}$. For $n \geq 1$, the Binet's formulas can be calculated as follows:*

$$\widetilde{Q}_n = \frac{\tilde{\alpha}^*\alpha^n - \tilde{\beta}^*\beta^n}{\alpha-\beta}, \qquad \widetilde{K}_n = \tilde{\alpha}^*\alpha^n + \tilde{\beta}^*\beta^n,$$

where $\alpha^* = 1+\alpha J+\alpha^2\varepsilon+\alpha^3J\varepsilon$, $\tilde{\alpha}^* = \alpha^*\left(1+\alpha e_1+\alpha^2e_2+\alpha^3e_3\right)$, $\beta^* = 1+\beta J+\beta^2\varepsilon+\beta^3J\varepsilon$ and $\tilde{\beta}^* = \beta^*\left(1+\beta e_1+\beta^2e_2+\beta^3e_3\right)$.

Proof. Using the Binet's formulas for the $\mathcal{DGC}$ Fibonacci and Lucas numbers (see (14) and (15), [14]), we have:

$$\begin{aligned}\widetilde{Q}_n &= \frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta} + \frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta}e_1 + \frac{\alpha^*\alpha^{n+2}-\beta^*\beta^{n+2}}{\alpha-\beta}e_2 + \frac{\alpha^*\alpha^{n+3}-\beta^*\beta^{n+3}}{\alpha-\beta}e_3\\ &= \frac{\alpha^*\alpha^n\left(1+\alpha e_1+\alpha^2e_2+\alpha^3e_3\right)-\beta^*\beta^n\left(1+\beta e_1+\beta^2e_2+\beta^3e_3\right)}{\alpha-\beta} = \frac{\tilde{\alpha}^*\alpha^n-\tilde{\beta}^*\beta^n}{\alpha-\beta}\end{aligned}$$

and

$$\begin{aligned}\widetilde{K}_n &= \alpha^*\alpha^n+\beta^*\beta^n+\left(\alpha^*\alpha^{n+1}+\beta^*\beta^{n+1}\right)e_1+\left(\alpha^*\alpha^{n+2}+\beta^*\beta^{n+2}\right)e_2+\left(\alpha^*\alpha^{n+3}+\beta^*\beta^{n+3}\right)e_3\\ &= \alpha^*\alpha^n\left(1+\alpha e_1+\alpha^2e_2+\alpha^3e_3\right)+\beta^n\left(1+\beta e_1+\beta^2e_2+\beta^3e_3\right) = \tilde{\alpha}^*\alpha^n+\tilde{\beta}^*\beta^n,\end{aligned}$$

where $\tilde{\alpha}^* = \alpha^*\left(1+\alpha e_1+\alpha^2e_2+\alpha^3e_3\right)$ and $\tilde{\beta}^* = \beta^*\left(1+\beta e_1+\beta^2e_2+\beta^3e_3\right)$. □

2 Matrix representations of $\mathcal{DGC}$ Fibonacci and Lucas quaternions

Theorem 6. *Any $\widetilde{Q}_n$ can be represented by 2×2 generalized complex matrices.*

Proof. We define linear transformation $\Lambda_{(a_1+a_2\varepsilon)} = \begin{bmatrix} a_1 & 0\\ a_2 & a_1\end{bmatrix}$ based on the isomorphism between dual numbers and the set $\left\{\begin{bmatrix} a_1 & 0\\ a_2 & a_1\end{bmatrix} : a_1, a_2 \in \mathbb{R}\right\}$. The 2×2 dual representation of $\widetilde{Q}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon$ with respect to the standard basis $\{1, \varepsilon\}$ can be obtained as below:

$$\Lambda_{\widetilde{Q}_n} = \begin{bmatrix} Q_n+Q_{n+1}J & 0\\ Q_{n+2}+Q_{n+3}J & Q_n+Q_{n+1}J\end{bmatrix},$$

where

$$\Lambda_{\widetilde{Q}_n}(1) = \widetilde{Q}_n = Q_n+Q_{n+1}J+\left(Q_{n+2}+Q_{n+3}J\right)\varepsilon, \qquad \Lambda_{\widetilde{Q}_n}(\varepsilon) = \widetilde{Q}_n\varepsilon = \left(Q_n+Q_{n+1}J\right)\varepsilon.$$

□

Theorem 7. *Any $\widetilde{Q}_n$ can be represented by 4×4 Fibonacci quaternion matrices.*

Proof. With the isomorphism Γ we define Fibonacci quaternion representation of $\widetilde{Q}_n$ as follows:

$$\Delta_{\widetilde{Q}_n} = \begin{bmatrix} \Gamma(Q_n + Q_{n+1}J) & \Gamma(0) \\ \Gamma(Q_{n+2} + Q_{n+3}J) & \Gamma(Q_n + Q_{n+1}J) \end{bmatrix}.$$

Here, Γ is a linear transformation such that $\Gamma(a_1 + a_2 J) = \begin{bmatrix} a_1 & \mathfrak{p} a_2 \\ a_2 & a_1 \end{bmatrix}$ based on the isomorphism between $\mathbb{C}_\mathfrak{p}$ and the set $\left\{ \begin{bmatrix} a_1 & \mathfrak{p} a_2 \\ a_2 & a_1 \end{bmatrix} : a_1, a_2 \in \mathbb{R} \right\}$. The matrix $\Delta_{\widetilde{Q}_n}$ is called the matrix representation of $\widetilde{Q}_n$. Thus

$$\Delta_{\widetilde{Q}_n} = \begin{bmatrix} Q_n & \mathfrak{p} Q_{n+1} & 0 & 0 \\ Q_{n+1} & Q_n & 0 & 0 \\ Q_{n+2} & \mathfrak{p} Q_{n+3} & Q_n & \mathfrak{p} Q_{n+1} \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \end{bmatrix}.$$

The columns of matrix $\Delta_{\widetilde{Q}_n}$ are represented by the coefficients in Fibonacci quaternions of the elements $\{Q_n, Q_n J, Q_n \varepsilon, Q_n J \varepsilon\}$. □

Theorem 8. *The 4×4 $\mathcal{DGC}$ Fibonacci right matrix representation of*

$$\widetilde{Q}_n = \widetilde{F}_n + \widetilde{F}_{n+1} e_1 + \widetilde{F}_{n+2} e_2 + \widetilde{F}_{n+3} e_3$$

with respect to the basis $\{1, e_1, e_2, e_3\}$ is

$$A_{\widetilde{Q}_n} = \begin{bmatrix} \widetilde{F}_n & -\widetilde{F}_{n+1} & -\widetilde{F}_{n+2} & -\widetilde{F}_{n+3} \\ \widetilde{F}_{n+1} & \widetilde{F}_n & \widetilde{F}_{n+3} & -\widetilde{F}_{n+2} \\ \widetilde{F}_{n+2} & -\widetilde{F}_{n+3} & \widetilde{F}_n & \widetilde{F}_{n+1} \\ \widetilde{F}_{n+3} & \widetilde{F}_{n+2} & -\widetilde{F}_{n+1} & \widetilde{F}_n \end{bmatrix}.$$

We remark that $\mathcal{DGC}$ Fibonacci left matrix representation of $\widetilde{Q}_n$ with respect to the basis $\{1, e_1, e_2, e_3\}$ is

$$B_{\widetilde{Q}_n} = \begin{bmatrix} \widetilde{F}_n & -\widetilde{F}_{n+1} & -\widetilde{F}_{n+2} & -\widetilde{F}_{n+3} \\ \widetilde{F}_{n+1} & \widetilde{F}_n & -\widetilde{F}_{n+3} & \widetilde{F}_{n+2} \\ \widetilde{F}_{n+2} & \widetilde{F}_{n+3} & \widetilde{F}_n & -\widetilde{F}_{n+1} \\ \widetilde{F}_{n+3} & -\widetilde{F}_{n+2} & \widetilde{F}_{n+1} & \widetilde{F}_n \end{bmatrix}.$$

Corollary 2. *The column matrix representation of an arbitrary $\mathcal{DGC}$ Fibonacci quaternion $\widetilde{Q}_n$ with respect to the basis $\{1, e_1, e_2, e_3\}$ is merely the collection of its coefficients:*

$$\widetilde{Q}_n = \begin{bmatrix} \widetilde{F}_n & \widetilde{F}_{n+1} & \widetilde{F}_{n+2} & \widetilde{F}_{n+3} \end{bmatrix}^T.$$

The left product of $\mathcal{DGC}$ Fibonacci quaternions $\widetilde{Q}_n$ and $\widetilde{Q}_m$ can also be expressed as:

$$\widetilde{Q}_n \widetilde{Q}_m = \begin{bmatrix} \widetilde{F}_m & -\widetilde{F}_{m+1} & -\widetilde{F}_{m+2} & -\widetilde{F}_{m+3} \\ \widetilde{F}_{m+1} & \widetilde{F}_m & \widetilde{F}_{m+3} & -\widetilde{F}_{m+2} \\ \widetilde{F}_{m+2} & -\widetilde{F}_{m+3} & \widetilde{F}_m & \widetilde{F}_{m+1} \\ \widetilde{F}_{m+3} & \widetilde{F}_{m+2} & -\widetilde{F}_{m+1} & \widetilde{F}_m \end{bmatrix} \begin{bmatrix} \widetilde{F}_n \\ \widetilde{F}_{n+1} \\ \widetilde{F}_{n+2} \\ \widetilde{F}_{n+3} \end{bmatrix}.$$

So, we can say that the multiplication of $\mathcal{DGC}$ Fibonacci quaternions can be calculated by matrix product.

Corollary 3. *The multiplication of $\widetilde{Q}_n$ as $\mathcal{DGC}$ number can also be given as*

$$\widetilde{Q}_n\widetilde{Q}_m = \begin{bmatrix} Q_m & \mathfrak{p}Q_{m+1} & 0 & 0 \\ Q_{m+1} & Q_m & 0 & 0 \\ Q_{m+2} & \mathfrak{p}Q_{m+3} & Q_m & \mathfrak{p}Q_{m+1} \\ Q_{m+3} & Q_{m+2} & Q_{m+1} & Q_m \end{bmatrix} \begin{bmatrix} Q_n \\ Q_{n+1} \\ Q_{n+2} \\ Q_{n+3} \end{bmatrix}.$$

Corollary 4. *The following statements are satisfied:*

- *$\mathcal{DGC}$ Fibonacci right matrix representation $A_{\widetilde{Q}_n}$ of $\widetilde{Q}_n = \widetilde{F}_n + \widetilde{F}_{n+1}e_1 + \widetilde{F}_{n+2}e_2 + \widetilde{F}_{n+3}e_3$ is also of the form $A_{\widetilde{Q}_n} = \widetilde{F}_n I_4 + \widetilde{F}_{n+1}E_1 + \widetilde{F}_{n+2}E_2 + \widetilde{F}_{n+3}E_3$, where*

$$E_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

  Similar form can be given using left matrix representation.

- *Fibonacci quaternion matrix representation $\Delta_{\widetilde{Q}_n}$ of $\widetilde{Q}_n = Q_n + Q_{n+1}J + Q_{n+2}\varepsilon + Q_{n+3}J\varepsilon$ is also in the form $\Delta_{\widetilde{Q}_n} = Q_n I_4 + Q_{n+1}\mathcal{J} + Q_{n+2}\mathcal{E} + Q_{n+3}\mathcal{J}\mathcal{E}$, where*

$$\mathcal{J} = \begin{bmatrix} 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{p} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Corollary 5. *Similar to the above notations and statements, the following matrix representations are obtained for $\widetilde{K}_n = K_n + K_{n+1}J + K_{n+2}\varepsilon + K_{n+3}J\varepsilon = \widetilde{L}_n + \widetilde{L}_{n+1}e_1 + \widetilde{L}_{n+2}e_2 + \widetilde{L}_{n+3}e_3$:*

$$\Lambda_{\widetilde{K}_n} = \begin{bmatrix} K_n + K_{n+1}J & 0 \\ K_{n+2} + K_{n+3}J & K_n + K_{n+1}J \end{bmatrix}, \quad \Delta_{\widetilde{K}_n} = \begin{bmatrix} K_n & \mathfrak{p}K_{n+1} & 0 & 0 \\ K_{n+1} & K_n & 0 & 0 \\ K_{n+2} & \mathfrak{p}K_{n+3} & K_n & \mathfrak{p}K_{n+1} \\ K_{n+3} & K_{n+2} & K_{n+1} & K_n \end{bmatrix},$$

$$A_{\widetilde{K}_n} = \begin{bmatrix} \widetilde{L}_n & -\widetilde{L}_{n+1} & -\widetilde{L}_{n+2} & -\widetilde{L}_{n+3} \\ \widetilde{L}_{n+1} & \widetilde{L}_n & \widetilde{L}_{n+3} & -\widetilde{L}_{n+2} \\ \widetilde{L}_{n+2} & -\widetilde{L}_{n+3} & \widetilde{L}_n & \widetilde{L}_{n+1} \\ \widetilde{L}_{n+3} & \widetilde{L}_{n+2} & -\widetilde{L}_{n+1} & \widetilde{L}_n \end{bmatrix}, \quad B_{\widetilde{K}_n} = \begin{bmatrix} \widetilde{L}_n & -\widetilde{L}_{n+1} & -\widetilde{L}_{n+2} & -\widetilde{L}_{n+3} \\ \widetilde{L}_{n+1} & \widetilde{L}_n & -\widetilde{L}_{n+3} & \widetilde{L}_{n+2} \\ \widetilde{L}_{n+2} & \widetilde{L}_{n+3} & \widetilde{L}_n & -\widetilde{L}_{n+1} \\ \widetilde{L}_{n+3} & -\widetilde{L}_{n+2} & \widetilde{L}_{n+1} & \widetilde{L}_n \end{bmatrix}.$$

Also the product of $\widetilde{K}_n$ and $\widetilde{K}_m$ can be calculated by matrix product as follows:

$$\widetilde{K}_n\widetilde{K}_m = \begin{bmatrix} \widetilde{L}_m & -\widetilde{L}_{m+1} & -\widetilde{L}_{m+2} & -\widetilde{L}_{m+3} \\ \widetilde{L}_{m+1} & \widetilde{L}_m & \widetilde{L}_{m+3} & -\widetilde{L}_{m+2} \\ \widetilde{L}_{m+2} & -\widetilde{L}_{m+3} & \widetilde{L}_m & \widetilde{L}_{m+1} \\ \widetilde{L}_{m+3} & \widetilde{L}_{m+2} & -\widetilde{L}_{m+1} & \widetilde{L}_m \end{bmatrix} \begin{bmatrix} \widetilde{L}_n \\ \widetilde{L}_{n+1} \\ \widetilde{L}_{n+2} \\ \widetilde{L}_{n+3} \end{bmatrix}$$

$$= \begin{bmatrix} K_m & \mathfrak{p}K_{m+1} & 0 & 0 \\ K_{m+1} & K_m & 0 & 0 \\ K_{m+2} & \mathfrak{p}K_{m+3} & K_m & \mathfrak{p}K_{m+1} \\ K_{m+3} & K_{m+2} & K_{m+1} & K_m \end{bmatrix} \begin{bmatrix} K_n \\ K_{n+1} \\ K_{n+2} \\ K_{n+3} \end{bmatrix}.$$

3 Conclusion

In this study, we introduce $\mathcal{DGC}$ Fibonacci and Lucas quaternions as an extension of complex/dual Fibonacci and Lucas quaternions. We extend the recurrence relations and well-known identities for these quaternions. Additionally, we give *quaternion*, $\mathcal{DGC}$ and *Fibonacci matrix forms* of them. One can also realized that this study is a continuation of the paper [13], considering Hamilton quaternion approach. By this way, this quaternion approach is also strictly linked to the papers [5, 12, 32]. Eventually, the striking part of this paper is that the identities and matrix forms of

- dual-complex Fibonacci and Lucas quaternions by taking $\mathfrak{p} = -1$,
- hyper-dual Fibonacci and Lucas quaternions by taking $\mathfrak{p} = 0$,
- dual-hyperbolic Fibonacci and Lucas quaternions by taking $\mathfrak{p} = 1$

can be found.

A Fibonacci and Lucas Numbers

Let F_n and L_n be nth Fibonacci and Lucas numbers, respectively. The following identities hold (see [3]):

$$F_{n+r} + F_{n-r} = \begin{cases} L_r F_n, & r = 2k, \\ F_r L_n, & r = 2k+1; \end{cases} \tag{1}$$

$$F_{n+r} - F_{n-r} = \begin{cases} F_r L_n, & r = 2k, \\ L_r F_n, & r = 2k+1; \end{cases} \tag{2}$$

$$L_{n+r} + L_{n-r} = \begin{cases} L_r L_n, & r = 2k, \\ 5F_r F_n, & r = 2k+1; \end{cases}$$

$$L_{n+r} - L_{n-r} = \begin{cases} 5F_r F_n, & r = 2k, \\ L_r L_n, & r = 2k+1. \end{cases}$$

B $\mathcal{DGC}$ Fibonacci and Lucas Numbers

The set of dual-generalized complex ($\mathcal{DGC}$) numbers $\mathbb{DC}_{\mathfrak{p}}$ is a vector space over $\mathbb{R}$ [13]. For $w_1 = z_{11} + z_{12}\varepsilon, w_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $\lambda \in \mathbb{R}$, the operations are given as follows (see [13]):

$$\begin{aligned} \text{equality}: &\quad w_1 = w_2 \;\Leftrightarrow z_{11} = z_{21}, z_{12} = z_{22}, \\ \text{addition}: &\quad w_1 + w_2 = (z_{11} + z_{21}) + (z_{12} + z_{22})\,\varepsilon, \\ \text{scalar multiplication}: &\quad \lambda w_1 = \lambda\,(z_{11} + z_{12}\varepsilon) = (\lambda z_{11}) + (\lambda z_{12})\,\varepsilon, \\ \text{multiplication}: &\quad w_1 w_2 = (z_{11} z_{21}) + (z_{11} z_{22} + z_{12} z_{21})\,\varepsilon. \end{aligned}$$

In [14], the nth $\mathcal{DGC}$ Fibonacci and Lucas numbers are respectively defined as follows:

$$\widetilde{F}_n = F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon \quad \text{and} \quad \widetilde{L}_n = L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon,$$

where J denotes the generalized complex unit ($J^2 = \mathfrak{p} \in \mathbb{R}$), ε represents the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), and $J\varepsilon$ represents the generalized complex-dual unit.

In [14], following identities are obtained:

$$\widetilde{F}_n + \widetilde{F}_{n+1} = \widetilde{F}_{n+2}, \tag{3}$$

$$\widetilde{F}_{n+r} + \widetilde{F}_{n-r} = \begin{cases} L_r\widetilde{F}_n, & r = 2k \\ F_r\widetilde{L}_n, & r = 2k+1, \end{cases} \tag{4}$$

$$\widetilde{F}_{n+r} - \widetilde{F}_{n-r} = \begin{cases} F_r\widetilde{L}_n, & r = 2k \\ L_r\widetilde{F}_n, & r = 2k+1, \end{cases} \tag{5}$$

$$\widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 = \widetilde{F}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J + (F_{2n+3} + 2\mathfrak{p}F_{2n+5})\,\varepsilon + 3F_{2n+4}J\varepsilon, \tag{6}$$

$$\widetilde{F}_m\widetilde{F}_n - \widetilde{F}_{m+r}\widetilde{F}_{n-r} = (-1)^{n-r}F_{m-n+r}F_r\;\left[(1-\mathfrak{p}) + J + 3\,(1-\mathfrak{p})\,\varepsilon + 3J\varepsilon\right], \tag{7}$$

$$\begin{aligned}\widetilde{F}_n\widetilde{F}_m + \widetilde{F}_{n+1}\widetilde{F}_{m+1} = \widetilde{F}_{n+m+1} &+ \mathfrak{p}F_{n+m+3} + F_{n+m+2}J \\ &+ (F_{n+m+3} + 2\mathfrak{p}F_{n+m+5})\,\varepsilon + 3F_{n+m+4}J\varepsilon,\end{aligned} \tag{8}$$

$$\widetilde{L}_n + \widetilde{L}_{n+1} = \widetilde{L}_{n+2}, \tag{9}$$

$$\widetilde{L}_{n+r} + \widetilde{L}_{n-r} = \begin{cases} L_r\widetilde{L}_n, & r = 2k \\ 5F_r\widetilde{F}_n, & r = 2k+1, \end{cases} \tag{10}$$

$$\widetilde{L}_{n+r} - \widetilde{L}_{n-r} = \begin{cases} 5F_r\widetilde{F}_n, & r = 2k \\ L_r\widetilde{L}_n, & r = 2k+1, \end{cases} \tag{11}$$

$$\widetilde{L}_n^2 + \widetilde{L}_{n+1}^2 = 5\left(\widetilde{F}_{2n+1} + \mathfrak{p}F_{2n+3} + F_{2n+2}J + (F_{2n+3} + 2\mathfrak{p}F_{2n+5})\,\varepsilon + 3F_{2n+4}J\varepsilon\right),$$

$$\widetilde{L}_m\widetilde{L}_n - \widetilde{L}_{m+r}\widetilde{L}_{n-r} = 5(-1)^{n-r+1}F_{m-n+r}F_r\;\left[(1-\mathfrak{p}) + J + 3\,(1-\mathfrak{p})\,\varepsilon + 3J\varepsilon\right],$$

$$\begin{aligned}\widetilde{L}_n\widetilde{L}_m + \widetilde{L}_{n+1}\widetilde{L}_{m+1} = 5\Big[&\widetilde{F}_{n+m+1} + \mathfrak{p}F_{n+m+3} + F_{n+m+2}J \\ &+ (F_{n+m+3} + 2\mathfrak{p}F_{n+m+5})\,\varepsilon + 3F_{n+m+4}J\varepsilon\Big].\end{aligned}$$

Furthermore, the nega$\mathcal{DGC}$ Fibonacci and Lucas numbers can be written as follows:

$$\widetilde{F}_{-n} = (-1)^{n+1}\widetilde{F}_n + (-1)^n L_n\,(J + \varepsilon + 2J\varepsilon)\,, \tag{12}$$

$$\widetilde{L}_{-n} = (-1)^n\widetilde{L}_n + 5(-1)^{n-1}F_n\,(J + \varepsilon + 2J\varepsilon)\,. \tag{13}$$

Moreover, the Binet's formulas for $\mathcal{DGC}$ Fibonacci and Lucas numbers are given by:

$$\widetilde{F}_n = \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha - \beta}, \tag{14}$$

$$\widetilde{L}_n = \alpha^*\alpha^n + \beta^*\beta^n, \tag{15}$$

where $\alpha^* = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$ and $\beta^* = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$ with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

References

[1] Akar M., Yüce S., Şahin S. *On the dual hyperbolic numbers and the complex hyperbolic numbers*. Journal of Computer Science and Computational Mathematics 2018, **8** (1), 1–6. doi:10.20967/jcscm.2018.01.001

[2] Alfsmann D. *On families of 2n-dimensional hypercomplex algebras suitable for digital signal processing*. In: 14th European Signal Processing Conf. (EUSIPCO 2006), Florence, Italy, September 4–8, 2006. doi:10.5281/ZENODO.52940

[3] Bergum G.E., Hoggat JR. V.E. *Sums and products for recurring sequences*. Fibonacci Quart. 1975, **13** (2), 115–120.

[4] Cheng H.H., Thompson S. *Dual polynomials and complex dual numbers for analysis of spatial mechanisms*. In: Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, August 19-22, 1996. doi:10.1115/96-DETC/MECH-1221

[5] Cihan A., Azak A.Z., Güngör M.A., Tosun M. *A study on dual hyperbolic Fibonacci and Lucas numbers*. Ann. Sci. Univ. "Ovidius" Constanta. Ser. Mat. 2019, **27** (1), 35–48. doi:10.2478/auom-2019-0002

[6] Cheng H.H., Thompson S. *Singularity analysis of spatial mechanisms using dual polynomials and complex dual numbers*. ASME. J. Mech. Des. 1999, **121** (2), 200–205. doi:10.1115/1.2829444

[7] Cockle J. *On a new imaginary in algebra*. Philosophical magazine. London-Dublin-Edinburgh 1849, **3** (34), 37–47. doi:10.1080/14786444908646169

[8] Cohen A., Shoham M. *Principle of transference-an extension to hyper-dual numbers*. Mech. Mach. Theory 2018, **125**, 101–110. doi:10.1016/j.mechmachtheory.2017.12.007

[9] Fike J.A., Alonso J.J. *The development of hyper-dual numbers for exact second- derivative calculations*. In: The 49th AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition, Orlando, Florida, January 4 - 7, 2011. doi:10.2514/6.2011-886

[10] Fike J.A., Alonso J.J. *Automatic differentiation through the use of hyper-dual numbers for second derivatives*. In: Forth S., Hovland P., Phipps E., Utke J., Walther A. (eds) Recent Advances in Algorithmic Differentiation. Lecture Notes in Computational Science and Engineering, **87**, 163–173. Springer, Berlin, Heidelberg, 2012. doi:10.1007/978-3-642-30023-3_15

[11] Fjelstad P., Sorin Gal G. *n-dimensional hyperbolic complex numbers*. Adv. Appl. Clifford Algebr. 1998, **8** (1), 47–68. doi:10.1007/BF03041925

[12] Güngör M.A., Azak A.Z. *Investigation of dual-complex Fibonacci, dual-complex Lucas numbers and their properties*. Adv. Appl. Clifford Algebr. 2017, **27** (4), 3083–3096. doi: 10.1007/s00006-017-0813-z

[13] Gürses N., Şentürk G.Y., Yüce S. *A study on dual-generalized complex and hyperbolic-generalized complex numbers*. Gazi University Journal of Science 2021, **34** (1), 180–194. doi: 10.35378/gujs.653906

[14] Gürses N., Şentürk G.Y., Yüce S. *A comprehensive survey of dual-generalized complex Fibonacci and Lucas numbers*. Sigma J Eng Nat Sci 2022, **40** (1), 179–187. doi:10.14744/sigma.2022.00014

[15] Halici S. *On Fibonacci quaternions*. Adv. Appl. Clifford Algebr. 2012, **22** (2), 321–327. doi:10.1007/s00006-011-0317-1

[16] Halici S. *On complex Fibonacci quaternions*. Adv. Appl. Clifford Algebr. 2013, **23** (1), 105–112. doi:10.1007/s00006-012-0337-5

[17] Hamilton W.R. *On quaternions; or on a new system of imaginaries in algebra*. The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science (3rd Series) **xxv-xxxvi**, 1844–1850.

[18] Harkin A.A., Harkin J.B. *Geometry of generalized complex numbers*. Math. Mag. 2004, **77** (2), 118–129. doi:10.1080/0025570X.2004.11953236

[19] Horadam A.F. *Complex Fibonacci numbers and Fibonacci quaternions*. Amer. Math. Monthly 1963, **70** (3), 289–291. doi:10.2307/2313129

[20] Horadam A.F. *Quaternion recurrence relations*. Ulam Quarterly 1993, **2** (2), 23–33.

[21] Iyer M.R. *Some results on Fibonacci quaternions*. Fibonacci Quart. 1969, **7** (2), 201–210.

[22] Iyer M.R. *A note on Fibonacci quaternions*. Fibonacci Quart. 1969, **3** (7), 225–229.

[23] Kantor I., Solodovnikov A. *Hypercomplex numbers: an elementary introduction to algebras*. Springer-Verlag, New York, 1989.

[24] Majernik V. *Multicomponent number systems*. Acta Phys. Polon. A 1996, **90** (3), 491–498. doi:10.12693/APhysPolA.90.491

[25] Messelmi F. *Dual-complex numbers and their holomorphic functions*. doi:10.5281/ZENODO.22961

[26] Nurkan S.K, Güven I.A. *Dual Fibonacci quaternions*. Adv. Appl. Clifford Algebr. 2015, **25** (2), 403–414. doi:10.1007/s00006-014-0488-7

[27] Pennestrì E., Stefanelli R. *Linear algebra and numerical algorithms using dual numbers*. Multibody Syst. Dyn. 2007, **18** (3), 323–344. doi:10.1007/s11044-007-9088-9

[28] Price G.B. An introduction to multicomplex spaces and functions. Monographs and textbooks in pure and applied mathematics, New-York. 1991. doi:10.1201/9781315137278

[29] Rochon D., Shapiro M. *On algebraic properties of bicomplex and hyperbolic numbers*. An. Univ. Oradea Fasc. Mat. 2004, **11**, 71–110.

[30] Sobczyk G. *The hyperbolic number plane*. College Math. J. 1995, **26** (4), 268–280. doi:10.1080/07468342.1995.11973712

[31] Study E. Geometrie der dynamen: Die zusammensetzung von kräften und verwandte gegenstände der geometrie bearb. Leipzig, B.G. Teubner, 1903.

[32] Tan E., Ait-Amrane N. R., Gök I. *Hyper-dual Horadam quaternions*. Miskolc Math. Notes 2021, **22** (2), 903–913. doi: 10.18514/MMN.2021.3747

[33] Toyoshima H. *Computationally efficient bicomplex multipliers for digital signal processing*. IEICE Trans Inf Syst. 1989, **E81-D** (2), 236–238.

[34] Yaglom I.M. Complex numbers in geometry. Academic Press, New York, 1968.

[35] Yaglom I.M. A simple non-Euclidean Geometry and its physical basis: An Elementary Account of Galilean Geometry and the Galilean Principle of Relativity. Heidelberg Science Library, 1979.

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Шентурк Г.Й., Ґурсес Н., Юце С. *Побудова дуально-узагальнених комплексних кватерніонів Фібоначчі та Люка* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 406–418.

Метою статті є побудова дуально-узагальнених кватерніонів Фібоначчі та Люка. Нами встановлено деякі властивості дуально-узагальнених комплексних чисел та їхніх кватерніонів. Зокрема, ми одержали загальні рекурентні співвідношення, формули Біне, тотожності Тагіурі, Гонсбергера, Оканя, Кассіні та Каталана. Також ми запровадили деякі матричні представлення цих спеціальних кватерніонів і виразили добуток дуально-узагальнених комплексних кватерніонів Фібоначчі і Люка у вигляді їхніх різних матричних представлень.

Ключові слова і фрази: кватерніон, дуально-узагальнене комплексне число, число Фібоначчі, число Люка.