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Invariant subspaces, exact solutions and stability analysis of nonlinear water wave equations

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Abstract

The key purpose of the present research is to derive the exact solutions of nonlinear water wave equations (NLWWEs) in oceans through the invariant subspace scheme (ISS). In this respect, the NLWWEs which describe specific nonlinear waves are converted to a number of systems of ordinary differential equations (ODEs) such that the resulting systems can be efficiently handled by computer algebra systems. As an accomplishment, the performance of the well-designed ISS in extracting a group of exact solutions is formally confirmed. In the end, the stability analysis for the NLWWE is investigated through the linear stability scheme.

Keywords: Nonlinear water wave equations; Invariant subspace scheme; Exact solutions; Stability analysis.

1. Introduction

One of the attractive topics in the area of mathematical physics is to search the exact solutions of nonlinear partial differential equations (NLPDEs). The importance of this topic of research comes from this fact that many information is obtained from the exact solutions. In the last decades, a lot of techniques such as Kudryashov scheme [1–6], sine-Gordon expansion scheme [7,8], Jacobi elliptic function scheme [9,10], ansatz scheme [11–14], sub-equation scheme [15–24], similarity transformation scheme [25] and invariant subspace scheme [26–33] have been used to acquire the exact solutions of NLPDEs.

The invariant subspace scheme is considered as one of the commonly used techniques, which has gained considerable notice owing to its performance in extracting the exact solutions of NLPDEs. Herein, a number of recently accomplished works which have benefited from advantages of the invariant subspace scheme are cited. Ma and Liu [31] utilized the ISS to seek the explicit solutions of a group of dispersive evolution equations. In another work, Sahadevan and Prakash [32] applied the ISS to search the exact solutions of a bunch of coupled time-fractional PDEs. Liu [33] employed the ISS to find the exact solutions of the generalized nonlinear diffusion–convection equation.

In the present paper, our purpose is extracting the exact solutions of the following nonlinear models

- Nonlinear water wave equation (NLWWE) [34,35]

$$u_t = \mathcal{F} = -(u_x + c_1 u u_x + c_2 u_{xxx} + c_3 u_x u_{xx} + c_4 u u_{xxx} + c_5 u_{xxxxx}), \tag{1}$$

and

- Nonlinear dispersive water wave equations (NLDWWEs) [36]

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$$u_t = \mathcal{G} = \frac{1}{2}\alpha_1 u_{xx} + \alpha_2 u v_x + \alpha_3 v u_x, \quad (2.1)$$

$$v_t = \mathcal{H} = -\frac{1}{2}\beta_1 v_{xx} + \beta_2 u_x + \beta_3 v v_x^2, \quad (2.2)$$

by applying the ISS. The models (1) and (2) have been solved by a number of different schemes; for example, the ansatz and tanh schemes [34,35] to obtain the exact solutions of NLWWE and the extended tanh scheme [36] to derive the exact solutions of NLDWWEs.

The rest of the current work is as follows: in Section 2, a short summary of the fundamental of invariant subspace scheme is presented. In Section 3, based on the use of various invariant subspaces, a wide range of exact solutions of nonlinear water wave models (1) and (2) is extracted. In Section 4, the stability analysis for the NLWWE is investigated through the linear stability scheme. The results of the current study are provided in the last section.

2. Invariant subspace scheme

Consider the given nonlinear PDE

$$\frac{\partial u}{\partial t} = \mathcal{P}(u), \quad (3)$$

in which $\mathcal{P}$ is a nonlinear operator of the differential type with respect to the variable x.

Definition 1. Let $\mathcal{V}_n = \text{span}\{\mathcal{S}_1(x), \mathcal{S}_2(x), \ldots, \mathcal{S}_n(x)\}$ and $\mathcal{P}(\mathcal{V}_n) \subseteq \mathcal{V}_n$. Then, the linear space $\mathcal{V}_n$ is identified as an invariant subspace with respect to (3).

Theorem 1. *If $\mathcal{V}_n = \text{span}\{\mathcal{S}_1(x), \mathcal{S}_2(x), \ldots, \mathcal{S}_n(x)\}$ is an invariant subspace with respect to (3), then there exist n functions $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ so that*

$$\mathcal{P}\left[\sum_{\mathcal{I}=1}^{n} \mathcal{X}_{\mathcal{I}}\mathcal{S}_{\mathcal{I}}(x)\right] = \sum_{\mathcal{I}=1}^{n} \mathcal{T}_{\mathcal{I}}(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n)\mathcal{S}_{\mathcal{I}}(x).$$

Additionally

$$u(x,t) = \sum_{\mathcal{I}=1}^{n} \mathcal{X}_{\mathcal{I}}(t)\mathcal{S}_{\mathcal{I}}(x),$$

is the solution of NLPDE (3), if $\mathcal{X}_{\mathcal{I}}(t),\ \mathcal{I} = 1, \ldots, n$ satisfy the following system of ODEs

$$\frac{d\mathcal{X}_{\mathcal{I}}(t)}{dt} = \mathcal{T}_{\mathcal{I}}(\mathcal{X}_1(t), \mathcal{X}_2(t), \ldots, \mathcal{X}_n(t)), \quad \mathcal{I} = 1, \ldots, n.$$

Theorem 2. *Presume that the functions $\mathcal{S}_1(x)$, $\mathcal{S}_2(x)$, $\ldots$, $\mathcal{S}_n(x)$ are the fundamental set of solutions of a linear ODE as*

$$\mathcal{L}[y] \equiv \mathfrak{D}^n y + a_1(x)\mathfrak{D}^{n-1}y + \cdots + a_{n-1}(x)\mathfrak{D}y + a_n(x)y = 0,$$

$$\mathfrak{D}^n = \frac{d^n}{dx^n}.$$

Then, the linear space $\mathcal{V}_n = \text{span}\{\mathcal{S}_1(x), \mathcal{S}_2(x), \ldots, \mathcal{S}_n(x)\}$ is an invariant subspace with respect to the nonlinear operator $\mathcal{P}$ if and only if

$$\mathcal{L}(\mathcal{P}(u))|_{\mathcal{L}[u]=0} = 0$$

3. Applications

In the current section, the exact solutions of nonlinear water wave equations in oceans are formally acquired through the invariant subspace scheme.

3.1. NLWWE and its invariant subspaces

This subsection deals with the nonlinear water wave equation and its invariant subspaces. By assuming $n = 2$, we arrive at an invariant subspace $\mathcal{V}_2$ given by a linear ODE of second-order as

$$\mathcal{V}_2 = \{y|\mathcal{L}[y] = y'' + a_1 y' + a_0 y = 0\}, \quad (4)$$

where a_0 and a_1 are constants to be calculated. The corresponding invariance condition is

$$(\mathfrak{D}^2\mathcal{F} + a_1\mathfrak{D}\mathcal{F} + a_0\mathcal{F})|_{u\in\mathcal{V}_2} = 0. \quad (5)$$

Setting the expression $\mathcal{F}$ in Eq. (5) and replacing $u''(x)$ by $-a_1u'(x) - a_0u(x)$ several times results in a nonlinear algebraic set as follows

$$\begin{aligned} &u^2: \ a_0^2 a_1(2c_3 + c_4) = 0,\\ &uu_x: -3a_0^2(c_3 + c_4) + a_0 a_1^2(4c_3 + 3c_4) + 3a_0c_1 = 0,\\ &u_x^2: \ 2a_1^3(c_3 + c_4) + 2a_1c_1 - a_0a_1(3c_3 + 4c_4) = 0. \end{aligned}$$

Solving the above system through MAPLE yields

$$\begin{aligned} &a_0 = 0, \quad a_1 = 0,\\ &a_0 = \frac{c_1}{c_3 + c_4}, \quad a_1 = 0, \quad c_3 + c_4 \neq 0,\\ &a_0 = 0, \quad a_1 = \pm\frac{\sqrt{-(c_3 + c_4)c_1}}{c_3 + c_4}, \quad c_3 + c_4 \neq 0. \end{aligned}$$

After substituting the above values into Eq. (4), we, respectively, find

$$\begin{aligned} &\mathcal{L}_1[y] = y'' = 0, \qquad \mathcal{V}_2^1 = \text{span}\{1, x\},\\ &\mathcal{L}_2[y] = y'' + \frac{c_1}{c_3 + c_4}y = 0,\\ &\mathcal{V}_2^2 = \text{span}\left\{\sin\left(\frac{\sqrt{c_1}}{\sqrt{c_3 + c_4}}x\right), \cos\left(\frac{\sqrt{c_1}}{\sqrt{c_3 + c_4}}x\right)\right\},\\ &\mathcal{L}_3[y],\ \mathcal{L}_4[y] = y'' \pm \frac{\sqrt{-(c_3 + c_4)c_1}}{c_3 + c_4}y' = 0,\\ &\mathcal{V}_2^3,\ \mathcal{V}_2^4 = \text{span}\left\{1, \exp\left(\mp\frac{\sqrt{-(c_3 + c_4)c_1}}{c_3 + c_4}x\right)\right\}. \end{aligned}$$

3.1.1. Exact solutions of the NLWWE

Due to the $\mathcal{L}_1[y] = y'' = 0$ and its invariant subspace, we assume the solution of the model (1) as

$$u(x,t) = \mathcal{X}_1(t) + \mathcal{X}_2(t)x.$$

Setting the above solution in Eq. (1) results in

$$\mathcal{X}_1'(t) + \mathcal{X}_2'(t)x = -c_1\mathcal{X}_1(t)\mathcal{X}_2(t) - \mathcal{X}_2(t) - c_1\mathcal{X}_2^2(t)x,$$

therefore

$$\begin{cases} \dfrac{d\mathcal{X}_1(t)}{dt} = -c_1\mathcal{X}_1(t)\mathcal{X}_2(t) - \mathcal{X}_2(t), \\ \dfrac{d\mathcal{X}_2(t)}{dt} = -c_1\mathcal{X}_2^2(t). \end{cases}$$

By solving the above system, the solution is procured as below

$$u(x,t) = \frac{x}{c_1 t + d_1} - \frac{t - d_2}{c_1 t + d_1}.$$

In the second case, owing to the $\mathcal{L}_2[y] = y'' + \frac{c_1}{c_3+c_4}y = 0$ and its invariant subspace, we suppose the solution of the model (1) as

$$u(x,t) = \mathcal{X}_1(t)\sin\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right) + \mathcal{X}_2(t)\cos\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right).$$

Substituting the above solution into Eq. (1) yields

$$\mathcal{X}_1'(t)\sin\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right) + \mathcal{X}_2'(t)\cos\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right)$$
$$= \mathcal{C}\left(\mathcal{X}_2(t)\sin\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right) - \mathcal{X}_1(t)\cos\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right)\right),$$
$$\mathcal{C} = \frac{\left((c_3+c_4)^2 - c_1c_2(c_3+c_4) + c_1^2c_5\right)\sqrt{c_1}}{(c_3+c_4)^{5/2}},$$

so

$$\begin{cases} \dfrac{d\mathcal{X}_1(t)}{dt} = \mathcal{C}\mathcal{X}_2(t), \\ \dfrac{d\mathcal{X}_2(t)}{dt} = -\mathcal{C}\mathcal{X}_1(t). \end{cases}$$

Through solving the above system of ODEs, the solution is gained as follows

$$u(x,t) = (-d_1\cos(\mathcal{C}t) + d_2\sin(\mathcal{C}t))\sin\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right)$$
$$+ (d_2\cos(\mathcal{C}t) + d_1\sin(\mathcal{C}t))\cos\left(\frac{\sqrt{c_1}}{\sqrt{c_3+c_4}}x\right),$$

in which

$$\mathcal{C} = \frac{\left((c_3+c_4)^2 - c_1c_2(c_3+c_4) + c_1^2c_5\right)\sqrt{c_1}}{(c_3+c_4)^{5/2}}.$$

In the third case, due to the $\mathcal{L}_3[y]$, $\mathcal{L}_4[y] = y'' \pm \frac{\sqrt{-(c_3+c_4)c_1}}{c_3+c_4}y' = 0$ and their invariant subspaces, we consider the solutions of the model (1) as below

$$u(x,t) = \mathcal{X}_1(t) + \mathcal{X}_2(t)\exp\left(\mp\frac{\sqrt{-(c_3+c_4)c_1}}{c_3+c_4}x\right).$$

Inserting the above solution into Eq. (1) gives

$$\mathcal{X}_1'(t) + \mathcal{X}_2'(t)\exp\left(\mp\frac{\sqrt{-(c_3+c_4)c_1}}{c_3+c_4}x\right)$$
$$= \pm\frac{\sqrt{-(c_3+c_4)c_1}}{(c_3+c_4)^3}\left(c_1^2c_5 + c_1(-c_2(c_3+c_4)\right.$$
$$\left. + \mathcal{X}_1(t)(c_3^2+c_3c_4)\right) + (c_3+c_4)^2\big)\mathcal{X}_2(t)$$
$$\times\exp\left(\mp\frac{\sqrt{-(c_3+c_4)c_1}}{c_3+c_4}x\right),$$

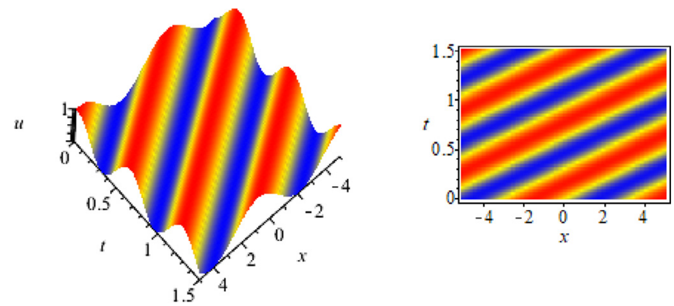

Fig. 1. The periodic solution for $d_1 = 1$, $d_2 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = -1$, $c_4 = 2$ and $c_5 = 10$.

thus

$$\begin{cases} \dfrac{d\mathcal{X}_1(t)}{dt} = 0, \\ \dfrac{d\mathcal{X}_2(t)}{dt} = \pm\dfrac{\sqrt{-(c_3+c_4)c_1}}{(c_3+c_4)^3}\left(c_1^2c_5 + c_1(-c_2(c_3+c_4)\right. \\ \quad \left. + \mathcal{X}_1(t)(c_3^2+c_3c_4)) + (c_3+c_4)^2\right)\mathcal{X}_2(t). \end{cases}$$

By solving the above system, the solutions are derived as

$$u(x,t) = d_1 + d_2\exp\left(\mp\frac{\sqrt{-(c_3+c_4)c_1}}{c_3+c_4}x \pm \mathcal{C}t\right),$$

in which

$$\mathcal{C} = \frac{\sqrt{-(c_3+c_4)c_1}}{(c_3+c_4)^3}\left(c_1^2c_5 + c_1(-c_2(c_3+c_4)\right.$$
$$\left. + d_1(c_3^2+c_3c_4)) + (c_3+c_4)^2\right).$$

The graphical representation of the periodic solution for appropriate values of parameters is presented in Fig. 1.

3.2. NLDWWEs and their invariant subspaces

Now, our aim is to classify the invariant subspaces of nonlinear dispersive water wave equations. Based on the theorem (2), when $n = 2$, we are able to get an invariant subspace $\mathcal{V}_{2,2} = \mathcal{V}_2^1 \times \mathcal{V}_2^2$ given by the following linear ODEs of second-order

$$\mathcal{V}_2^1 = \left\{y|\mathcal{L}_1[y] = y'' + a_1y' + a_0y = 0\right\},$$
$$\mathcal{V}_2^2 = \left\{z|\mathcal{L}_2[z] = z'' + b_1z' + b_0z = 0\right\}, \quad (6)$$

where a_0, a_1, b_0 and b_1 are the constants to be calculated. The corresponding invariance conditions are

$$(\mathfrak{D}^2\mathcal{G} + a_1\mathfrak{D}\mathcal{G} + a_0\mathcal{G})|_{u\in\mathcal{V}_2^1, v\in\mathcal{V}_2^2} = 0,$$
$$(\mathfrak{D}^2\mathcal{H} + b_1\mathfrak{D}\mathcal{H} + b_0\mathcal{H})|_{u\in\mathcal{V}_2^1, v\in\mathcal{V}_2^2} = 0. \quad (7)$$

By setting the expressions $\mathcal{G}$ and $\mathcal{H}$ in Eq. (7) and replacing $u''(x)$ by $-a_1u'(x) - a_0u(x)$ and $v''(x)$ by $-b_1v'(x) -$

$b_0v(x)$ several times, we arrive at a nonlinear algebraic system as

$$
\begin{aligned}
&uv : b_0(a_1-b_1)\alpha_2=0,\\
&uv_x : 2a_0\alpha_3+\left(a_1b_1+b_0-b_1^2\right)\alpha_2=0,\\
&u_xv : b_0(2\alpha_2+\alpha_3)=0,\\
&u_xv_x : a_1\alpha_3+b_1(2\alpha_2+\alpha_3)=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&u : a_0(a_1-b_1)\beta_2=0,\\
&u_x : \left(-a_0+b_0-a_1b_1+a_1^2\right)\beta_2=0,\\
&v^3 : 2b_0^2\beta_3=0,\\
&v^2v_x : 4b_0b_1\beta_3=0,\\
&vv_x^2 : 2\left(3b_0-b_1^2\right)\beta_3=0,\\
&v_x^3 : 4b_1\beta_3=0.
\end{aligned}
$$

After solving the above system, the following sets are gained

$$
\begin{aligned}
&a_0=0,\quad a_1=b_1,\quad b_0=0,\quad \alpha_2=-\alpha_3,\quad \beta_3=0,\\
&a_0=0,\quad a_1=0,\quad b_0=0,\quad b_1=0,\\
&b_0=0,\quad b_1=0,\quad \alpha_3=0,\quad \beta_2=0,\\
&a_1=0,\quad b_0=4a_0,\quad b_1=0,\quad \alpha_2=-\frac{1}{2}\alpha_3,\quad \beta_2=0,\quad \beta_3=0,\\
&a_0=\frac{1}{4}\left(a_1^2-b_1^2\right),\quad b_0=0,\quad \alpha_2=-\frac{1}{2}\frac{a_1+b_1}{b_1}\alpha_3,\quad \beta_2=0,\quad \beta_3=0.
\end{aligned}
$$

By inserting the obtained values into Eq. (6), we, respectively, derive

$$
\begin{cases}\mathcal{L}_1[y]=y''+b_1y'=0,\\ \mathcal{L}_2[z]=z''+b_1z'=0,\end{cases}
\qquad
\begin{cases}\mathcal{V}_2^1=\operatorname{span}\{1,\exp(-b_1x)\},\\ \mathcal{V}_2^2=\operatorname{span}\{1,\exp(-b_1x)\},\end{cases}
$$

$$
\begin{cases}\mathcal{L}_1[y]=y''=0,\\ \mathcal{L}_2[z]=z''=0,\end{cases}
\qquad
\begin{cases}\mathcal{V}_2^1=\operatorname{span}\{1,x\},\\ \mathcal{V}_2^2=\operatorname{span}\{1,x\},\end{cases}
$$

$$
\begin{cases}\mathcal{L}_1[y]=y''+a_1y'+a_0y=0,\\ \mathcal{L}_2[z]=z''=0,\end{cases}
\qquad
\begin{cases}\mathcal{V}_2^1=\operatorname{span}\left\{\exp\left(-\frac{1}{2}\left(a_1\mp\sqrt{a_1^2-4a_0}\right)x\right)\right\},\\ \mathcal{V}_2^2=\operatorname{span}\{1,x\},\end{cases}
$$

$$
\begin{cases}\mathcal{L}_1[y]=y''+a_0y=0,\\ \mathcal{L}_2[z]=z''+4a_0z=0,\end{cases}
\qquad
\begin{cases}\mathcal{V}_2^1=\operatorname{span}\left\{\sin\left(\sqrt{a_0}x\right),\cos\left(\sqrt{a_0}x\right)\right\},\\ \mathcal{V}_2^2=\operatorname{span}\left\{\sin\left(2\sqrt{a_0}x\right),\cos\left(2\sqrt{a_0}x\right)\right\},\end{cases}
$$

$$
\begin{cases}\mathcal{L}_1[y]=y''+a_1y'+\frac{1}{4}\left(a_1^2-b_1^2\right)y=0,\\ \mathcal{L}_2[z]=z''+b_1z'=0,\end{cases}
\qquad
\begin{cases}\mathcal{V}_2^1=\operatorname{span}\left\{\exp\left(-\frac{1}{2}(a_1\mp b_1)x\right)\right\},\\ \mathcal{V}_2^2=\operatorname{span}\{1,\exp(-b_1x)\}.\end{cases}
$$

3.2.1. Exact solutions of the NLDWWEs

Due to the

$$
\begin{cases}\mathcal{L}_1[y]=y''+b_1y'=0,\\ \mathcal{L}_2[z]=z''+b_1z'=0,\end{cases}
$$

and its invariant subspace, namely

$$
\begin{cases}\mathcal{V}_2^1=\operatorname{span}\{1,\exp(-b_1x)\},\\ \mathcal{V}_2^2=\operatorname{span}\{1,\exp(-b_1x)\},\end{cases}
$$

the solution is considered as

$$
\begin{aligned}
&u(x,t)=\mathcal{X}_1(t)+\mathcal{X}_2(t)\exp(-b_1x),\\
&v(x,t)=\mathcal{X}_3(t)+\mathcal{X}_4(t)\exp(-b_1x).
\end{aligned}
$$

By substituting the above solution along with the given parameters into the model (2), we find

$$
\begin{cases}
\mathcal{X}_1'(t)+\mathcal{X}_2'(t)\exp(-b_1x)=(b_1\alpha_3\mathcal{X}_1(t)\mathcal{X}_4(t)\\
\quad+\left(-b_1\alpha_3\mathcal{X}_3(t)+\frac{1}{2}b_1^2\alpha_1\right)\mathcal{X}_2(t))\exp(-b_1x),\\
\mathcal{X}_3'(t)+\mathcal{X}_4'(t)\exp(-b_1x)\\
\quad=\left(-b_1\beta_2\mathcal{X}_2(t)-\frac{1}{2}b_1^2\beta_1\mathcal{X}_4(t)\right)\exp(-b_1x),
\end{cases}
$$

and so

$$
\begin{cases}
\dfrac{d\mathcal{X}_1(t)}{dt}=0,\\
\dfrac{d\mathcal{X}_2(t)}{dt}=b_1\alpha_3\mathcal{X}_1(t)\mathcal{X}_4(t)\\
\quad+\left(-b_1\alpha_3\mathcal{X}_3(t)+\dfrac{1}{2}b_1^2\alpha_1\right)\mathcal{X}_2(t),\\
\dfrac{d\mathcal{X}_3(t)}{dt}=0,\\
\dfrac{d\mathcal{X}_4(t)}{dt}=-b_1\beta_2\mathcal{X}_2(t)-\frac{1}{2}b_1^2\beta_1\mathcal{X}_4(t).
\end{cases}
$$

By solving the above system, the solution is derived as

$$
\begin{aligned}
u(x,t)=d_1&+d_3\exp\left(b_1\left(-\frac{1}{4}(\mathcal{C}_1-\mathcal{C}_2)t-x\right)\right)\\
&+d_4\exp\left(b_1\left(-\frac{1}{4}(\mathcal{C}_1+\mathcal{C}_2)t-x\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
v(x,t)=d_2&\\
&+\frac{1}{4}\left(2\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1-\mathcal{C}_2)t\right)d_2d_3\alpha_3\right.\\
&-\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1-\mathcal{C}_2)t\right)b_1d_3\alpha_1\\
&-\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1-\mathcal{C}_2)t\right)b_1d_3\beta_1\\
&+2\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1+\mathcal{C}_2)t\right)d_2d_4\alpha_3\\
&-\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1+\mathcal{C}_2)t\right)b_1d_4\alpha_1\\
&-\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1+\mathcal{C}_2)t\right)b_1d_4\beta_1
\end{aligned}
$$

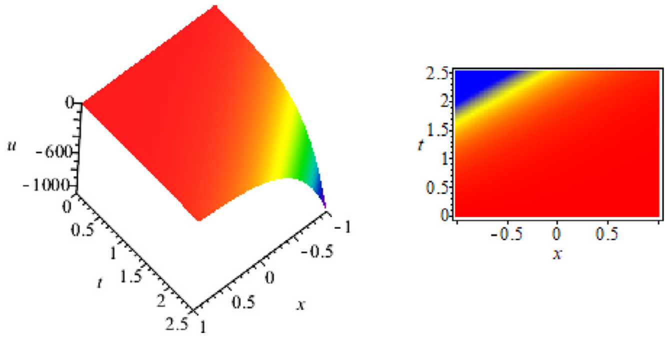

Fig. 2. The exponential solution for $d_1 = 1,\ d_2 = -1,\ d_3 = -1,\ d_4 = 1,\ \alpha_1 = 1,\ \alpha_3 = 1,\ \beta_1 = 1,\ \beta_2 = 2$ and $b_1 = 2$.

$$+\exp\left(-\frac{1}{4}b_1(\mathcal{C}_1-\mathcal{C}_2)t\right)\mathcal{C}_2 d_3 - \exp\left(-\frac{1}{4}b_1(\mathcal{C}_1+\mathcal{C}_2)t\right)\mathcal{C}_2 d_4\Bigg)\frac{\exp(-b_1 x)}{d_1\alpha_3},$$

in which

$$\begin{cases}\mathcal{C}_1 = 2d_2\alpha_3 + b_1(\beta_1-\alpha_1),\\ \mathcal{C}_2 = \sqrt{-16d_1\alpha_3\beta_2 + 4d_2^2\alpha_3^2 - 4(\alpha_1+\beta_1)b_1d_2\alpha_3 + b_1^2\alpha_1^2 + 2b_1^2\alpha_1\beta_1 + b_1^2\beta_1^2}.\end{cases}$$

Fig. 2 shows the graphical representation of the exponential solution for appropriate values of parameters.

Remark. It should be stated that other exact solutions can be gained for the nonlinear dispersive water wave equations owing to the above invariant subspaces.

4. Stability analysis and phase and group velocities for the NLWWE

Suppose that the NLWWE (1) has the following steady-state solution [37–41]

$$u(x,\ t) = \mathfrak{p} + s\mathcal{W}(x,\ t), \tag{8}$$

which $\mathfrak{p}$ indicates the incident power. Inserting Eq. (8) into the NLWWE (1) yields

$$s\mathcal{W}_t + s\mathcal{W}_x + c_1 s\mathfrak{p}\mathcal{W}_x + c_1 s^2\mathcal{W}\mathcal{W}_x + c_2 s\mathcal{W}_{xxx} + c_3 s^2\mathcal{W}_x\mathcal{W}_{xx} + c_4 s\mathfrak{p}\mathcal{W}_{xxx} + c_4 s^2\mathcal{W}\mathcal{W}_{xxx} + c_5 s\mathcal{W}_{xxxxx} = 0. \tag{9}$$

By linearizing Eq. (9), we acquire

$$\mathcal{W}_t + \mathcal{W}_x + c_1\mathfrak{p}\mathcal{W}_x + c_2\mathcal{W}_{xxx} + c_4\mathfrak{p}\mathcal{W}_{xxx} + c_5\mathcal{W}_{xxxxx} = 0. \tag{10}$$

Now, by considering

$$\mathcal{W}(x,\ t) = \exp(i\kappa x - \omega t),$$

in which κ illiustrates the normalized wave number and setting it in Eq. (10), we derive the following dispersion relation

$$\omega(\kappa) = i\kappa\left(c_5\kappa^4 - (c_2 + c_4\mathfrak{p})\kappa^2 + c_1\mathfrak{p} + 1\right).$$

It is obvious that the real part of the above dispersion relation is exactly zero and consequently, the steady-state solution is marginally stable.

To determine the phase and group velocities, by considering

$$\mathcal{W}(x,\ t) = \exp(i(\kappa x - \delta t)),$$

and substituting it into Eq. (10), we, respectively, procure the phase and group velocities as below

$$\frac{\delta(\kappa)}{\kappa} = c_5\kappa^4 - (c_2 + c_4\mathfrak{p})\kappa^2 + c_1\mathfrak{p} + 1,$$

$$\frac{d\delta(\kappa)}{d\kappa} = 5c_5\kappa^4 - 3(c_2 + c_4\mathfrak{p})\kappa^2 + c_1\mathfrak{p} + 1.$$

Now, if $\mathcal{W}(x,\ t) = \exp(i(\kappa x - \delta t))$ with $\delta = \delta(\kappa)$ satisfies Eq. (10), then the full-time evolution can be written as

$$\mathcal{W}(x,\ t) = \int_{-\infty}^{\infty}\mathcal{A}(\kappa)e^{i(\kappa x - \delta(\kappa)t)}d\kappa,$$

in which

$$\mathcal{A}(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{W}(x,\ 0)e^{-i\kappa x}dx.$$

5. Conclusion

The under investigation in the present work was to study the nonlinear water wave equations in oceans by means of the invariant subspace scheme. In this regard, the governing models which describe specific nonlinear waves were converted to systems of ODEs such that their exact solutions could be obtained by computer algebra systems. As a direct outcome, a number of exact solutions to the NLWWEs were procured, approving the effectiveness of invariant subspace scheme. The correctness of the solutions listed in the current study was checked by substituting them into their corresponding equations.

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