

POLYNOMIAL BASED DIFFERENTIAL QUADRATURE FOR NUMERICAL SOLUTIONS OF KURAMOTO-SIVASHINSKY EQUATION

by

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In this study, a numerical discrete derivative technique for solutions of Kuramoto-Sivashinsky equation is considered. According to the procedure, differential quadrature algorithm is adapted in space by using Chebyshev polynomials and explicit scheme is constructed to discretize time derivative. Sample problems are presented to support the idea. Numerical solutions are compared with exact solutions and also previous works. It is observed that the numerical solutions are well matched with the exact or existing solutions.

Key words: Kuramoto-Sivashinsky equation, differential quadrature, Chebyshev polynomials

Introduction

Numerical techniques to solve PDE which model many real life phenomenon, are widely used due to their fast and effective outcomes. Today, solving these kinds of problems both analytically or numerically attract many scientist. In this content differential quadrature (DQ) method is considered to solve Kuramoto-Sivasinsky (KS) equation.

The KS equation has been presented as models of phase turbulence in reaction-diffusion systems [1, 2], plasma instabilities and flame front propagation [3]. The model equation has been widely studied analytically and numerically. Collocation methods based on Chebyshev spectral scheme [4], quintic B-splines [5], exponential cubic B-splines [6], have been considered.

Solutions of KS equation analyzed by using various methods such as finite difference methods [7], discontinuous Galerkin method [8], numeric meshless method for space derivatives using radial basis function [9], He's variational iteration method [10]. Rademacher and Wattenberg [11] studied on viscous shocks for the model equation. Also, some control results of the equation are presented [12, 13].

The DQ is discrete derivative method to solve ODE or PDE which gives numerical results effectively. The method presented by Bellman and Casti [14] and Bellman *et al.* [15]. The idea was to give a new perspective for previous numerical techniques in solving problems. Since then, the way has been adapted in a wide range of applications. As to the idea, the derivative of a function is defined as a weighted linear sum of the function values at all grid points related to the used direction. So, the term weighting coefficients occurs, and to obtain

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these coefficients, generally polynomials are chosen as test functions which can be obtained by polynomial approximation theory.

In the beginning, Bellman proposed the idea of two polynomial-based methods for computation of the weighting coefficients for the first order derivative. Power function was used as test function for the first approach and for the second one he choose test function as Legendre polynomials [14, 15]. Shortly after, many new different approaches have been presented with applications to many different kind of engineering problems. Popular ways have been suggested such as polynomials, Spline functions or Fourier series expansion. Quan and Chang [16, 17] used Legendre interpolation polynomials as test functions then obtained explicit formulation to find the weighting coefficients. Shu [18], gave a powerful way as a combination Bellman's and Quang-Chang's approaches. Also, first and higher order derivative formulations were analyzed in detail based on polynomial approaches and Fourier series expansion approaches using different kind of grid points. Stability analysis based on eigenvalue distribution was explained together with different time integration schemes. Shu [18] also presented the relationships between finite difference and collocation methods with the DQ method.

The DQ method has been effectively used in areas such as material science, thermal and structural mechanical analysis, physics and biology. It can be seen that this technique gives accurate solutions with time saving computations [19]. Civan and Spliepevich [20] applied this method to both Poisson equation, and to multi-dimensional problems [21]. Saka *et al.* [22] considered equal width equation (EW) by using three methods including cosine expansion based differential quadrature. Korkwaz and Dag [23, 24], studied on a wide of range of problems using Spline functions or polynomials. For time discretization they used fourth order Runge-Kutta scheme and stability analysis is examined. Sari and Guraslan [25] investigated the polynomial based method for generalized Burgers-Huxley equation together third order third-order Runge-Kutta scheme for temporal discretization, without using linearization. Mittal and Arora [26] used Bernstein polynomials to acquire the weighting coefficients. In our work, we consider eigenvalue distribution to check the stability, also, several theoretical works have been established for stability analysis of non-linear PDE [27, 28].

The KS equation is a non-linear PDE given:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \alpha \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^4 u}{\partial x^4} = 0, \quad x \in [x_0, x_N], \quad t \in (0, T) \quad (1)$$

along with the boundary conditions:

$$\begin{aligned} u(x_0, t) &= g_0, & u(x_N, t) &= g_1, \\ u_x(x_0, t) &= 0, & u_x(x_N, t) &= 0, \\ u_{xx}(x_0, t) &= 0, & u_{xx}(x_N, t) &= 0, \end{aligned} \quad (2)$$

and initial condition:

$$u(x, 0) = u_0 \quad (3)$$

where α represents growth of the linear stability and ν shows surface tension. When $\nu = 0$, the term surface tension is removed from the equation, then the equation becomes Burgers' equation [5].

In this work, it is used Chebyshev polynomial approximations to obtain numeric solutions. When the method is applied for the derivatives, differential equation is reduced to linear

system of equation, with the implementation of boundary or initial conditions, matrix equation can be solved to obtain the desired solution.

The differential quadrature method

Consider a sufficiently smooth function $f(x)$ on a closed interval $[a, b]$. Derivative of the function at a grid point x_i , is approximated by a linear sum of all functional values on the whole domain and the quadrature formula for first derivative is given:

$$f_x(x_i) = \left. \frac{df}{dx} \right|_{x_i} = \sum_{j=1}^N w_{ij} f(x_j), \quad i = 1, 2, \dots, N \tag{4}$$

where w_{ij} represents the weighting coefficients to be evaluated, N is the number of grid points [18]. The n^{th} order derivative is defined as same idea given:

$$f_x^{(n)}(x_i) = \left. \frac{d^n f}{dx^n} \right|_{x_i} = \sum_{j=1}^N w_{ij}^{(n)} f(x_j), \quad i = 1, 2, \dots, N \tag{5}$$

where $w_{ij}^{(n)}$ represents the weighting coefficients, N is the number of grid points. The main idea according to the procedure is to determine weighting coefficients. The DQ method offers using uniform or non-uniform selection of grid points but, it gives more effective and stable solutions using Chebyshev-Gauss Labotto points [18, 19]. Here, we choose grid points as the Chebyshev collocation points defined:

$$x_i = \cos(\theta_i), \quad \theta = \frac{i\pi}{N}, \quad i = 0, 1, 2, \dots, N \tag{6}$$

which is applicable for only interval $[1, -1]$. If the problem is given on interval $[a, b]$ to obtain x_i following transformation is used [18]:

$$x_i = \frac{b-a}{2}(1 - \xi_i) + a \tag{7}$$

When Lagrange interpolating polynomials are considered as test function:

$$r_k(x) = \frac{K(x)}{(x - x_k)K'(x_k)}, \quad i = 1, 2, \dots, N \tag{8}$$

where $r_k(x)$ represents the test function and

$$K(x) = (x - x_1)(x - x_2) \dots (x - x_N), \quad K'(x) = \prod_{k=1, k \neq j}^N (x_j - x_k) \tag{9}$$

Using polynomial approach theory, for first order derivative the weighting coefficients, given in eq. (4) becomes [18]:

$$a_{ij}(x) = \frac{K'(x_i)}{(x_i - x_j)K'(x_j)}, \quad j \neq i. \tag{10}$$

For the entries on main diagonal, the following relation becomes:

$$a_{ii}(x) = \frac{K''(x_i)}{2K'(x_i)} \tag{11}$$

By using the linear vector space spanning property as being represented by different kind of bases, on diagonal entries following formula can be used which is obtained by power functions, x^{k-1} , $k = 1, 2, \dots, N$ when $k = 1$:

$$\sum_{j=1}^N a_{ij} = 0, \quad a_{ii} = - \sum_{j=1, j \neq i}^N a_{ij} \quad (12)$$

Now, to obtain quadrature solutions of the model problem Chebyshev polynomial is used as a basis together with Chebyshev collocation points, the function $K(x)$ can be obtained:

$$K(x) = (1-x^2)T_N^{(1)}(x) \quad (13)$$

where $T_N^{(1)}(x)$ represents first derivative of $T_N(x) = \cos(N\theta)$, and $\theta = \arccos(x)$. Thus:

$$T_N^{(1)}(x) = [\cos(N\theta)]' = -N \sin(N\theta) d\theta \quad (14)$$

where

$$d\theta = -\frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sin(\theta)} \quad (15)$$

The expression can be rewritten:

$$K(x) = K(\theta) = N \sin(\theta) \sin(N\theta) \quad (16)$$

Since, derivative approximation is needed according to the structure of the method, by differentiating eq. (16):

$$K^{(1)}(x) = [N \sin(\theta) \sin(N\theta)]' \quad (17)$$

or

$$K^{(1)}(x) = -\frac{N \sin(N\theta) \cos(\theta) + N^2 \sin(\theta) \cos(N\theta)}{\sin(\theta)} \quad (18)$$

Since $N\theta_i = i\pi$, when $\sin(\theta_i) \neq 0$ that is $i \neq 0, N$, then eq. (18) becomes:

$$K^{(1)}(x_i) = (-1)^{i+1} N^2 \quad (19)$$

When $\sin(\theta_i) = 0$, to remove the undetermined form L'Hospital's rule is applied:

$$\begin{aligned} K^{(1)}(x_0) &= -2N^2 \\ K^{(1)}(x_N) &= (-1)^{N+1} 2N^2 \end{aligned} \quad (20)$$

The reduced formulation related to find first derivative matrix interpreted as follows [18, 23]:

$$\begin{aligned} w_{ij} &= \frac{\bar{c}_i (-1)^{i+j}}{\bar{c}_j (x_i - x_j)}, \quad 0 \leq i, j \leq N, i \neq j \\ w_{ii} &= -\frac{x_i}{2(1-x_i^2)}, \quad 1 \leq i \leq N-1 \\ w_{00} &= -w_{NN} = \frac{2N^2 + 1}{6} \end{aligned} \quad (21)$$

where $\bar{c}_0 = \bar{c}_N = 2$ and $\bar{c}_j = 1$, $1 \leq i \leq N - 1$. The model equation requires rewriting higher order derivatives in DQ formulations. In this manner, matrix multiplication method is used which mentioned in [18].

Quadrature discretization of model equation

The KS equation is rewritten:

$$U_t = -(UU_x + \alpha U_{xx} + \nu U_{xxxx}) \tag{22}$$

To obtain the approximated solution, we apply the method for each grid points, as follows:

$$U_t(x_i) = -[U(x_i)U_x(x_i) + \alpha U_{xx}(x_i) + \nu U_{xxxx}(x_i)] \tag{23}$$

Then, spatial derivatives are replaced by the DQ equality:

$$\frac{\partial U(x_i)}{\partial t} = - \left[U(x_i) \sum_{j=1}^N w_{ij}^{(1)} U(x_j) + \alpha \sum_{j=1}^N w_{ij}^{(2)} U(x_j) + \nu \sum_{j=1}^N w_{ij}^{(4)} U(x_j) \right] \tag{24}$$

First order temporal discretization is obtained by forward Euler scheme:

$$U_t(x_i, t) = \frac{U^{n+1} - U^n}{\Delta t} \tag{25}$$

Matrix stability has been studied for the DQ discretized systems. Discrete time-dependent problem is of the form:

$$\frac{\partial U}{\partial t} = \ell(U) \tag{26}$$

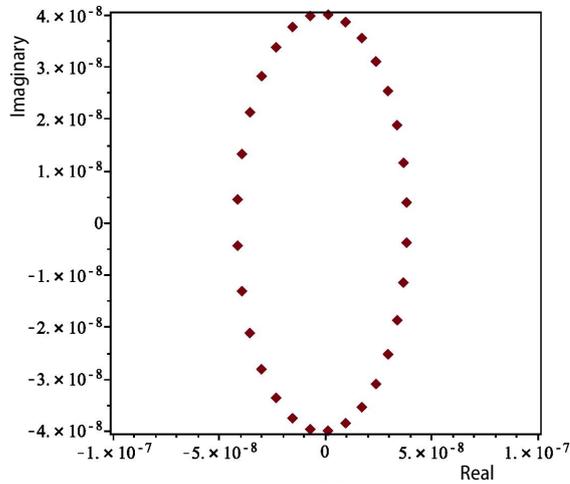
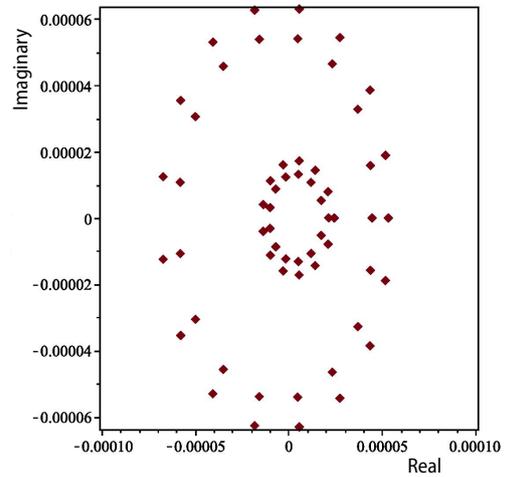
with proper initial and boundary conditions. Here, ℓ represents spatial non-linear differential operator. After applying DQ method and linearization of the non-linear term $U(x)U_x(x)$ the equation becomes:

$$\frac{d\{U\}}{dt} = [A]\{U\} + \{g\} \tag{27}$$

where $\{U\}$ is an unknown vector of the function values in the domain, $\{g\}$ is the vector containing the non-homogeneous part and the boundary conditions and $[A]$ is the discretized coefficient matrix. The stability of the numerical discretized system depends on eigenvalues distribution [18]. The condition for absolute stability of the forward scheme is given:

$$|1 + \lambda \Delta t| \leq 1 \tag{28}$$

The stability region for the scheme is the circle with radius 1 and center $(-1,0)$ on the complex $\lambda \Delta t$ plane [29]. When the idea is implemented for the test *problem 1*, the maximum real parts of the eigenvalues are determined as $3.8338 \cdot 10^{-8}$, and $5.1920 \cdot 10^{-4}$, for $N = 30$, and $N = 60$, respectively. Eigenvalue distributions for each grid points are given by figs. 1-2. The approximated solutions are obtained by combining the explicit scheme and quadrature scheme by reducing the model to an algebraic system of equations. The solution of the matrix equation gives desired solution. We used two sample problems to illustrate the efficiency of the presented method and all the results in terms of error norms are given in tables.

Figure 1. Eigenvalue distribution when $N = 30$ Figure 2. Eigenvalue distribution when $N = 60$

In the end, solutions are also compared with previous studies. Solutions show approximately same accuracy.

Numerical illustrations

Efficiency of the method is demonstrated using L_2 error norm which is given:

$$L_2 = \|U_{ex} - U_{nu}\| = \left[\sum_{j=1}^N |(U_{ex})_j - (U_{nu})_j|^2 \right]^{1/2} \quad (29)$$

and maximum error norm given:

$$L_\infty = \|U_{ex} - U_{nu}\|_\infty = \max_j |(U_{ex})_j - (U_{nu})_j| \quad (30)$$

To compare the accuracy with previous studies it is also measured global relative error (GRE) which is given:

$$GRE = \frac{\sum_{j=1}^N |(U_{ex})_j - (U_{nu})_j|}{\sum_{j=1}^N |(U_{ex})_j|} \quad (31)$$

where U_{ex} and U_{nu} represents the analytical and numerical solutions, respectively.

Test Problem 1. As a first case, KS equation is considered for $\alpha = 1$ and $\nu = 1$. Exact solution of the problem is [5]:

$$u(x, t) = b + \frac{15}{19} \sqrt{\frac{11}{19}} \left\{ -9 \tanh[k(x - bt - x_0)] + 11 \tanh^3[k(x - bt - x_0)] \right\} \quad (32)$$

The initial and boundary conditions can be computed by using exact solution given by eq (32). Number of partitions are considered 15, 30, 60, and 150, and $b = 5$, $k = (1/2)(11/19)^{1/2}$, and $x_0 = -12$. Comparisons between the exact and numerical solutions are tabulated for the interval $[-30, 30]$. The solution models the shock wave propagation with speed b and initial position x_0 [6]. Solutions are given by tab. 1 and fig. 3.

Table 1. Error norms for Problem 1

Error norms	N	$t = 0.001$	$t = 0.01$	$t = 1.0$
L_2	15	3.5043E-05	4.4490E-03	9.7979E-03
	30	5.1571E-05	5.0774E-04	4.1705E-03
	60	1.4117E-04	7.4159E-04	1.7145E-02
	150	2.2567E-04	1.1758E-03	2.7408E-02
L_∞	15	1.3189E-05	1.1675E-03	3.6482E-03
	30	1.3286E-05	1.3080E-03	1.0742E-03
	60	2.2529E-04	1.3102E-04	3.0704E-03
	150	2.2529E-04	1.3108E-04	3.0718E-03

Test Problem 2. Now, we consider the problem for $\alpha = -1$ and $\nu = 1$. Exact solution of the problem is given by [5]:

$$u(x, t) = b + \frac{15}{19} \sqrt{\frac{1}{19}} \left\{ -3 \tanh[k(x - bt - x_0)] + 11 \tanh^3[k(x - bt - x_0)] \right\} \quad (33)$$

The initial and boundary conditions are obtained by using the exact solution given by eq. (33). We have computed the algorithm with parameters $b = 5$, $k = 1/[2(19)^{1/2}]$, $x_0 = -25$. Number of partitions are considered 15, 30, and 200. Comparisons between the exact and numerical solutions are tabulated for the interval $[-50, 50]$ (tab. 2 and fig. 4).

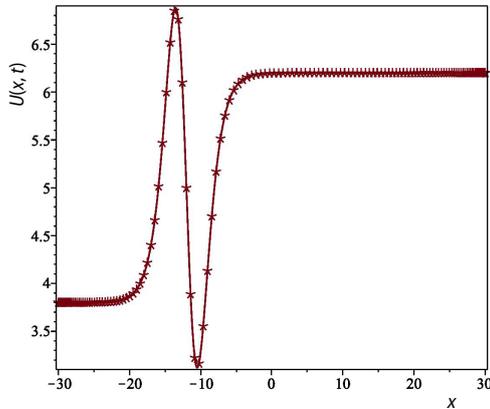


Figure 3. Comparison between numerical and exact solutions of KS equation when $N = 150$ Problem 1 (dotted line represents numerical solutions)

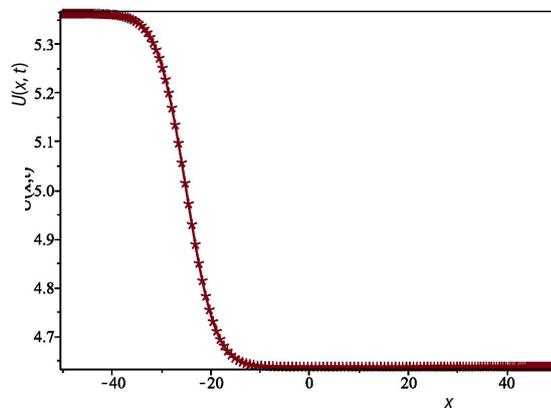


Figure 4. Comparison between numerical and exact solutions of KS equation when $N = 200$ for Problem 2 (dotted line represents numerical solutions)

Table 2. Error norms for Problem 2

Error norms	N	$t = 0.001$	$t = 0.01$	$t = 1.0$
L_2	15	1.3980E-07	1.3823E-06	1.2138E-05
	30	2.0538E-07	2.0309E-06	1.7832E-05
	200	2.0538E-07	5.4047E-06	4.7457E-05
L_∞	15	4.7292E-08	4.6764E-07	4.1065E-06
	30	4.6969E-08	1.3080E-06	4.1240E-06
	200	4.7561E-08	4.7030E-07	4.1300E-06

Table 3. Comparison of GRE for Problem 1

Method	GRE
Chebyshev DQ	1.1942E-03
Exponential B-spline collocation [6]	8.7463E-04
Quintic B-spline collocation [5]	3.8172E-04
Lattice Boltzmann [30]	6.7923E-04

Table 4. Comparison of GRE for Problem 2

Method	GRE
Chebyshev DQ	1.6812E-05
Exponential B-spline collocation [6]	3.6467E-05
Quintic B-spline collocation [5]	6.5092E-06
Lattice Boltzmann [30]	7.8088E-06

Conclusion

In this study, the Chebyshev based DQ method is used for solutions of KS equation. The efficiency of the approach is examined by two examples. According to the tabs. 1 and 2 as t increases, accuracy decreases, but as the number of partitions increases, we can see approximately same accuracy. Also by tabs. 3 and 4, comparison with other methods shows that effectiveness is approximately same for similar numerical techniques. It can be also seen that from the figures numerical and exact solutions are in good agreement. The method is easy to implement by using small number of grid points.

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