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On Conformable Double Laplace Transform and One Dimensional Fractional Coupled Burgers' Equation

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Abstract: In the present work we introduced a new method and name it the conformable double Laplace decomposition method to solve one dimensional regular and singular conformable functional Burger's equation. We studied the existence condition for the conformable double Laplace transform. In order to obtain the exact solution for nonlinear fractional problems, then we modified the double Laplace transform and combined it with the Adomian decomposition method. Later, we applied the new method to solve regular and singular conformable fractional coupled Burgers' equations. Further, in order to illustrate the effectiveness of present method, we provide some examples.

Keywords: conformable fractional derivative; conformable partial fractional derivative; conformable double Laplace decomposition method; conformable Laplace transform; singular one dimensional coupled Burgers' equation

1. Introduction

The fractional partial differential equations play a crucial role in mathematical and physical sciences. In [1], the authors studied the solution of some time-fractional partial differential equations by using a method known as simplest equation method. In this work, we deal with Burgers' equation, these type of equations have appeared in the area of applied sciences such as fluid mechanics and mathematical modeling. In fact, Burgers' equation was first proposed in [2], where the steady state solutions were discussed. Later it was modified by Burger, in order to solve the descriptive certain viscosity of flows. Today in the literature it is widely known as Burgers' equation, see [3]. Several researchers focused and concentrated to study the exact as well as the numerical solutions of this type of equation. In the present work, we considered and modified the conformable double Laplace transform method which was introduced in [4] in order to solve the fractional partial differential equations. The authors in [5] applied the first integral method to establish the exact solutions for time-fractional Burgers' equation. In [6], the researchers applied the generalized two-dimensional differential transform method (DTM) and obtained the solution for the coupled Burgers' equation with space- and time-fractional derivatives. Recently in [7], the conformable fractional Laplace transform method was applied to solve the coupled system of conformable fractional differential equations. Thus the aim of this study is to propose an analytic solution for the one dimensional regular and singular conformable fractional coupled Burgers' equation by using conformable double Laplace decomposition method (CDLDM). In [8], the following space-time fractional order coupled Burgers' equation, were considered

$$\begin{aligned} \frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \eta u \frac{\partial^\alpha}{\partial x^\alpha} u + \zeta \frac{\partial^\alpha}{\partial x^\alpha}(uv) &= f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \\ \frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + \eta v \frac{\partial^\alpha}{\partial x^\alpha} v + \mu \frac{\partial^\alpha}{\partial x^\alpha}(uv) &= g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right). \end{aligned} \tag{1}$$

Conformable fractional derivatives were studied in [9] and extended in [10]. Next, we recall the definition of conformable fractional derivatives, which are used in this study.

Definition 1. *Let $f : (0, \infty) \to R$ then the conformable fractional derivative of f order β is defined by*

$$\frac{d^\beta}{dt^\beta} f\left(\frac{t^\beta}{\beta}\right) = \lim_{\epsilon \to 0} \frac{f\left(\frac{t^\beta}{\beta} + \epsilon t^{1-\beta}\right) - f\left(\frac{t^\beta}{\beta}\right)}{\epsilon}, \quad \frac{t^\beta}{\beta} > 0,\ 0 < \beta \leq 1,$$

see [9,11,12].

Conformable Partial Derivatives:

Definition 2. *([13]): Given a function $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) : R \times (0, \infty) \to R$. Then, the conformable space fractional partial derivative of order α a function $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ is defined as:*

$$\frac{\partial^\alpha}{\partial x^\alpha} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = \lim_{\epsilon \to 0} \frac{f\left(\frac{x^\alpha}{\alpha} + \epsilon x^{1-\alpha}, t\right) - f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)}{\epsilon}, \quad \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} > 0, 0 < \alpha, \beta \leq 1.$$

Definition 3. *([13]): Given a function $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) : R \times (0, \infty) \to R$. Then, the conformable time fractional partial derivative of order β a function $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ is defined as:*

$$\frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = \lim_{\sigma \to 0} \frac{f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} + \sigma t^{1-\beta}\right) - f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)}{\sigma}, \quad \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} > 0,\ 0 < \alpha, \beta \leq 1.$$

Conformable fractional derivatives of certain functions:

Example 1. *We have the following*

$$\begin{aligned} \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{t^\beta}{\beta}\right) &= \left(\frac{t^\beta}{\beta}\right), \quad \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\right)^n\left(\frac{t^\beta}{\beta}\right) = n\left(\frac{x^\alpha}{\alpha}\right)^{n-1}\left(\frac{t^\beta}{\beta}\right) \\ \frac{\partial^\beta}{\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\right)\left(\frac{t^\beta}{\beta}\right) &= \left(\frac{x^\alpha}{\alpha}\right), \quad \frac{\partial^\beta}{\partial t^\beta}\left(\frac{x^\alpha}{\alpha}\right)^n\left(\frac{t^\beta}{\beta}\right)^m = m\left(\frac{x^\alpha}{\alpha}\right)^n\left(\frac{t^\beta}{\beta}\right)^{m-1} \\ \frac{\partial^\beta}{\partial t^\beta}\left(\sin\left(\frac{x^\alpha}{\alpha}\right)\sin\left(\frac{t^\beta}{\beta}\right)\right) &= \sin\left(\frac{x^\alpha}{\alpha}\right)\cos\left(\frac{t^\beta}{\beta}\right), \\ \frac{\partial^\alpha}{\partial x^\alpha}\left(\sin a\left(\frac{x^\alpha}{\alpha}\right)\sin\left(\frac{t^\beta}{\beta}\right)\right) &= a\cos\left(\frac{x^\alpha}{\alpha}\right)\sin\left(\frac{t^\beta}{\beta}\right) \\ \frac{\partial^\alpha}{\partial x^\alpha}\left(e^{\lambda\frac{x^\alpha}{\alpha} + \frac{\tau t^\beta}{\beta}}\right) &= \lambda e^{\lambda\frac{x^\alpha}{\alpha} + \frac{\tau t^\beta}{\beta}}, \quad \frac{\partial^\beta}{\partial t^\beta}\left(e^{\lambda\frac{x^\alpha}{\alpha} + \frac{\tau t^\beta}{\beta}}\right) = \tau e^{\lambda\frac{x^\alpha}{\alpha} + \frac{\tau t^\beta}{\beta}}. \end{aligned}$$

Conformable Laplace transform:

Definition 4. *([14]): Let $f : [0,\infty) \to \mathbb{R}$ be a real valued function. The conformable Laplace transform of f is defined by*

$$L_t^{\beta}\left(f\left(\frac{t^{\beta}}{\beta}\right)\right) = \int_0^{\infty} e^{-s\frac{t^{\beta}}{\beta}} f\left(\frac{t^{\beta}}{\beta}\right) t^{\beta-1} dt$$

for all values of s, provided the integral exists.

Definition 5. *([4]): Let $u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$ be a piecewise continuous function on the interval $[0,\infty) \times [0,\infty)$ having exponential order. Consider for some $a, b \in \mathbb{R}$ $\sup \frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta} > 0$, $\frac{\left|u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right|}{e^{\frac{ax^{\alpha}}{\alpha}+\frac{bt^{\beta}}{\beta}}}$. Under these conditions the conformable double Laplace transform is given by*

$$L_x^{\alpha} L_t^{\beta}\left(u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right) = U(p,s) = \int_0^{\infty}\int_0^{\infty} e^{-p\frac{x^{\alpha}}{\alpha}-s\frac{t^{\beta}}{\beta}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx$$

where $p, s \in \mathbb{C}$, $0 < \alpha, \beta \leq 1$ and the integrals are by means of conformable fractional with respect to $\frac{x^{\alpha}}{\alpha}$ and $\frac{t^{\beta}}{\beta}$ respectively.

Example 2. *The double fractional Laplace transform for certain functions given by*

1. $L_x^{\alpha} L_t^{\beta}\left[\left(\frac{x^{\alpha}}{\alpha}\right)^n \left(\frac{t^{\beta}}{\beta}\right)^m\right] = L_x L_t\left[(x)^n (t)^m\right] = \frac{n!m!}{p^{n+1}s^{m+1}}.$
2. $L_x^{\alpha} L_t^{\beta}\left[e^{\lambda\frac{x^{\alpha}}{\alpha}+\frac{\tau t^{\beta}}{\beta}}\right] = L_x L_t\left[e^{\lambda x+\tau t}\right] = \frac{1}{(p-\lambda)(s-\tau)}.$
3. $L_x^{\alpha} L_t^{\beta}\left[\left(\sin(\lambda\frac{x^{\alpha}}{\alpha}\right)\sin\left(\tau\frac{t^{\beta}}{\beta}\right)\right] = L_x L_t\left[\sin(x)\sin(t)\right] = \frac{1}{p^2+\lambda^2}\frac{1}{s^2+\tau^2}.$
4. *If $a(>-1)$ and $b(>-1)$ are real numbers, then double fractional Laplace transform of the function $f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) = \left(\frac{x^{\alpha}}{\alpha}\right)^a \left(\frac{t^{\beta}}{\beta}\right)^b$ is given by*

$$L_x^{\alpha} L_t^{\beta}\left[(\frac{x^{\alpha}}{\alpha})^a (\frac{t^{\beta}}{\beta})^b\right] = \frac{\Gamma(a+1)\Gamma(b+1)}{p^{a+1}s^{b+1}}.$$

Theorem 1. *Let $0 < \alpha, \beta \leq 1$ and $m, n \in \mathbb{N}$ such that $u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) \in C^l(\mathbb{R}^+ \times \mathbb{R}^+)$, $l = \max(m,n)$. Further let the conformable Laplace transforms of the functions given as $u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$, $\frac{\partial^{m\alpha}u}{\partial x^{m\alpha}}$ and $\frac{\partial^{n\beta}u}{\partial t^{n\beta}}$. Then*

$$L_x^{\alpha} L_t^{\beta}\left(\frac{\partial^{m\alpha}u}{\partial x^{m\alpha}}\right) = p^m U(p,s) - p^{m-1}U(0,s) - \sum_{i=1}^{m-1} p^{m-1-i} L_t^{\beta}\left(\frac{\partial^{i\alpha}}{\partial x^{i\alpha}} u\left(0, \frac{t^{\beta}}{\beta}\right)\right)$$

$$L_x^{\alpha} L_t^{\beta}\left(\frac{\partial^{n\beta}}{\partial t^{n\beta}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right) = s^n U(p,s) - s^{n-1}U(p,0) - \sum_{j=1}^{n-1} s^{n-1-j} L_x^{\alpha}\left(\frac{\partial^{j\beta}}{\partial t^{j\beta}} u\left(\frac{x^{\alpha}}{\alpha}, 0\right)\right)$$

where $\frac{\partial^{m\alpha}u}{\partial x^{m\alpha}}$ and $\frac{\partial^{n\beta}v}{\partial t^{n\beta}}$ denotes m, n times conformable fractional derivatives of function $u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$, for more details see [4].

In the following theorem, we study double Laplace transform of the function $\left(\frac{x^{\alpha}}{\alpha}\right)^n \frac{\partial^{\beta}}{\partial t^{\beta}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$ as follows:

Theorem 2. *If conformable double Laplace transform of the partial derivatives* $\frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ *is given by Equation (27), then double Laplace transform of* $\left(\frac{x^\alpha}{\alpha}\right)^n \frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ *and* $\left(\frac{x^\alpha}{\alpha}\right)^n g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ *are given by*

$$(-1)^n \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right]\right) = L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n \frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right] \tag{2}$$

and

$$(-1)^n \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right]\right) = L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right], \tag{3}$$

where $n = 1, 2, 3, \ldots$.

Proof. Using the definition of double Laplace transform of the fractional partial derivatives one gets

$$L_x^\alpha L_t^\beta\left[\frac{\partial^\beta}{\partial t^\beta} f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})\right] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} \left(\frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right) t^{\beta-1} x^{\alpha-1} dt\, dx, \tag{4}$$

by taking the ^{n}th derivative with respect to p for both sides of Equation (4), we have

$$\begin{aligned} \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right]\right) &= \int_0^\infty \int_0^\infty \frac{d^n}{dp^n}\left(e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} \frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right) t^{\beta-1} x^{\alpha-1} dt dx \\ &= (-1)^n \int_0^\infty \int_0^\infty \left(\frac{x^\alpha}{\alpha}\right)^n e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} t^{\beta-1} x^{\alpha-1} \frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) dt\, dx \\ &= (-1)^n L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n \frac{\partial^\beta}{\partial t^\beta} f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})\right], \end{aligned}$$

thus we obtain

$$(-1)^n \frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right]\right) = L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n \frac{\partial^\beta}{\partial t^\beta} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right].$$

Similarly, we can prove Equation (3). □

Existence Condition for the conformable double Laplace transform:

If $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ is an exponential order a and b as $\frac{x^\alpha}{\alpha} \to \infty$, $\frac{t^\beta}{\beta} \to \infty$, if there exists a positive constant K such that for all $x > X$ and $t > T$

$$\left|f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right| \leq K e^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}, \tag{5}$$

it is easy to get,

$$f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = O\left(e^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}\right) \quad \text{as } \frac{x^\alpha}{\alpha} \to \infty, \frac{t^\beta}{\beta} \to \infty.$$

Or, equivalently,

$$\lim_{\substack{\frac{x^\alpha}{\alpha} \to \infty \\ \frac{t^\beta}{\beta} \to \infty}} e^{-\mu\frac{x^\alpha}{\alpha} - \eta\frac{t^\beta}{\beta}} \left|f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right| = K \lim_{\substack{\frac{x^\alpha}{\alpha} \to \infty \\ \frac{t^\beta}{\beta} \to \infty}} e^{-(\mu-a)\frac{x^\alpha}{\alpha} - (\eta-b)\frac{t^\beta}{\beta}} = 0,$$

where $\mu > a$ and $\eta > b$. The function $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ is called an exponential order as $\frac{x^\alpha}{\alpha} \to \infty$, $\frac{t^\beta}{\beta} \to \infty$, and clearly, it does not grow faster than $Ke^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}}$ as $\frac{x^\alpha}{\alpha} \to \infty$, $\frac{t^\beta}{\beta} \to \infty$.

Theorem 3. *If a function $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ is a continuous function in every finite intervals $(0, X)$ and $(0, T)$ and of exponential order $e^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}}$, then the conformable double Laplace transform of $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ exists for all $Re(p) > \mu, Re(s) > \eta$.*

Proof. From the definition of the conformable double Laplace transform of $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$, we have

$$\begin{aligned} |U(p,s)| &= \left| \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f(\tfrac{x^\alpha}{\alpha}, \tfrac{t^\beta}{\beta}) t^{\beta-1} x^{\alpha-1} dt\, dx \right| \\ &\leq K \left| \int_0^\infty \int_0^\infty e^{-(p-a)\frac{x^\alpha}{\alpha}-(s-b)\frac{t^\beta}{\beta}} t^{\beta-1} x^{\alpha-1} dt\, dx \right| \\ &= \frac{K}{(p-a)(s-b)}. \end{aligned} \tag{6}$$

For $Re(p) > \mu, Re(s) > \eta,$ from Equation (6), we have

$$\lim_{\substack{p\to\infty \\ s\to\infty}} |U(p,s)| = 0 \text{ or } \lim_{\substack{p\to\infty \\ s\to\infty}} U(p,s) = 0.$$

□

2. One Dimensional Fractional Coupled Burgers' Equation

In this section, we discuss the solution of regular and singular one dimensional conformable fractional coupled Burgers' equation by using conformable double Laplace decomposition methods (CDLDM). We note that if $\alpha = 1$ and $\beta = 1$ in the following problems, one can obtain the problems which was studied in [15]:

The first problem: One dimensional conformable fractional coupled Burgers' equation is given by

$$\begin{aligned} \frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \eta u \frac{\partial^\alpha}{\partial x^\alpha} u + \zeta \frac{\partial^\alpha}{\partial x^\alpha}(uv) &= f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \\ \frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + \eta v \frac{\partial^\alpha}{\partial x^\alpha} v + \mu \frac{\partial^\alpha}{\partial x^\alpha}(uv) &= g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right), \end{aligned} \tag{7}$$

subject to

$$u\left(\frac{x^\alpha}{\alpha}, 0\right) = f_1\left(\frac{x^\alpha}{\alpha}\right), \quad v\left(\frac{x^\alpha}{\alpha}, 0\right) = g_1\left(\frac{x^\alpha}{\alpha}\right). \tag{8}$$

for $t > 0$. Here, $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$, $g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$, $f_1\left(\frac{x^\alpha}{\alpha}\right)$ and $g_1\left(\frac{x^\alpha}{\alpha}\right)$ are given functions, η, ζ and μ are arbitrary constants depend on the system parameters such as; Peclet number, Stokes velocity of particles due to gravity and Brownian diffusivity, see [16]. By taking conformable double Laplace transform for both sides of Equation (7) and conformable single Laplace transform for Equation (8), we have

$$U(p,s) = \frac{F_1(p)}{s} + \frac{F(p,s)}{s} + \frac{1}{s} L_x^\alpha L_t^\beta \left[\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - \eta u \frac{\partial^\alpha}{\partial x^\alpha} u - \zeta \frac{\partial^\alpha}{\partial x^\alpha}(uv) \right], \tag{9}$$

and

$$V(p,s) = \frac{G_1(p)}{s} + \frac{G(p,s)}{s} + \frac{1}{s} L_x^\alpha L_t^\beta \left[\frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} - \eta v \frac{\partial^\alpha}{\partial x^\alpha} v - \mu \frac{\partial^\alpha}{\partial x^\alpha}(uv) \right]. \tag{10}$$

The conformable double Laplace decomposition methods (CDLDM) defines the solution of one dimensional conformable fractional coupled Burgers' equation as $u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)$ and $v\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)$ by the infinite series

$$u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\sum_{n=0}^{\infty}u_n\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right),\quad v\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\sum_{n=0}^{\infty}v_n\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right). \tag{11}$$

We can give Adomian's polynomials A_n, B_n and C_n respectively as follows

$$A_n=\sum_{n=0}^{\infty}u_nu_{xn},\ B_n=\sum_{n=0}^{\infty}v_nv_{xn},\ \ C_n=\sum_{n=0}^{\infty}u_nv_n. \tag{12}$$

In particular, the Adomian polynomials for the nonlinear terms uu_x, vv_x and uv can be computed by the following equations

$$\begin{aligned}A_0&=u_0u_{0x}\\A_1&=u_0u_{1x}+u_1u_{0x}\\A_2&=u_0u_{2x}+u_1u_{1x}+u_2u_{0x},\\A_3&=u_0u_{3x}+u_1u_{2x}+u_2u_{1x}+u_3u_{0x},\\A_4&=u_0u_{4x}+u_1u_{3x}+u_2u_{2x}+u_3u_{1x}+u_4u_{0x},\end{aligned} \tag{13}$$

$$\begin{aligned}B_0&=v_0v_{0x}\\B_1&=v_0v_{1x}+v_1v_{0x},\\B_2&=v_0v_{2x}+v_1v_{1x}+v_2v_{0x},\\B_3&=v_0v_{3x}+v_1v_{2x}+v_2v_{1x}+v_3v_{0x},\\B_4&=v_0v_{4x}+v_1v_{3x}+v_2v_{2x}+v_3v_{1x}+v_4v_{0x}.\end{aligned} \tag{14}$$

and

$$\begin{aligned}C_0&=u_0v_0\\C_1&=u_0v_1+u_1v_0\\C_2&=u_0v_2+u_1v_1+u_2v_0.\\C_3&=u_0v_3+u_1v_2+u_2v_1+u_3v_0,\\C_4&=u_0v_4+u_1v_3+u_2v_2+u_3v_1+u_4v_0.\end{aligned} \tag{15}$$

By applying the inverse conformable double Laplace transform on both sides of Equations (9) and (10), making use of Equation (12), we have

$$\begin{aligned}\sum_{n=0}^{\infty}u_n\left(\tfrac{x^{\alpha}}{\alpha},\tfrac{t^{\beta}}{\beta}\right)&=f_1\left(x\right)+L_p^{-1}L_s^{-1}\left[\tfrac{F(p,s)}{s}\right]+L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\tfrac{\partial^{2\alpha}u_n}{\partial x^{2\alpha}}\right]\right]\\&\quad-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\eta A_n\right]\right]-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\zeta\left(C_n\right)\right]\right],\end{aligned} \tag{16}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}v_n\left(\tfrac{x^{\alpha}}{\alpha},\tfrac{t^{\beta}}{\beta}\right)&=g_1\left(x\right)+L_p^{-1}L_s^{-1}\left[\tfrac{G(p,s)}{s}\right]+L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\tfrac{\partial^{2\alpha}v_n}{\partial x^{2\alpha}}\right]\right]\\&\quad-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\eta B_n\right]\right]-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\mu\left(C_n\right)\right]\right].\end{aligned} \tag{17}$$

On comparing both sides of the Equations (16) and (17) we have

$$\begin{aligned} u_0 &= f_1(x) + L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s}\right], \\ v_0 &= g_1(x) + L_p^{-1}L_s^{-1}\left[\frac{G(p,s)}{s}\right]. \end{aligned} \tag{18}$$

In general, the recursive relation is given by the following equations

$$u_{n+1} = L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}u_n}{\partial x^{2\alpha}}\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\eta A_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\zeta\left(C_n\right)\right]\right], \tag{19}$$

and

$$v_{n+1} = L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}v_n}{\partial x^{2\alpha}}\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\eta B_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\mu\left(C_n\right)\right]\right], \tag{20}$$

provided that the double inverse Laplace transform with respect to p and s exist in the above equations. In order to illustrate this method for one dimensional conformable fractional coupled Burgers' equation we provide the following example:

Example 3. *Consider the homogeneous one dimensional conformable fractional coupled Burgers' equation*

$$\begin{aligned} \frac{\partial^{\beta}u}{\partial t^{\beta}} - \frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} - 2u\frac{\partial^{\alpha}}{\partial x^{\alpha}}u + \frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv) &= 0 \\ \frac{\partial^{\beta}v}{\partial t^{\beta}} - \frac{\partial^{2\alpha}v}{\partial x^{2\alpha}} - 2v\frac{\partial^{\alpha}}{\partial x^{\alpha}}v + \frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv) &= 0, \end{aligned} \tag{21}$$

with initial condition

$$u\left(\frac{x^{\alpha}}{\alpha},0\right) = \sin\left(\frac{x^{\alpha}}{\alpha}\right),\ v\left(\frac{x^{\alpha}}{\alpha},0\right) = \sin\left(\frac{x^{\alpha}}{\alpha}\right). \tag{22}$$

By using Equations (18)–(20) we have

$$\begin{aligned} u_0 &= \sin\left(\frac{x^{\alpha}}{\alpha}\right),\ v_0 = \sin\left(\frac{x^{\alpha}}{\alpha}\right) \\ u_1 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}u_0}{\partial x^{2\alpha}} + 2u_0\frac{\partial^{\alpha}u_0}{\partial x^{\alpha}} - \frac{\partial^{\alpha}}{\partial x^{\alpha}}(u_0v_0)\right]\right] \\ &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[-\sin\left(\frac{x^{\alpha}}{\alpha}\right)\right]\right] = L_p^{-1}L_s^{-1}\left[\frac{1}{s^2(p^2+1)}\right] = -\frac{t^{\beta}}{\beta}\sin\left(\frac{x^{\alpha}}{\alpha}\right), \\ v_1 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}v_0}{\partial x^{2\alpha}} + 2v_0\frac{\partial^{\alpha}v_0}{\partial x^{\alpha}} - \frac{\partial^{\alpha}}{\partial x^{\alpha}}(u_0v_0)\right]\right] \\ &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[-\sin\left(\frac{x^{\alpha}}{\alpha}\right)\right]\right] = L_p^{-1}L_s^{-1}\left[\frac{1}{s^2(p^2+1)}\right] = -\frac{t^{\beta}}{\beta}\sin\left(\frac{x^{\alpha}}{\alpha}\right) \end{aligned}$$

$$\begin{aligned} u_2 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}u_1}{\partial x^{2\alpha}} + 2\left(u_0\frac{\partial^{\alpha}u_1}{\partial x^{\alpha}} + u_1\frac{\partial^{\alpha}u_0}{\partial x^{\alpha}}\right) - \frac{\partial^{\alpha}}{\partial x^{\alpha}}(u_0v_1 + u_1v_0)\right]\right] \\ &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{t^{\beta}}{\beta}\sin\left(\frac{x^{\alpha}}{\alpha}\right)\right]\right] = L_p^{-1}L_s^{-1}\left[\frac{1}{s^3(p^2+1)}\right] = \frac{\left(\frac{t^{\beta}}{\beta}\right)^2}{2}\sin\left(\frac{x^{\alpha}}{\alpha}\right), \end{aligned}$$

$$\begin{aligned} v_2 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}v_1}{\partial x^{2\alpha}} + 2\left(v_0\frac{\partial^{\alpha}v_1}{\partial x^{\alpha}} + v_1\frac{\partial^{\alpha}v_0}{\partial x^{\alpha}}\right) - \frac{\partial^{\alpha}}{\partial x^{\alpha}}(u_0v_1 + u_1v_0)\right]\right] \\ &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{t^{\beta}}{\beta}\sin\left(\frac{x^{\alpha}}{\alpha}\right)\right]\right] = L_p^{-1}L_s^{-1}\left[\frac{1}{s^3(p^2+1)}\right] = \frac{\left(\frac{t^{\beta}}{\beta}\right)^2}{2}\sin\left(\frac{x^{\alpha}}{\alpha}\right), \end{aligned}$$

and

$$
\begin{aligned}
u_3 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}u_2}{\partial x^{2\alpha}}+2\left(u_0\frac{\partial^{\alpha}u_2}{\partial x^{\alpha}}+u_1\frac{\partial^{\alpha}u_1}{\partial x^{\alpha}}+u_2\frac{\partial^{\alpha}}{\partial x^{\alpha}}u_0\right)\right]\right] \\
&\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(u_0v_2+u_1v_1+u_2v_0\right)\right]\right] \\
&= -\frac{\left(\frac{t^{\beta}}{\beta}\right)^3}{6}\sin\left(\frac{x^{\alpha}}{\alpha}\right), \\
v_3 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{2\alpha}v_2}{\partial x^{2\alpha}}+2\left(v_0\frac{\partial^{\alpha}v_2}{\partial x^{\alpha}}+v_1\frac{\partial^{\alpha}v_1}{\partial x^{\alpha}}+v_2\frac{\partial^{\alpha}}{\partial x^{\alpha}}v_0\right)\right]\right] \\
&\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(u_0v_2+u_1v_1+u_2v_0\right)\right]\right] \\
&= -\frac{\left(\frac{t^{\beta}}{\beta}\right)^3}{6}\sin\left(\frac{x^{\alpha}}{\alpha}\right),
\end{aligned}
$$

and similar to the other components. Therefore, by using Equation (11), the series solutions are given by

$$
\begin{aligned}
u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) &= u_0+u_2+u_3+\ldots=\left(1-\left(\frac{t^{\beta}}{\beta}\right)+\frac{\left(\frac{t^{\beta}}{\beta}\right)^2}{2!}-\frac{\left(\frac{t^{\beta}}{\beta}\right)^3}{3!}+\ldots\right)\sin\left(\frac{x^{\alpha}}{\alpha}\right) \\
v\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) &= v_0+v_2+v_3+\ldots=\left(1-\left(\frac{t^{\beta}}{\beta}\right)+\frac{\left(\frac{t^{\beta}}{\beta}\right)^2}{2!}-\frac{\left(\frac{t^{\beta}}{\beta}\right)^3}{3!}+\ldots\right)\sin\left(\frac{x^{\alpha}}{\alpha}\right)
\end{aligned}
$$

and hence the exact solutions become

$$
u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=e^{-\frac{t^{\beta}}{\beta}}\sin\left(\frac{x^{\alpha}}{\alpha}\right),\quad v\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=e^{-\frac{t^{\beta}}{\beta}}\sin\left(\frac{x^{\alpha}}{\alpha}\right).
$$

By taking $\alpha=1$ and $\beta=1$, the fractional solution become

$$
u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=e^{-t}\sin x,\quad v\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=e^{-t}\sin x.
$$

The second problem: Now consider the singular one dimensional conformable fractional coupled Burgers' equation with Bessel operator

$$
\begin{aligned}
\frac{\partial^{\beta}u}{\partial t^{\beta}}-\frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)+\eta u\frac{\partial^{\alpha}}{\partial x^{\alpha}}u+\zeta\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv) &= f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) \\
\frac{\partial^{\beta}v}{\partial t^{\beta}}-\frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)+\eta u\frac{\partial^{\alpha}}{\partial x^{\alpha}}v+\mu\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv) &= g\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right),
\end{aligned}
\tag{23}
$$

and with initial conditions

$$
u\left(\frac{x^{\alpha}}{\alpha},0\right)=f_1\left(\frac{x^{\alpha}}{\alpha}\right),\quad v\left(\frac{x^{\alpha}}{\alpha},0\right)=g_1\left(\frac{x^{\alpha}}{\alpha}\right),
\tag{24}
$$

where the linear terms $\frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\right)$ is known as conformable Bessel operator where ζ, μ and η are real constants. Now to obtain the solution of Equation (23), First, we multiply both sides of Equation (23) by $\frac{x^{\alpha}}{\alpha}$ and obtain

$$\begin{aligned} \frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}u}{\partial t^{\beta}}-\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)+\eta\frac{x^{\alpha}}{\alpha}u\frac{\partial^{\alpha}}{\partial x^{\alpha}}u+\zeta\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv) &= \frac{x^{\alpha}}{\alpha}f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) \\ \frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}v}{\partial t^{\beta}}-\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)+\eta\frac{x^{\alpha}}{\alpha}v\frac{\partial^{\alpha}}{\partial x^{\alpha}}v+\mu\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv) &= \frac{x^{\alpha}}{\alpha}g\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right). \end{aligned} \tag{25}$$

Second: we apply conformable double Laplace transform on both sides of Equation(25) and single conformable Laplace transform for initial condition, we get

$$\begin{aligned} L_x^{\alpha}L_t^{\beta}\left[\frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}u}{\partial t^{\beta}}\right] &= L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)-\eta\frac{x^{\alpha}}{\alpha}u\frac{\partial^{\alpha}}{\partial x^{\alpha}}u-\zeta\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv)+\frac{x^{\alpha}}{\alpha}f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right], \\ L_x^{\alpha}L_t^{\beta}\left[\frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}v}{\partial t^{\beta}}\right] &= L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)-\eta\frac{x^{\alpha}}{\alpha}v\frac{\partial^{\alpha}}{\partial x^{\alpha}}v-\mu\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv)+\frac{x^{\alpha}}{\alpha}g\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right] \end{aligned} \tag{26}$$

by applying Theorems 1 and 2, we have

$$\begin{aligned} -s\frac{d}{dp}U(p,s)+\frac{d}{dp}L_x^{\alpha}\left[f_1(x)\right] &= L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)-\eta\frac{x^{\alpha}}{\alpha}u\frac{\partial^{\alpha}}{\partial x^{\alpha}}u-\zeta\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv)\right] \\ &\quad -\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}\left[f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]\right), \\ -s\frac{d}{dp}V(p,s)+\frac{d}{dp}L_x^{\alpha}\left[g_1(x)\right] &= L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)-\eta\frac{x^{\alpha}}{\alpha}v\frac{\partial^{\alpha}}{\partial x^{\alpha}}v-\mu\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv)\right] \\ &\quad -\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]\right), \end{aligned} \tag{27}$$

simplifying Equation (27), we obtain

$$\begin{aligned} \frac{d}{dp}U(p,s) &= \frac{1}{s}\frac{d}{dp}L_x^{\alpha}\left[f_1(x)\right]-\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)-\eta\frac{x^{\alpha}}{\alpha}u\frac{\partial^{\alpha}}{\partial x^{\alpha}}u-\zeta\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv)\right] \\ &\quad +\frac{1}{s}\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}\left[f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]\right). \\ \frac{d}{dp}V(p,s) &= \frac{1}{s}\frac{d}{dp}L_x^{\alpha}\left[g_1(x)\right]-\frac{1}{s}L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)-\eta\frac{x^{\alpha}}{\alpha}v\frac{\partial^{\alpha}}{\partial x^{\alpha}}v-\mu\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}(uv)\right] \\ &\quad +\frac{1}{s}\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]\right). \end{aligned} \tag{28}$$

Third: integrating both sides of Equation (28) from 0 to p respect to p, we have

$$\begin{aligned} U(p,s) &= \frac{1}{s}\int_0^p\left(\frac{d}{dp}L_x^{\alpha}\left[f_1(x)\right]\right)dp-\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\right)-\eta\frac{x^{\alpha}}{\alpha}N_1-\zeta\frac{x^{\alpha}}{\alpha}N_2\right]dp \\ &\quad +\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}\left[f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]\right)\right)dp, \\ V(p,s) &= \frac{1}{s}\int_0^p\left(\frac{d}{dp}L_x^{\alpha}\left[g_1(x)\right]\right)dp-\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}v\right)-\eta\frac{x^{\alpha}}{\alpha}N_3-\mu\frac{x^{\alpha}}{\alpha}N_2\right]dp \\ &\quad +\frac{1}{s}\int_0^p\left(\frac{d}{dp}\left(L_x^{\alpha}L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]\right)\right)dp. \end{aligned} \tag{29}$$

Using conformable double Laplace decomposition method to define a solution of the system as $u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)$ and $v(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta})$ by infinite series

$$u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\sum_{n=0}^{\infty}u_n\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right),\quad v\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\sum_{n=0}^{\infty}v_n\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right). \tag{30}$$

Here the nonlinear operators can be defined as

$$N_1=\sum_{n=0}^{\infty}A_n,\ \ N_2=\sum_{n=0}^{\infty}C_n\ N_3=\sum_{n=0}^{\infty}B_n \tag{31}$$

$$\begin{aligned}\sum_{n=0}^{\infty} u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dF(p,s)\right]\\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty}u_n\right)\right)\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\eta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}A_n\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\zeta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}C_n\right]\right)dp\right],\end{aligned} \tag{32}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} v_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= g_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dG(p,s)\right]\\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty}v_n\right)\right)\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\eta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}B_n\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\mu\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}C_n\right]\right)dp\right].\end{aligned} \tag{33}$$

The first few components can be written as

$$\begin{aligned}u_0 &= f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dF(p,s)\right],\\ v_0 &= g_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dG(p,s)\right],\end{aligned} \tag{34}$$

and

$$\begin{aligned}u_{n+1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty}u_n\right)\right)\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\eta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}A_n\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\zeta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}C_n\right]\right)dp\right],\end{aligned} \tag{35}$$

and

$$\begin{aligned}v_{n+1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty}v_n\right)\right)\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\eta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}B_n\right]\right)dp\right]\\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\zeta\frac{x^\alpha}{\alpha}\sum_{n=0}^{\infty}C_n\right]\right)dp\right].\end{aligned} \tag{36}$$

Provided the double inverse Laplace transform with respect to p and s exist for Equations (34)–(36).

Example 4. *Singular one dimensional conformable fractional coupled Burgers' equation*

$$\begin{aligned}\frac{\partial^\beta u}{\partial t^\beta}-\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\right)-2u\frac{\partial^\alpha}{\partial x^\alpha}u+\frac{\partial^\alpha}{\partial x^\alpha}(uv) &= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}-4e^{\frac{t^\beta}{\beta}}\\ \frac{\partial^\beta v}{\partial t^\beta}-\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}v\right)-2v\frac{\partial^\alpha}{\partial x^\alpha}v+\frac{\partial^\alpha}{\partial x^\alpha}(uv) &= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}-4e^{\frac{t^\beta}{\beta}},\end{aligned} \tag{37}$$

subject to

$$u(x,0)=\left(\frac{x^\alpha}{\alpha}\right)^2,\quad v(x,0)=\left(\frac{x^\alpha}{\alpha}\right)^2. \tag{38}$$

By following similar steps, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n\left(\tfrac{x^\alpha}{\alpha},\tfrac{t^\beta}{\beta}\right) &= \left(\tfrac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4 \\ &-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\tfrac{\partial^\alpha}{\partial x^\alpha}\left(\tfrac{x^\alpha}{\alpha}\tfrac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty} v_n\right)\right)\right]\right)dp\right] \\ &-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[2\tfrac{x^\alpha}{\alpha}\sum_{n=0}^{\infty} A_n\right]\right)dp\right] \\ &+L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\tfrac{x^\alpha}{\alpha}\tfrac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty} C_n\right)\right]\right)dp\right],\end{aligned} \tag{39}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} v_n\left(\tfrac{x^\alpha}{\alpha},\tfrac{t^\beta}{\beta}\right) &= \left(\tfrac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4 \\ &-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\tfrac{\partial^\alpha}{\partial x^\alpha}\left(\tfrac{x^\alpha}{\alpha}\tfrac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty} v_n\right)\right)\right]\right)dp\right] \\ &-L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[2\tfrac{x^\alpha}{\alpha}\sum_{n=0}^{\infty} B_n\right]\right)dp\right] \\ &+L_p^{-1}L_s^{-1}\left[\tfrac{1}{s}\int_0^p\left(L_x^\alpha L_t^\beta\left[\tfrac{x^\alpha}{\alpha}\sum_{n=0}^{\infty} C_n\right]\right)dp\right]\end{aligned} \tag{40}$$

where A_n, B_n and C_n are defined in Equations (14)–(16) respectively. On using Equations (34)–(36) the components are given by

$$\begin{aligned}u_0 &= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4,\ v_0 = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^{\frac{t^\beta}{\beta}} + 4, \\ u_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_0}{\partial x^\alpha}\right) + 2\frac{x^\alpha}{\alpha}u_0\frac{\partial^\alpha u_0}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}(u_0v_0)\right]dp\right] \\ u_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\left(4\frac{x^\alpha}{\alpha}e^{\frac{t^\beta}{\beta}}\right)\right]dp\right] = 4e^{\frac{t^\beta}{\beta}} - 4, \\ v_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha v_0}{\partial x^\alpha}\right) + 2\frac{x^\alpha}{\alpha}v_0\frac{\partial^\alpha v_0}{\partial x^\alpha} - \frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}(u_0v_0)\right]dp\right] \\ v_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\left(4\frac{x^\alpha}{\alpha}e^{\frac{t^\beta}{\beta}}\right)\right]dp\right] = 4e^{\frac{t^\beta}{\beta}} - 4.\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}u_2 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_0}{\partial x^\alpha}\right)\right]dp\right] \\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[2\frac{x^\alpha}{\alpha}\left(u_0\frac{\partial^\alpha u_1}{\partial x^\alpha} + u_1\frac{\partial^\alpha u_0}{\partial x^\alpha}\right)\right]dp\right] \\ &+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}(u_0v_1 + u_1v_0)\right]dp\right] \\ u_2 &= 0, \\ v_2 &= 0.\end{aligned}$$

Thus it is obvious that the self-canceling some terms appear among various components and following terms, then we have,

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = u_0 + u_1 + u_2 + ...,\quad v\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = v_0 + v_1 + v_2 + ...$$

Therefore, the exact solution is given by

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} \text{ and } v\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}.$$

By taking $\alpha = 1$ *and* $\beta = 1$*, the fractional solution becomes*

$$u\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = x^2 e^t,$$
$$v\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = x^2 e^t.$$

3. Conclusions

In this work some properties and conditions for existence of solutions for the conformable double Laplace transform are discussed. We give a solution to the one dimensional regular and singular conformable fractional coupled Burgers' equation by using the conformable double Laplace decomposition method, which is the combination between the conformable double Laplace and Adomian decomposition methods. Further, two examples were given to validate the present method. This method can also be applied to solve some nonlinear time-fractional differential equations having conformable derivatives. The present method can also be used to approximate the solutions of the nonlinear differential equations with the linearization of non-linear terms by using Adomian polynomials.

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