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RESEARCH ARTICLE

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Theory and application for the time fractional Gardner equation with Mittag-Leffler kernel

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ABSTRACT

In this work, the time fractional Gardner equation is presented as a new fractional model for Atangana–Baleanu fractional derivative with Mittag-Leffler kernel. The approximate consequences are analysed by applying a recurrent process. The existence and uniqueness of solution for this system is discussed. To explain the effects of several parameters and variables on the movement, the approximate results are shown in graphics and tables.

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1. Introduction

In the last few years, there has been considerable interest and significant theoretical developments in fractional calculus used in many fields and in fractional differential equations and its applications [1–7]. Abdeljawad and Baleanu [8] used discrete fractional differences with non-singular discrete Mittag-Leffler kernels; Owolabi and Atangana [9] investigated the mathematical analysis and numerical simulation of pattern formation in a subdiffusive multicomponent fractional-reaction diffusion system; in [10], Abdeljawad and Baleanu introduced non-local fractional derivative with Mittag-Leffler kernel; Abdeljawad [11] defined a Lyapunov type inequality for fractional operators with non-singular Mittag-Leffler kernel; Abdeljawad and Al-Mdallal studied the Caputo and Riemann–Liouville type discrete fractional in [12]; in [13], Abdeljawad and Madjidi investigated Lyapunov-type inequalities for fractional difference operators with discrete Mittag-Leffler kernel of order $2 < \alpha < 5/2$; Zhang et al. [14] applied the series expansion process with local fractional operator to find the solutions of transport equations; Khan et al. investigated the advection–reaction diffusion model involving fractional-order derivatives with Mittag-Leffler kernel in [15]; Khan et al. [16] deal with two core aspects of fractional calculus in Caputo sense; Gómez-Aguilar et al. [17] considered three-dimensional cancer model using the Caputo–Fabrizio–Caputo type and with Mittag-Leffler kernel in Liouville–Caputo sense and Khan et al. [18] studied fractional order nonlinear Klein–Gordon equations with the help of the Sumudu decomposition method. Many more research studies related to fractional derivatives can be seen in [19–28].

In this study, we apply the fractional homotopy perturbation transform method (FHPTM) to find numerical solution for a fractional equation. The FHPTM is a combination of HPM and Laplace transform process [19–21]. Besides, the solution is in the form of a convergent series. An iterative process is composed for the shape of the infinite numerical solution. In [22], Kumar et al. analysed the numerical solution for fractional RLW equation by using this method, and, in [23], this method is used to find the series solutions of logarithmic KdV equation.

In this work, we analysed the time fractional Gardner equation (FGE). The Gardner equation is an advantageous example for the definition of interior solitary waves in shallow water , while Buckmaster's equation is applied in thin viscous fluid sheet flows and has been generally examined by several methods (see [24–26]).

This equation is given by [26],

$$
\begin{aligned}
&D_\tau^\alpha p(\varkappa,\tau) + 6(p(\varkappa,\tau) - \varepsilon^2 p(\varkappa,\tau)^2) p_\varkappa(\varkappa,\tau) \\
&\quad + p_{\varkappa\varkappa\varkappa}(\varkappa,\tau) = 0, \\
&\varkappa \in R, \quad \tau > 0, \quad 0 < \alpha \le 1,
\end{aligned}
$$

with the primary situation

$$p(\varkappa,0) = \frac{1}{2} + \frac{1}{2}\tanh\frac{\varkappa}{2}.$$

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The analytical solution to this model, for $\varepsilon = 1$ and $\alpha = 1$, is

$$p(\varkappa,\tau) = \frac{1}{2} + \frac{1}{2}\tanh\frac{\varkappa-\tau}{2}.$$

Some fractional derivatives contain singular kernels. Two of them are Riemann and Caputo and they have their own restrictions due to their singular kernels. However, recently some fractional operators such as Atangana–Baleanu (AB) have defeated these restrictions and deficiencies. In particular, AB used a new fractional derivative with non-singular, non-local and ML kernel and cleared its significant effects [27,29]. In [30], Yadav et al. investigated numerical schemes to compute ABC derivative; Chatibi et al. applied variational calculus involving non-local fractional derivative with Mittag-Leffler kernel in [31] and Koca obtained numerical solutions the fractional partial differential equations with non-singular kernel derivatives in [32].

We analyse FGE for AB fractional operator with Mittag-Leffler kernel due to the great importance of AB fractional derivative in scientific and engineering fields.

The FGE with AB fractional derivative is given as

$$\begin{aligned}&{}^{ABC}_{a}D^{\alpha}_{\tau}p(\varkappa,\tau) + 6(p(\varkappa,\tau) - \varepsilon^2 p(\varkappa,\tau)^2)p_{\varkappa}(\varkappa,\tau)\\ &\quad + p_{\varkappa\varkappa\varkappa}(\varkappa,\tau) = 0, \quad 0 < \alpha \le 1.\end{aligned}$$

The main purpose of this article is to analyse FGE with Mittag-Leffler kernel. The existence and uniqueness analysis of the solutions for FGE has been viewed by using the fixed-point theorem.

In Section 2 of this study, various basic knowledge regarding the AB fractional order derivative are defined. In the next section, FGE with AB fractional derivative is investigated and the existence and uniqueness of solutions for these systems has been investigated by using the fixed-point theorem. In the next section, the FHPTM is applied to construct the solutions of the FGE for AB fractional derivative with Mittag-Leffler kernel. In Section 5, some graphical representations of the solutions are shown to display the accuracy and efficiency of the method. Moreover, some results are pointed out in Section 6.

2. Preliminaries

In this part, we will present the basic definitions and several properties for AB fractional order derivative [8,28,29,33–35].

Definition 2.1: When $p \in H^1(\varkappa, y), \alpha \in [0,1],\ y > \varkappa$ and differentiable, AB fractional order derivative with arbitrary order in the case of Caputo is given as

$$\begin{aligned}&{}^{ABC}_{a}D^{\alpha}_{\tau}(p(\tau))\\ &\quad = \frac{B(\alpha)}{1-\alpha}\int_{\varkappa}^{\tau} p'(s)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(\tau-s)^{\alpha}\right]\mathrm{d}s, \quad (2.1)\end{aligned}$$

where $B(\alpha)$ provides the requirement $B(0) = B(1) = 1$.

Definition 2.2: When $p \in H^1(\varkappa, y), \alpha \in [0,1],\ y > \varkappa$ and is not necessarily differentiable, the AB derivative of arbitrary order in the case of Riemann–Liouville is given as

$$\begin{aligned}&{}^{ABR}_{a}D^{\alpha}_{\tau}(p(\tau))\\ &\quad = \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\varkappa}^{\tau} p(s)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(\tau-s)^{\alpha}\right]\mathrm{d}s. \quad (2.2)\end{aligned}$$

Definition 2.3: When $0 < \alpha < 1$, and $p = p(\tau)$, the fractional integral operator of order α is given as [8]

$$\begin{aligned}&{}^{ABR}_{a}I^{\alpha}_{\tau}(p(\tau))\\ &\quad = \frac{1-\alpha}{B(\alpha)}p(\tau) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\varkappa}^{\tau} p(l)(\tau-l)^{\alpha-1}\,\mathrm{d}l. \quad (2.3)\end{aligned}$$

3. Analysis of the FGE with AB fractional derivative

The FGE is written as: $0 < \alpha < 1$,

$$\begin{aligned}&{}^{ABC}_{a}D^{\alpha}_{\tau}p(\varkappa,\tau) + 6(p(\varkappa,\tau) - \varepsilon^2 p(\varkappa,\tau)^2)p_{\varkappa}(\varkappa,\tau)\\ &\quad + p_{\varkappa\varkappa\varkappa}(\varkappa,\tau) = 0, \quad (3.1)\end{aligned}$$

with the initial condition

$$p(\varkappa,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\varkappa}{2}\right).$$

Using the fractional integral operator produced by AB [8,35] in Equation (3.1), we obtain

$$\begin{aligned}&p(\varkappa,\tau) - p(\varkappa,0)\\ &\quad = \frac{1-\alpha}{B(\alpha)}K(\varkappa,\tau,p)\\ &\qquad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}K(\varkappa,l,p)\,\mathrm{d}l, \quad (3.2)\end{aligned}$$

where

$$\begin{aligned}K(\varkappa,\tau,p) &= -6(p(\varkappa,\tau) - \varepsilon^2 p(\varkappa,\tau)^2)p_{\varkappa}(\varkappa,\tau)\\ &\quad - p_{\varkappa\varkappa\varkappa}(\varkappa,\tau)\\ &= -6p(\varkappa,\tau)p_{\varkappa}(\varkappa,\tau) + 6\varepsilon^2 p(\varkappa,\tau)^2 p_{\varkappa}(\varkappa,\tau)\\ &\quad - p_{\varkappa\varkappa\varkappa}(\varkappa,\tau).\end{aligned}$$

The kernel $K(\varkappa,\tau,p)$ has the Lipschitz state, which justified that the function $p(\varkappa,\tau)$ has upper bound. So,

$$\begin{aligned}&\|K(\varkappa,\tau,p) - K(\varkappa,\tau,P)\|\\ &\quad = \left\|-6\left(pp_{\varkappa} - PP_{\varkappa}\right) + 6\varepsilon^2\left(p^2p_{\varkappa} - P^2P_{\varkappa}\right)\right.\\ &\qquad \left. - \left(p_{\varkappa\varkappa\varkappa} - P_{\varkappa\varkappa\varkappa}\right)\right\|. \quad (3.3)\end{aligned}$$

By applying the triangular inequality of norm in Equation (3.3),

$$\begin{aligned}\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\| &\le -6\,\|pp_{\varkappa}-PP_{\varkappa}\|+6\varepsilon^2\,\|p^2p_{\varkappa}-P^2P_{\varkappa}\| \\ &\quad -\|p_{\varkappa\varkappa\varkappa}-P_{\varkappa\varkappa\varkappa}\| \\ &\le -3\left\|\frac{\partial}{\partial\varkappa}(p^2-P^2)\right\|+2\varepsilon^2\left\|\frac{\partial}{\partial\varkappa}(p^3-P^3)\right\| \\ &\quad -\left\|\frac{\partial^3}{\partial\varkappa^3}(p-P)\right\| \\ &\le -3\delta(a+b)\,\|p-P\| \\ &\quad +2\varepsilon^2\gamma(a^2+ab+b^2)\,\|p-P\|-\kappa^3\,\|p-P\| \\ &\le (-3\delta(a+b)+2\varepsilon^2\gamma(a^2+ab+b^2)-\kappa^3) \\ &\quad \times\|p-P\|. \end{aligned} \tag{3.4}$$

Setting $\Phi=-3\delta(a+b)+2\varepsilon^2\gamma(a^2+ab+b^2)-\kappa^3$, where p and P are limited functions, we can say $\|p\|\le a$, $\|P\|\le b$ and we have

$$\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\|\le\Phi\,\|p-P\|.$$

Then, the Lipschitz state is justified for the kernel $K(\varkappa,\tau,p)$.

3.1. Existence and uniqueness analysis for solutions

In this part, we will present the existence and uniqueness of the solution of FGE for arbitrary order (3.1). From Equation (3.2), we have

$$\begin{aligned}p_{n+1}(\varkappa,\tau)&=\frac{1-\alpha}{B(\alpha)}K(\varkappa,\tau,p_n)\\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-l)^{\alpha-1}K(\varkappa,l,p_n)\,\mathrm{d}l,\end{aligned} \tag{3.5}$$

and $p_0(\varkappa,\tau)=p(\varkappa,0)$.

The difference of the successive terms is represented as follows:

$$\begin{aligned}Y_n(\varkappa,\tau)&=p_n(\varkappa,\tau)-p_{n-1}(\varkappa,\tau)\\ &=\frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,p_{n-1})\\ &\quad-K(\varkappa,\tau,p_{n-2})\}+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\\ &\quad\times\int_0^{\tau}(\tau-l)^{\alpha-1}\{K(\varkappa,l,p_{n-1})\\ &\quad-K(\varkappa,l,p_{n-2})\}\,\mathrm{d}l,\end{aligned} \tag{3.6}$$

where we say that,

$$p_n(\varkappa,\tau)=\sum_{k=0}^{n}Y_k(\varkappa,\tau). \tag{3.7}$$

From Equation (3.7), we get

$$\begin{aligned}&\|Y_n(\varkappa,\tau)\|\\ &\quad=\|p_n(\varkappa,\tau)-p_{n-1}(\varkappa,\tau)\|\\ &\quad=\left\|\begin{array}{l}\dfrac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,p_{n-1})-K(\varkappa,\tau,p_{n-2})\}\\ +\dfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\displaystyle\int_0^{\tau}(\tau-l)^{\alpha-1}\{K(\varkappa,l,p_{n-1})\\ \qquad -K(\varkappa,l,p_{n-2})\}\,\mathrm{d}l\end{array}\right\|\end{aligned} \tag{3.8}$$

Using the triangular inequality in Equation (3.8), we have

$$\begin{aligned}&\|Y_n(\varkappa,\tau)\|\\ &\quad\le\begin{array}{l}\dfrac{1-\alpha}{B(\alpha)}\,\|K(\varkappa,\tau,p_{n-1})-K(\varkappa,\tau,p_{n-2})\|\\ +\dfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\displaystyle\int_0^{\tau}(\tau-l)^{\alpha-1}\left\|\begin{array}{l}K(\varkappa,l,p_{n-1})\\ -K(\varkappa,l,p_{n-2})\end{array}\right\|\,\mathrm{d}l.\end{array}\end{aligned} \tag{3.9}$$

As the kernel justifies the Lipschitz state, they give

$$\begin{aligned}\|Y_n(\varkappa,\tau)\|&\le\frac{1-\alpha}{B(\alpha)}\Phi\,\|p_{n-1}-p_{n-2}\|+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\\ &\quad\times\int_0^{\tau}(\tau-l)^{\alpha-1}\Phi\,\|p_{n-1}-p_{n-2}\|\,\mathrm{d}l,\end{aligned} \tag{3.10}$$

or

$$\begin{aligned}\|Y_n(\varkappa,\tau)\|&\le\frac{1-\alpha}{B(\alpha)}\Phi\,\|Y_{n-1}(\varkappa,\tau)\|+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\\ &\quad\times\Phi\int_0^{\tau}(\tau-l)^{\alpha-1}\,\|Y_{n-1}(\varkappa,\tau)\|\,\mathrm{d}l.\end{aligned} \tag{3.11}$$

Theorem 3.1: *The FGE given as Equation* (3.1) *has the solutions that provide the following conditions that is found with* ξ_0*:*

$$\frac{1-\alpha}{B(\alpha)}\Phi+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Phi\xi_0^{\alpha}<1. \tag{3.12}$$

Proof: Let us consider that the function $p(\varkappa,\tau)$ is limited. Additionally, it has already been stated that the kernel provides the Lipschitz state; hence, from Equation (3.12), Equation (3.11) is written as follows:

$$\|Y_n(\varkappa,\tau)\|\le\left[\frac{1-\alpha}{B(\alpha)}\Phi+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Phi\xi^{\alpha}\right]^n\|p(\varkappa,0)\| \tag{3.13}$$

Therefore, the function

$$p_n(\varkappa,\tau)=\sum_{k=0}^{n}Y_k(\varkappa,\tau) \tag{3.14}$$

exists and is smooth. Now, we examine that the function given in the above equation is the solution of

Equation (3.1). Let us consider

$$p(\varkappa,\tau)-p(\varkappa,0)=p_n(\varkappa,\tau)-D_n(\varkappa,\tau).$$

Therefore, we have

$$\begin{aligned}\|D_n(\varkappa,\tau)\| &= \left\| \begin{array}{c} \frac{1-\alpha}{B(\alpha)}\left[K(\varkappa,\tau,p)-K(\varkappa,\tau,p_{n-1})\right] \\ +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1} \\ \left[\begin{array}{c}K(\varkappa,l,p)\\ -K(\varkappa,l,p_{n-1})\end{array}\right]dl \end{array}\right\| \\ &\le \frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,p)-K(\varkappa,\tau,p_{n-1})\| \\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1} \\ &\quad\times\left\|\begin{array}{c}K(\varkappa,l,p)\\ -K(\varkappa,l,p_{n-1})\end{array}\right\|dl \\ &\le \frac{1-\alpha}{B(\alpha)}\Phi\|p-p_{n-1}\| \\ &\quad+\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\|p-p_{n-1}\|\xi^\alpha. \end{aligned} \tag{3.15}$$

■

By continuing the same process, we have

$$\|D_n(\varkappa,\tau)\|\le\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)\Gamma(\alpha)}\xi^\alpha\right)^{n+1}\Phi^{n+1}d.$$

Then, at $\xi=\xi_0$, we have

$$\|D_n(\varkappa,\tau)\|\le\left(\frac{1-\alpha}{B(\alpha)}+\frac{1}{B(\alpha)\Gamma(\alpha)}\xi_0^\alpha\right)^{n+1}\Phi^{n+1}d,$$

where when $n\to\infty$, we have

$$\|D_n(\varkappa,\tau)\|\to 0.$$

Then, the proof of existence is completed.

Now, we analyse the uniqueness of solution for FGE (3.1). Let us assume that $p(\varkappa,\tau)$ gets another solution for Equation (3.1),

$$\begin{aligned}&p(\varkappa,\tau)-P(\varkappa,\tau)\\ &=\frac{1-\alpha}{B(\alpha)}\{K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\}\\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1}\left\{\begin{array}{c}K(\varkappa,l,p)\\ -K(\varkappa,l,P)\end{array}\right\}dl.\end{aligned} \tag{3.16}$$

Taking the norm on Equation (3.18) gives

$$\begin{aligned}&\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\\ &\le\frac{1-\alpha}{B(\alpha)}\|K(\varkappa,\tau,p)-K(\varkappa,\tau,P)\|\\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^\tau(\tau-l)^{\alpha-1}\left\|\begin{array}{c}K(\varkappa,l,p)\\ -K(\varkappa,l,P)\end{array}\right\|dl.\end{aligned}$$

Since the kernel justifies the Lipschitz states, we have

$$\begin{aligned}&\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\\ &\le\frac{1-\alpha}{B(\alpha)}\Phi\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\\ &\quad+\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\|p(\varkappa,\tau)-P(\varkappa,\tau)\|.\end{aligned} \tag{3.17}$$

This gives

$$\begin{aligned}&\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\\ &\times\left(1-\frac{1-\alpha}{B(\alpha)}\Phi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\right)\le 0.\end{aligned} \tag{3.18}$$

Theorem 3.2: *If the following inequality is provided, there is a unique solution of FGE* (3.1),

$$\left(1-\frac{1-\alpha}{B(\alpha)}\Phi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\right)>0. \tag{3.19}$$

Proof: If the (3.19) condition is satisfied, then

$$\begin{aligned}&\|p(\varkappa,\tau)-P(\varkappa,\tau)\|\\ &\times\left(1-\frac{1-\alpha}{B(\alpha)}\Phi-\frac{1}{B(\alpha)\Gamma(\alpha)}\Phi\xi^\alpha\right)\le 0\end{aligned} \tag{3.20}$$

implies that

$$\|p(\varkappa,\tau)-P(\varkappa,\tau)\|=0.$$

Then, we get

$$p(\varkappa,\tau)=P(\varkappa,\tau).$$

It completes the proof of the uniqueness of the solution for Equation (3.1). ■

4. FHPTM for the time fractional Gardner equation with AB fractional derivative

In this part, first of all, we consider the Laplace transform for FGE with AB fractional operator (3.1) by using FHPTM and use the following initial condition:

$$p(\varkappa,0)=\frac{1}{2}\left(1+\tanh\left(\frac{\varkappa}{2}\right)\right),$$

which yields

$$\begin{aligned}L[p(\varkappa,\tau)]&=\frac{\frac{1}{2}\left(1+\tanh\left(\frac{\varkappa}{2}\right)\right)}{s}-\left(\frac{s^\alpha+\alpha(1-s^\alpha)}{s^\alpha}\right)\\ &\quad L[-6pp_\varkappa+6\varepsilon^2p^2p_\varkappa-p_{\varkappa\varkappa\varkappa}].\end{aligned} \tag{4.1}$$

By using the inverse of Laplace transform in Equation (4.1), we have

$$\begin{aligned}p(\varkappa,\tau)&=\frac{1}{2}\left(1+\tanh\left(\frac{\varkappa}{2}\right)\right)-L^{-1}\\ &\quad\times\left[\begin{array}{c}\left(\frac{s^\alpha+\alpha(1-s^\alpha)}{s^\alpha}\right)\\ L\left[-6pp_\varkappa+6\varepsilon^2p^2p_\varkappa-p_{\varkappa\varkappa\varkappa}\right]\end{array}\right],\end{aligned} \tag{4.2}$$

by applying the HPM, we have

$$\sum_{n=0}^{\infty} z^n p_n = \frac{1}{2}\left(1+\tanh\left(\frac{\varkappa}{2}\right)\right) - z\left(L^{-1}\left[\left(\frac{s^\alpha+\alpha(1-s^\alpha)}{s^\alpha}\right) \times L\left[-6\sum_{n=0}^{\infty} z^n H_n(p) + 6\varepsilon^2\sum_{n=0}^{\infty} z^n K_n(p) - \sum_{n=0}^{\infty} z^n p_{\varkappa\varkappa\varkappa}\right]\right]\right). \quad (4.3)$$

In Equation (4.3), $H_n(p)$ and $K_n(p)$ are He's polynomials as follows:

$$\sum_{n=0}^{\infty} z^n H_n(p) = pp_\varkappa, \quad \sum_{n=0}^{\infty} z^n K_n(p,q) = p^2 p_\varkappa.$$

The initial elements of the He's polynomials are described as

$$\begin{aligned} H_0(p) &= p_0 p_{0_\varkappa}, \\ H_1(p) &= p_0 p_{1_\varkappa} + p_1 p_{0_\varkappa}, \\ H_2(p) &= p_0 p_{2_\varkappa} + p_1 p_{1_\varkappa} + p_2 p_{0_\varkappa}, \\ &\vdots \\ K_0(p) &= p_0^2 p_{0_\varkappa}, \\ K_1(p) &= p_0^2 p_{1_\varkappa} + 2p_0 p_1 p_{0_\varkappa}, \\ K_2(p) &= p_0^2 p_{2_\varkappa} + 2p_0 p_1 p_{1_\varkappa} + 2p_0 p_2 p_{0_\varkappa} + p_1^2 p_{0_\varkappa} \\ &\vdots \end{aligned}$$

Comparing the coefficients of the power of z, we obtain

$$z^0:$$

$$p_0(\varkappa,\tau) = \frac{1}{2}\left(1+\tanh\frac{\varkappa}{2}\right),$$

$$z^1:$$

$$p_1(\varkappa,\tau) = -\frac{1}{8\Gamma(1+\alpha)}(\tau^\alpha\alpha-(-1+\alpha)\,\Gamma(1+\alpha)) \times \operatorname{sec}h\left(\frac{\varkappa}{2}\right)^4(-1+(-4+3\varepsilon^2) \times \cosh\varkappa + 3(-1+\varepsilon^2)\sinh\varkappa),$$

$$z^2:$$

$$\begin{aligned} p_2(\varkappa,\tau) = &-\frac{1}{64\Gamma(1+\alpha)\Gamma(1+2\alpha)} \\ &\times(-2\tau^\alpha(-1+\alpha)\alpha\Gamma(1+2\alpha) \\ &+\Gamma(1+\alpha)(t^{2\alpha}\alpha^2+(-1+\alpha)^2\Gamma(1+2\alpha))) \\ &\times \operatorname{sec}h\left(\frac{\varkappa}{2}\right)^7\Big(-24(-1+\varepsilon^2)\cosh\frac{\varkappa}{2} \\ &-6(22-37\varepsilon^2+15\varepsilon^4)\cosh\frac{3\varkappa}{2} \\ &24\cosh\frac{5\varkappa}{2}-42\varepsilon^2\cosh\frac{5\varkappa}{2} \\ &+18\varepsilon^4\cosh\frac{5\varkappa}{2}+206\sinh\frac{\varkappa}{2} \\ &-204\varepsilon^2\sinh\frac{\varkappa}{2}-129\sinh\frac{3\varkappa}{2} \\ &+222\varepsilon^2\sinh\frac{3\varkappa}{2}-90\varepsilon^4\sinh\frac{3\varkappa}{2} \\ &+25\sinh\frac{5\varkappa}{2}-42\varepsilon^2\sinh\frac{5\varkappa}{2} \\ &+18\varepsilon^4\sinh\frac{5\varkappa}{2}\Big), \end{aligned}$$

$$\vdots$$

Continuing the same process, we obtain $p_n(\varkappa,\tau)$. Then, the solutions can be presented as

$$p(\varkappa,\tau) = p_0(\varkappa,\tau)+p_1(\varkappa,\tau)+p_2(\varkappa,\tau)+\cdots. \quad (4.4)$$

5. Graphical representation of the solutions

The graphical illustrations of the solutions are given in the figures and tables with the aid of Mathematica.

In Table 1, we present the comparison between the approximate results for integer order FGE. The approximate results obtained are fractional AB derivative, familiar fractional Caputo–Fabrizio (CF) derivative and fractional Liouville–Caputo (LC) derivative [29].

Table 1. Comparison of numerical solutions with Liouville–Caputo (LC), Caputo–Fabrizio (CF) and fractional Atangana–Baleanu (AB) derivative at $\varkappa = 2$ for $p(\varkappa,\tau).\epsilon = 1$.

		LC		CF		AB	
τ	$\alpha = 1$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 0.85$	$\alpha = 0.95$
0.01	0.879743	0.882992	0.882139	0.895462	0.886766	0.89622	0.887026
0.02	0.878681	0.884724	0.883377	0.896147	0.887678	0.897363	0.888114
0.03	0.877611	0.886301	0.884572	0.896827	0.888583	0.898404	0.889163
0.04	0.876533	0.88778	0.885736	0.897502	0.88948	0.899379	0.890185
0.05	0.875447	0.889184	0.886874	0.898172	0.890371	0.900306	0.891185
0.06	0.874352	0.89053	0.887991	0.898836	0.891255	0.901193	0.892166
0.07	0.873249	0.891826	0.889088	0.899496	0.892133	0.902048	0.89313
0.08	0.872138	0.893078	0.890168	0.90015	0.893003	0.902874	0.894078
0.09	0.871019	0.894292	0.891231	0.9008	0.893867	0.903675	0.895012
0.1	0.869892	0.895472	0.892279	0.901444	0.894725	0.904453	0.895933

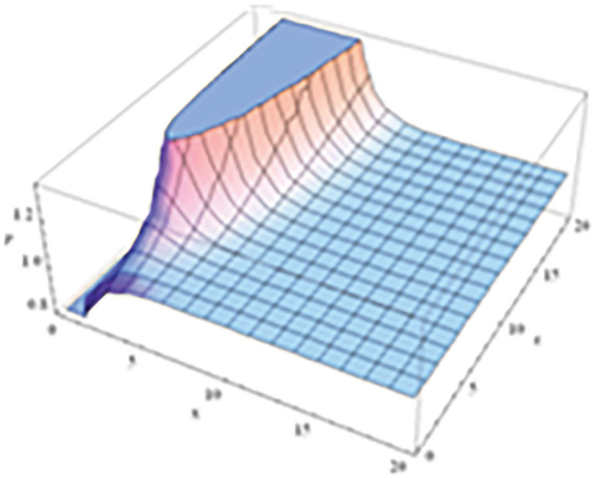

Figure 1. The 3D graphic for the FGE with AB fractional operator when $\alpha = 0.85$. $\epsilon = 1$.

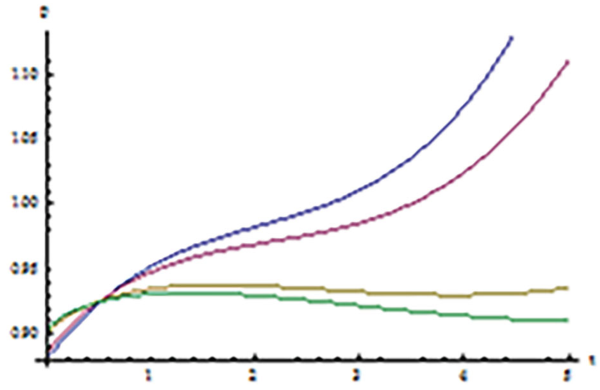

Figure 2. The 2D graphic of the FGE for different value of α when $\varkappa = 2$. $\epsilon = 1$.

In Figure 1, we draw 3D graphic for the FGE with AB fractional operator and in Figure 2, we plot the approximate solution $p(\varkappa, \tau)$ by using FHPTM for $\alpha = 0.75, 0.8, 0.95, 1$. These figures show that the converging of the numerical solutions to the analytical solution connected to the exact error and the order of the solution becomes smaller as the order of the solution is increasing.

6. Final remarks

In this study, the time fractional Gardner equation is analysed for Atangana–Baleanu fractional operator with Mittag-Leffler kernel. We applied the fractional homotopy perturbation transform method for the time fractional Gardner equation with Caputo–Fabrizio, Liouville–Caputo and Atangana–Baleanu fractional-order derivatives. We obtained approximate solutions of the equation with these different fractional-order derivatives. We showed the existence and uniqueness of the solutions for FGE. We compared these approximate solutions with each other via graphical and numerical consequences. From these conclusions, we can say that the FGE with fractional AB derivative is suitable for examining many problems in the fields of science and engineering.

Disclosure statement

No potential conflict of interest was reported by the authors.

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